

IRREDUCIBILITY ALGORITHM FOR THE WEIERSTRASS
POLYNOMIALS OF TWO COMPLEX VARIABLES AND
THE PUISEUX EXPANSIONS: PART[A], PART[B], PART[C]

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(For brevity, Weierstrass polynomials may be written by W-polys.)

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ABSTRACT

It is very fundamental to study irreducible plane curve singularities in algebraic geometry. The contents of the paper consist of three parts, called Part[A], Part[B] and Part[C] with Good Appendix. Our aim is to prove by Part[B] and Part[C] that a complete irreducibility algorithm for the Weierstrass polynomial of two complex variables and the Puiseux expansions in Part[A] can be explicitly and rigorously computable in an elementary way, as follows. For brevity, Weierstrass polynomials may be written by W-polys throughout this paper.

By The 1st Algorithm of Part[A], we can find explicit algorithm for computing a one-to-one function from Family(1)(the family of the standard irreducible W-polys of two complex variables of the recursive type) onto Family(2)(the family of the standard Puiseux expansions).

Then, each element of Family(1) is called the standard Puiseux W-poly of the recursive type throughout this paper, because each element of Family(1) among the irreducible W-polys satisfies the same kind of property as the standard Puiseux expansion among the Puiseux expansion does. The proof for **The 1st Algorithm** will be completely done in **Part[B]**.

By The 2nd Algorithm of Part[A], we can find a complete and explicit algorithm for computing the irreducibility criterion of all the W-polys of two complex variables. To compute the irreducibility criterion for germs of analytic functions of two complex variables, without loss of generality, in **Part[A]** it suffices to find **The 2nd Algorithm** for computing completely irreducible W-polys from all the W-polys of two complex variables by using the Weierstrass preparation theorem and the Weierstrass division theorem. The proof for **The 2nd Algorithm** will be completely done in **Part[C]**.

By The 3rd Algorithm of Part[A] again, we can find explicit algorithm for computing the standard Puiseux expansion which has the same multiplicity sequence as the zero set of any given irreducible W-poly of two complex variables does. Equivalently, as soon as any given W-poly $f(y, z)$ of two complex variables is found to be irreducible in $\mathbb{C}\{y, z\}$ by **The 2nd Algorithm, in Part[A]** we can find **The 3rd Algorithm** for computing the standard Puiseux expansion, denoted by $C(t)$, such that $C(t)$ and the zero set $f(y, z) = 0$ at $0 \in \mathbb{C}^2$ have the same multiplicity sequence, using **The 1st Algorithm**. The proof for **The 3rd Algorithm** will be completely done in **Part[C]**.

In Appendix, as very good applications of Part[A], we can find an elementary explicit algorithm for computing a one-to-one function from Family(2) onto Family(3)(the family of all the multiplicity sequences defining irreducible plane curve singularities) and so on.

Key words and phrases.

the standard Puiseux expansions, the divisors under the standard resolutions, the multiplicity sequences, the standard irreducible W-polys of two complex variables of the recursive type, the Euclidean multiplicity sequence for two positive integers, quasisingularity, the Weierstrass preparation theorem, and the Weierstrass division theorem, the division algorithm for the W-polys.

INTRODUCTION

It is very fundamental to study what problems can be computed in irreducible plane curve singularities in Algebraic geometry? For brevity, Weierstrass polynomials may be written by W-polys throughout this paper.

In general, irreducible plane curve singularities are given by either complex parametrization or germs of analytic functions of two complex variables.

Whenever all the irreducible plane curve singularities are given by the complex parametrization, it has been well-known that the irreducible plane curve singularities can be classified by the standard Puiseux expansions as far as the multiplicity sequences of irreducible plane curve singularities are concerned.

(*) For example, it has been well-known by Theorem 8.3(Enriques-Chisini) that an algorithm for finding a one-to-one correspondence between Family(2)(the family of the standard Puiseux expansions) and Family(3)(the family of all the multiplicity sequences of irreducible plane curves with isolated singularity under the standard resolution) is computable. But, even if it is small, **as an application of the Algorithms in this paper**, we can get a complete, explicit and elementary algorithm for computing a one-to-one function from Family(2) onto Family(3)(See Theorem A.4 and Theorem A.5 of Appendix A).

On the other hand, it has been not known at all that the classification of the family of germs of analytic functions at $0 \in \mathbb{C}^2$, which are irreducible in $\mathbb{C}\{y, z\} = {}_2\mathcal{O}$, the ring of convergent power series at $0 \in \mathbb{C}^2$, with isolated singularity at the origin, can be computed as far as the multiplicity sequences of irreducible plane curve singularities are concerned. Also, it has been not known yet how any equivalence of the irreducible germs of analytic functions in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin and the Puiseux expansions can be computed.

In fact, when I stayed at Purdue University in 1988, S. S. Abhyankar gave me his long manuscript[Ab3], called "Irreducibility criterion for germs of analytic functions of two complex variables", which was his answer to T. C. Kuo's question. It was very difficult for me to find any algorithm for computing the irreducibility criterion from his manuscript. But, for a long time, the theme of the above manuscript has been made me very interesting.

To compute irreducibility criterion for germs of analytic functions of two complex variables, by the Weierstrass preparation theorem it suffices to find explicit algorithms for computing irreducibility criterion of all the W-polys of two complex variables, without using a nonsingular change of coordinates of the point in \mathbb{C}^2 .

The contents of the paper consist of three parts, called Part[A], Part[B] and Part[C]. In more detail, Part[A] is divided by Chapter I and Chapter II; and Part[B] is divided by Chapter III, ..., Chapter VII; and Part[C] is divided by Chapter VIII, ..., Chapter XI.

In order to solve the above problems completely, computationally and rigorously, the aim of this paper is to prove by Part[B] and Part[C] that three explicit algorithms themselves, which are called The 1st Algorithm, The 2nd Algorithm and The 3rd Algorithm in Part[A], can be completely computable rigorously in an elementary way, as follows:

By The 1st Algorithm of Part[A], we can find explicit algorithm for computing the correspondence between the irreducible W-polys of two complex variables and the Puiseux expansions. Rigorously speaking, **in Part[A]** we can find **The 1st Algorithm** for computing a one-to-one function from the family of the standard irreducible W-polys of two complex variables of the recursive type(Family(1)) onto the family of the standard Puiseux expansions(Family(2)). Then, each element of Family(1) is called the standard Puiseux W-poly of the recursive type throughout this paper, because each element of Family(1) among the irreducible W-polys satisfies the same type of property as the standard Puiseux expansion among the Puiseux expansion does. The proof for **The 1st Algorithm** will be completely done **in Part[B]**.

By The 2nd Algorithm of Part[A], we can find a complete and explicit algorithm for computing the irreducibility criterion of all the W-polys of two complex variables.

To compute the irreducibility criterion for germs of analytic functions of two complex variables, without loss of generality, **in Part[A]** it suffices to find **The 2nd Algorithm** for

computing completely irreducible W-polys from all the W-polys of two complex variables by using the Weierstrass preparation theorem and the Weierstrass division theorem. The proof for **The 2nd Algorithm** will be completely done in **Part[C]**.

By The 3rd Algorithm of Part[A] again, we can find explicit algorithm for computing the standard Puiseux expansion which has the same multiplicity sequence as the zero set of any given irreducible W-poly of two complex variables does. Equivalently, as soon as any given W-poly $f(y, z)$ of two complex variables is found to be irreducible in $\mathbb{C}\{y, z\}$ by **The 2nd Algorithm, in Part[A]** we can find **The 3rd Algorithm** for computing the standard Puiseux expansion, denoted by $C(t)$, such that $C(t)$ and the zero set $f(y, z) = 0$ at $0 \in \mathbb{C}^2$ have the same multiplicity sequence, using **The 1st Algorithm**. The proof for **The 3rd Algorithm** will be completely done in **Part[C]**.

Rigorously in Chapter II of Part[A], we will write down completely **The 1st Algorithm, The 2nd Algorithm and The 3rd Algorithm**, by [I], [II] and [III] respectively, and after then using the above algorithms, we will show how to compute irreducibility criterion of all the W-polys $f(y, z) \in \mathbb{C}\{y, z\}$ of two complex variables, as follows: Note that the proofs of these algorithms will be completely finished in Part[B] and in Part[C].

Moreover, we can classify all the irreducible W-polys of two complex variables with respect to the standard Puiseux expansions, using **three explicit algorithms in Part[A]**.

[I] The 1st Algorithm(The algorithm for finding a one-to-one function between Family(1) and Family(2)) in Part[A]

Note that The 1st Algorithm consists of the first half of The 1st Algorithm and the second half of The 1st Algorithm.

[I-1] The first half of The 1st Algorithm(Theorem 1.4) and Example 1.4.1

Theorem 1.4(The first half of The 1st Algorithm: an algorithm for finding a one-to-one function ϕ from Family(1) into Family(2)).

Assumptions Let $g_r \in \mathbb{C}\{y, z\}$ be the standard Puiseux W-poly of the recursive r -type in z in Family(1) satisfying six conditions with the same notations as in Definition 1.1.

Conclusions

By explicit algorithm in (1.4.1), we can compute the standard Puiseux expansion $C_r(t)$ for the curve C such that the zero set $(V(g_r))$ and $C_r(t)$ have the same multiplicity sequence.

(Algorithm 1.4.1 for Theorem 1.4)

$$(1.4.1) \quad C_r(t) := \begin{cases} y = t^n \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$$

such that (1.4.1.1) $n = n_1 n_2 \dots n_r$ and $\alpha_1 = \beta_{1,1} n_2 \dots n_r$,

(1.4.1.2) $\alpha_j = \alpha_{j-1} + \widehat{\Delta}_j n_{j+1} n_{j+2} \dots n_r$,

where $\widehat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} > 0$ for $2 \leq j \leq r$ and $\Delta_1(t) = t$. \square

Example 1.4.1 for Theorem 1.4:

Let the polynomial g_3 in $\mathbb{C}[y, z]$ be given as follows:

$$(1.4.2) \quad g_1 = z^3 + y^4, \quad g_2 = g_1^5 + y^{18} z^2, \quad g_3 = g_2^3 + y^{61} z^1.$$

Then, it is clear by either Definition 1.1 or Definition 1.2 that either g_3 is the standard Puiseux W-poly of the recursive 3rd type or $g_3 \in \text{Family}[1]$. Now, it is easy to compute by (1.4.1) in Algorithm 1.4.1 for Theorem 1.4 that the standard Puiseux expansion for $C_3(t)$ such that the zero set $(V(g_3))$ and $C_3(t)$ have the same multiplicity sequence, is given by

$$(1.4.3) \quad C_3(t) := \begin{cases} y = t^{45} \\ z = t^{60} + t^{66} + t^{71}. \end{cases}$$

[I-2] The second half of The 1st Algorithm(Theorem 1.6) and Example 1.6.3

To use the the algorithm in Theorem 1.6, we need to prove Sublemma 1.5, whose proof is trivial, because there is a well-known theorem for the Euclidean algorithm that for any two positive integers A and B , we find two integers γ and δ such that $\gcd(A, B) = \gamma A + \delta B$, noting that the proof of Sublemma 1.5 just follows from a well-known theorem.

Sublemma 1.5.(Corollary 7.6) for Theorem 1.6.

Assumptions Let $A \geq 2$ and $B \geq 2$ be integers with $\gcd(A, B) = 1$. Let p be an integer such that $p > nAB$ for some integer $n \geq 2$.

Conclusions We can compute a unique pair of two integers s_1 and t_1 such that $p = s_1A + t_1B$ with $0 \leq s_1 < B$ and $t_1 > A$. \square

Theorem 1.6(Theorem 11.4:Algorithm for finding the unique element of Family(1) corresponding to any given standard Puiseux expansion of Family(2)).

Assumptions Let the standard Puiseux expansion of the r -type for the curve $C_r(t)$ be given by

$$(1.6.1) \quad C_r(t) := \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$$

where $2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_r$ and
 $n > d_1 > d_2 > \dots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

Conclusions To compute the standard Puiseux W -poly g_r of the recursive r -type such that the zero set $V(g_r)$ and $C_r(t)$ have the same multiplicity sequence, is to find **explicit algorithm(Algorithm 1.6.2)**, using a finite number $\frac{r(r-1)}{2}$ of Sublemma 1.5(Corollary 7.6), as soon as the standard Puiseux W -poly g_r of the recursive r -type satisfies the same kind of properties and notations as in Definition 1.1, and also as in (1.4.1.1) and (1.4.1.2) of Algorithm 1.4.1 for Theorem 1.4, for notation.

Example 1.6.3 for Theorem 1.6:

Let the parametrization $C(t)$ for the curve C be given as follows:

$$(1.6.2) \quad y = t^{100} \text{ and } z = t^{250} + t^{375} + t^{410} + t^{417} \text{ (See page 517 of [Bri-Kn])}$$

It is easy to compute that $C(t)$ is the standard Puiseux expansion. Then, it can be proved by following explicit algorithm(**Algorithm 1.6.2**) for Theorem 1.6 that we can compute the unique standard irreducible W -poly $f(y, z)$ in z with coefficients in $\mathbb{C}\{y\}$ such that the zero set $f(y, z) = 0$ and the parametrization $C(t)$ have the same multiplicity sequence at the origin, where the construction of $f(y, z) = g_4(y, z)$ is as follows:

$f = g_4 = g_3^5 + y^{300}z g_1 g_2$ where $g_1 = g_1(y, z) = z^2 + y^5$, $g_2 = g_2(y, z) = g_1^2 + y^{10}z$ and $g_3 = g_3(y, z) = g_2^5 + y^{58}g_1$ and $g_4 = g_4(y, z)$ and $f = f(y, z)$.

[II] The 2nd Algorithm(A complete and explicit algorithm for computing the irreducibility criterion of all the W -polys of two complex variables)

Observe the following:

(a) **A generalized representation of irreducible W -polys of two complex variables(The Irreducibility criterion of W -polys of two complex variables) is found by Theorem 1.13 with Theorem 1.14, completely.**

(b) To find a generalized representation of irreducible W -polys of two complex variables in (a), we can use Lemma 1.12, Theorem 15.4(Theorem 1.8: The Division Algorithm for the W -polys), and Theorem 12.0(The generalized standard irreducible W -polys of the recursive r -type).

(c) Note that **The 2nd Algorithm consists of Theorem 1.15 and Corollary 1.15.1.**

As an example for The 2nd Algorithm in Part[A], let $f(y, z)$ be a W -poly of two complex variables, which is given by the following:

$$(*1) \quad f(y, z) = z^{16} + 4y^3z^{14} + \{4y^5 + 6y^6\}z^{12} + \{12y^8 + 4y^9\}z^{10} + \{6y^{10} + 12y^{11} + y^{12}\}z^8 \\ + \{12y^{13} + 4y^{14} + y^{17}\}z^6 + \{6y^{10} + 4y^{15} + y^{20}\}z^4 + \{4y^{18} + y^{22}\}z^2 \\ + y^{24}z + \{y^{20} + y^{29}\}.$$

In this paper, note that The 2nd Algorithm with Theorem 1.15 is much more computational than The 2nd Algorithm with Corollary 1.15.1, and that The 2nd Algorithm with

Corollary 1.15.1 is in the sense of Theorem 1.13 rather than The 2nd Algorithm with Theorem 1.15 is in the sense of Theorem 1.13 **with respect to A generalized representation of irreducible W-polys of two complex variables**(The irreducibility criterion of **W-polys of two complex variables**).

Observe by either Theorem 1.15 or Corollary 1.15.1 that The 2nd Algorithm can be solved as follows:

[II-1] Firstly, using Theorem 1.15 in §1.8, as it is seen in Example 1.10.1 of §1.10, we can compute irreducibility criterion of the above example directly as follows:

(i) We use (Eq.4)(Eq.4.1) with (Eq.4.1.1) and **with proving that (Eq.4.1.2) and (Eq.4.1.3) are true.**

(ii) We use (Eq.5)(Eq.5.1) with (Eq.5.1.1) and **with proving that (Eq.5.1.2) and (Eq.5.1.3) are true.**

(iii) We use (Eq.5)(Eq.5.2) with (Eq.5.2.1) and **with proving that (Eq.5.2.2) and (Eq.5.2.3) are true.**

By (i), (ii) and (iii), it can be proved whether $f(y, z)$ of (Eq.1) is irreducible or not in $\mathbb{C}\{y, z\}$.

[II-2] Secondly, using Corollary 1.15.1 in §1.8, as it is seen in Example 1.10.1 of §1.10, we can compute irreducibility criterion of the above example directly as follows:

(i) We use (Eq.4)(Eq.4.1) with (Eq.4.1.1) and **without mentioning that (Eq.4.1.2) and (Eq.4.1.3) are true or not.**

(ii) We use (Eq.5)(Eq.5.1) with (Eq.5.1.1) and **without mentioning that (Eq.5.1.2) and (Eq.5.1.3) are true or not.**

(iii) We use (Eq.5)(Eq.5.2) with (Eq.5.2.1) and **with proving that (Eq.5.2.2) and (Eq.5.2.3) are true.**

By (i), (ii) and (iii), it can be proved whether $f(y, z)$ of (*1) is irreducible or not in $\mathbb{C}\{y, z\}$.

[III] **The 3rd Algorithm(A complete and explicit algorithm for computing the corresponding standard Puiseux expansion from any irreducible W-poly of two complex variables) in Part[A]**

Note that **The 3rd Algorithm consists of Theorem 1.16 together with Theorem 1.4. As an example for The 3rd Algorithm in Part[A],** let $f(y, z)$ be a W-poly of two complex variables, which is given by the above (*1). By the same method just as above, it can be computed that $f(y, z)$ of (*1) is irreducible in $\mathbb{C}\{y, z\}$.

To find the standard Puiseux expansion $C(t)$ such that that $C(t)$ and the zero set $f(y, z) = 0$ at $0 \in \mathbb{C}^2$ have the same multiplicity sequence, we use **The 1st Algorithm.**

In preparation, (f_1, f) is a generalized representation of f in the sense of Theorem 16.5 because it was just known that $f(y, z)$ of (*1) can be computed as follows:

$$(*2) \quad f = f_1^4 + y^{17}z^2f_1 + y^{24}z + y^{29} = f_1^{d_2} + \sum_{i=0}^{d_2-2} R_{2,i}^{(3)}f_1 \quad \text{with } f_1 = h_{1,3},$$

$$f_1 = z^4 + y^3z^2 + y^5 = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(3)}z^i,$$

where $f_1 = h_{1,3}$ and $R_{2,3}^{(3)} = 0$, $R_{2,2}^{(3)} = 0$, $R_{2,1}^{(3)} = y^{17}z^2$, $R_{2,0}^{(3)} = y^{24}z + y^{29}$ and $R_{1,3}^{(3)} = 0$, $R_{1,2}^{(3)} = y^3$, $R_{1,1}^{(3)} = 0$, $R_{1,0}^{(3)} = y^5$, as we have seen in (Eq.5.2.1) of (Eq.5) in §1.10.

Next, by Theorem 1.16 of §1.9, as it is seen in Example 1.10.2 of §1.10, we can get that $f(y, z) = 0$ and $\phi(y, z) = 0$ have the same multiplicity sequence at $0 \in \mathbb{C}^2$ where $\phi(y, z) = \phi_1^4 + y^{24}z$ where $\phi_1 = \phi_1(y, z) = (z^4 + y^5)$. Since $\phi(y, z)$ is the standard irreducible W-poly of two complex variables of the recursive 2-type, then it is easy to compute by the 1st algorithm(Theorem 1.4) that the standard Puiseux expansion $C(t)$, which defined by $y(t) = t^{16}$ and $z(t) = t^{20} + t^{41}$, and the zero set $\phi(y, z) = 0$ have the same multiplicity sequence at $0 \in \mathbb{C}^2$. So, $C(t)$ and the zero set $f(y, z) = 0$ have the same multiplicity sequence at $0 \in \mathbb{C}^2$.

By **Appendix(as an application of the Algorithms in this paper)**, for the above example $f \in \mathbb{C}\{y, z\}$ as it was seen in an example for The 2nd Algorithm in Part[A], we can show how to compute the following problems (A), (B), (C),(D) with corresponding solutions:

(A): The problem is to compute $g_2 \in \mathbf{Family}[1]$ such that g_2 and f have the same multiplicity sequence. As a unique solution, it was found by The 3rd Algorithm that $g_2 = \phi_2$.

(B): The problem is to compute $C(t) \in \mathbf{Family}[2]$ such that $C(t)$ and f have the same multiplicity sequence. By The 3rd Algorithm in Part[A], the standard Puiseux expansion $C(t)$ can be found by $y(t) = t^{16}$ and $z(t) = t^{20} + t^{41}$.

(C): The problem is to compute $(g_2 \circ \tau_\xi)_{divisor} \in \mathbf{Family}(4)$ such that $(g_2 \circ \tau_\xi)_{divisor} = V^{(\xi)}(g_2) + \sum_{i=1}^{\xi} e_i E_i$ where $\tau = \tau_\xi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ is the composition of a finite number ξ of successive blow-ups π_i at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singularity of $V(g_2)$ where $\tau_\xi^{-1}(0, 0) = E = \cup_{i=1}^{\xi} E_i$, called the exceptional curves of the first kind, is the decomposition into irreducible components.

For any $g_r \in \mathbf{Family}(1)$, it can be proved by Theorem 14.0 of §14 of Part[C] that $(g_r \circ \tau_\xi)_{divisor}$ can be rewritten by

$$(C.1) \quad (g_r \circ \tau_\xi)_{divisor} = V^{(\xi)}(g_r) + \sum_{i=1}^{\lambda_1} e_i E_i + \sum_{i=\lambda_1+1}^{\lambda_2} e_i E_i + \cdots + \sum_{i=\lambda_{r-1}+1}^{\lambda_r} e_i E_i,$$

with $\lambda_r = \xi$ where each e_i is the multiplicity of $g_r \circ \tau_\xi$ along E_i for $1 \leq i \leq \xi = \lambda_r$ and $V^{(\xi)}(g_r)$ is the proper transform of $V(g_r)$ under τ_ξ and write $\lambda_0 = 0$ for notation, satisfying two properties:

Write $L = V^{(\xi)}(g_r) \cup (\cup_{i=1}^{\xi} E_i)$, and for any $A \subset M^{(\xi)}$ \overline{A} is the closure of A in $M^{(\xi)}$.

- (i) For each $i = 1, \dots, \xi$, $E_i \cap (\overline{L - E_i})$ has at most three distinct points under τ_ξ in L .
- (ii)(iia) There is a strictly increasing finite sequence $\{\lambda_i : 1 \leq i \leq r\}$ such that $E_{\lambda_i} \cap (\overline{L - E_{\lambda_i}})$ has exactly three distinct points under τ_{λ_r} in L for each $i = 1, 2, \dots, r$.
- (iib) For any $j \notin \{\lambda_i : 1 \leq i \leq r\}$ where $1 \leq j \leq \xi$, $E_j \cap (\overline{L - E_j})$ has at most two distinct points under τ_{λ_r} in L .

Since $\{e_i : i = 1, 2, \dots, \xi = \lambda_r\}$ is a strictly increasing sequence and $\{\lambda_i : 1 \leq i \leq r\}$ is a unique sequence where $\lambda_i < \lambda_j$ for all $i < j$, using the same properties and notations as in (C.1) it is clear by Definition 9.1 and Definition 9.2 that $\{(g_r \circ \tau_{\lambda_r})_{divisor}\}_{seq.}$ is well-defined by

$$(C.2) \quad \{(g_r \circ \tau_{\lambda_r})_{divisor}\}_{seq.} \equiv \{e_i : i = 1, 2, \dots, \lambda_r\} \equiv \text{Join}\{B_1, B_2, \dots, B_r\}, \text{ as sequence,}$$

where for $j = 1, 2, \dots, r$, the j -th subsequence B_j is written respectively as follows:

$$(C.3) \quad \begin{aligned} B_1 &= \{e_i : i = 1, 2, \dots, \lambda_1\} \quad \text{with } 1 < \lambda_1 < \lambda_2 < \cdots < \lambda_r, \\ B_j &= \{e_{\lambda_{j-1}+i} : i = 1, 2, \dots, (\lambda_j - \lambda_{j-1})\} \quad \text{for } j = 2, 3, \dots, r. \end{aligned}$$

So, **Family(4)** of(C.1) can be identified with **Family(4)_{seq.}** of (C.4)

$$(C.4) \quad \underline{\mathbf{Family}(4)_{seq.} = \{\{(g_r)_{divisor}\}_{seq.} : g_r \in \mathbf{Family}(1) \text{ where } \tau_\xi : M \rightarrow \mathbb{C}^2 \text{ is the standard resolution of the singularity of } V(g_r)\}}.$$

For example, if $f = g_2$, the problem is to compute $(g_2 \circ \tau_\xi)_{divisor} = V^{(\xi)}(g_2) + \sum_{i=1}^{\lambda_1} e_i E_i + \sum_{i=\lambda_1+1}^{\lambda_2} e_i E_i$ by (C.1). So, $\{(g_2 \circ \tau_{\lambda_2})_{divisor}\}_{seq.} = \{e_i : i = 1, 2, \dots, \lambda_2\} = \text{Join}\{B_1, B_2\}$, where $B_1 = \{e_i : 1 \leq i \leq \lambda_1 = 5\} = \{16, 20, 40, 60, 80\}$ with $e_5 = 80$, and $B_2 = \{e_i : \lambda_1 + 1 = 4 \leq i \leq \lambda_2 = 14\} = \{84, 88, 92, 96, 100, 101, 202, 303, 404\}$ with $e_{14} = 404$.

(D): The problem is to compute $\text{Multiseq}(g_r(y, z)) \in \mathbf{Family}[3]$ for any $g_r \in \mathbf{Family}(1)$ as we have seen in (C), the problem is to compute $\text{Multiseq}(g_r(y, z)) \in \mathbf{Family}[3]$ where $\text{Multiseq}(g_r(y, z))$ is defined by the multiplicity sequence of the zero set $V(g_r(y, z))$ with isolated singularity under the standard resolution.

For notation, we may assume that $g_r \in \text{Family}(1)$ satisfies the same properties and notations as in (C.1). For any given $g_r \in \text{Family}(1)$, note that $\{\lambda_j : 1 < \lambda_1 < \dots < \lambda_r, 1 \leq j \leq r\}$ is a uniquely determined finite strictly increasing sequence in (C.1). By the same way as we have used in (C.4), $\text{Multiseq}(V(g_r))$ in $\text{Family}(3)$ is well-defined by

$$(D.1) \quad \text{Multiseq}(V(g_r)) \equiv \{c_i : i = 1, 2, \dots, \lambda_r\} \equiv \text{Join}\{P_1, P_2, \dots, P_r\}, \quad \text{as sequence}$$

for $j = 1, 2, \dots, r$, each subsequence P_j of which can be uniquely written as follows:

$$(D.2) \quad \begin{aligned} P_1 &= \{c_i : i = 1, 2, \dots, \lambda_1\} \quad \text{and} \\ P_j &= \{c_{\lambda_{j-1}+i} : i = 1, 2, \dots, (\lambda_j - \lambda_{j-1})\} \quad \text{for } j = 2, 3, \dots, r. \quad \square \end{aligned}$$

By Theorem 1.4 and Theorem 1.6, for any given standard Puiseux W-poly g_r of the recursive r -type we can find an algorithm how to compute the standard Puiseux expansion $C_r(t)$ such that the zero set $V(g_r)$ and $C_r(t)$ have the same multiplicity sequence. By Theorem 10.1 or Appendix, it is easy to find an algorithm for computing $\text{Multiseq}(C_r(t)) \in \text{Family}[3]$ where $\text{Multiseq}(C_r(t))$ is defined by the multiplicity sequence of the zero set of the standard Puiseux expansion $C_r(t)$ with isolated singularity under the standard resolution. Then, it is clear by Definition 9.1, Definition 9.2 and Theorem 10.1 that $\text{Multiseq}(g_2(y, z)) = \{[16, 20], [4, 21]\}$.

(E): the problem is to compute $(g_r \circ \tau_{\lambda_r})_{\text{singular part of divisor}} \in \text{Family}(5)$ for any $g_r \in \text{Family}(1)$, called the new family defined by this paper only, such that $(g_r \circ \tau_{\lambda_r})_{\text{singular part of divisor}}$ can be written as follows:

$$(E.1) \quad (g_r \circ \tau_{\xi})_{\text{singular part of the divisor}} = \sum_{i=0}^{r-1} \{e_{\lambda_i+1} E_{\lambda_i+1} + e_{\lambda_i+s_i} E_{\lambda_i+s_i}\}$$

where $\lambda_0 = 0$, satisfying the following properties: For convenience of notation, let $\Omega^{(1)} = \bigcup_{i=1}^{\lambda_1} E_i$, $\Omega^{(2)} = \bigcup_{i=\lambda_1+1}^{\lambda_2} E_i, \dots$, and $\Omega^{(r)} = \bigcup_{i=\lambda_{r-1}+1}^{\lambda_r} E_i$.

Property(1) For any $E_t \subset \Omega^{(w+1)}$ with $w+1 \leq r$, $E_t \cap \overline{\Omega^{(w+1)} - E_t}$ have at most two distinct points in $\Omega^{(w+1)}$.

Property(2) Let w be with $w+1 \leq r$, and $\Omega^{(w+1)} = \bigcup_{i=\lambda_w+1}^{\lambda_{w+1}} E_i$.

(i) There are two distinct exceptional curves of the first kind in $\Omega^{(w+1)}$, denoted by E_{λ_w+1} and $E_{\lambda_w+s_w}$ with $1 < s_w \leq \lambda_{w+1} - \lambda_w$, each of which satisfies the following property: Note that $\Omega^{(w+1)} = \bigcup_{i=\lambda_w+1}^{\lambda_{w+1}} E_i$.

(ia) $E_{\lambda_w+1} \cap \overline{\Omega^{(w+1)} - E_{\lambda_w+1}}$ has one and only one intersection point in $\Omega^{(w+1)}$.

(ib) $E_{\lambda_w+s_w} \cap \overline{\Omega^{(w+1)} - E_{\lambda_w+s_w}}$ has one and only one intersection point in $\Omega^{(w+1)}$.

(ii) $E_{\lambda_w+j} \cap \overline{\Omega^{(w+1)} - E_{\lambda_w+j}}$ has two distinct intersection points in $\Omega^{(w+1)}$ for any j where $1 < j \leq \lambda_{w+1} - \lambda_w$ and $j \neq s_w$.

In Appendix C, it will be proved that we can find an explicit algorithm for finding a one-to-one function from $\text{Family}(2)$ onto $\text{Family}(5)$ (the family of the singular parts of the divisors defined by the total transforms of irreducible plane curves with isolated singularity under the standard resolution), without mentioning anything about $\text{Family}[4]$, because $(g_r \circ \tau_{\lambda_r})_{\text{singular part of divisor}}$ can be identified with a strictly increasing sequence $\{e_1, e_{s_1}; e_{\lambda_1+1}, e_{\lambda_1+s_2}; \dots; e_{\lambda_{r-1}+1}, e_{\lambda_{r-1}+s_r}\}$, consisting of $2r$ elements in $\{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}}$. For any $g_2 \in \text{Family}[1]$, we can compute $(g_2 \circ \tau_{\xi})_{\text{singular part of the divisor}}$ as follows:

$$(E.2) \quad (g_2 \circ \tau_{\xi})_{\text{singular part of the divisor}} = V^{(\xi)}(g_2) + 16E_1 + 20E_{s_1} + 84E_{\lambda_1+1} + 101E_{\lambda_1+s_2}$$

by Appendix C where $s_1 = 2$ and $s_2 = 6$. Note that $\lambda_1 = 5$ and $\lambda_2 = 14$.

Conversely, if $\{(g_2 \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}}$ is given by a sequence $Q = \{16, 20, 84, 101\}$, then we can compute by Appendix C that the standard Puiseux expansion defined by $y = t^{16}$ and $z = t^{20} + t^{41}$ and Q are equivalent as far as the multiplicity sequences are concerned.

Part[A](Chapter I, Chapter II)

How to use a complete and explicit irreducibility algorithm for the W-polys of two complex variables without proofs and related topics in the Puiseux expansions

Chapter I: In preparation for the representation of irreducibility algorithms for the W-polys of two complex variables

§1.0. The definition of the new terminology for three explicit algorithms for all the irreducible W-polys of two complex variables and some remarks

To solve all the problems in **Introduction** completely, the aim in §1.0 is to represent explicit algorithms, consisting of three algorithms, completely, computationally and rigorously in an elementary way without proofs, as follows:

By Part[A], we can find The 1st Algorithm for computing a one-to-one map from the family of the standard irreducible W-polys of two complex variables of the recursive type(Family(1)) onto the family of the standard Puiseux expansions(Family(2)).

In §1.4 the aim is how to use The 1st Explicit Algorithm for computing a one-to-one map between Family(1) and Family(2) by Theorem 1.4 and Theorem 1.6 of §1.4. Note by Definition 1.1 of §1.1 that each element of Family(1) is called the standard Puiseux W-poly of the recursive type throughout this paper.

By Part[A] again, we can find The 2nd Algorithm for computing irreducibility criterion of all the W-polys of two complex variables in Theorem 1.15 of §1.8, by using Theorem 1.8(The Division Algorithm for the W-polys) of §1.5.

To compute irreducibility criterion for germs of analytic functions of two complex variables, without loss of generality, we can find The 2nd Algorithm for computing completely irreducible Weierstrass polynomials from all the Weierstrass polynomials of two complex variables by the Weierstrass preparation theorem and the Weierstrass division theorem. The proof of The 2nd Algorithm can be given by Theorem 16.6 together with Proposition 16.7 and Proposition 16.8 in Part[B].

By Part[A] again, we can find The 3rd Algorithm for computing the standard Puiseux expansion which has the same multiplicity sequence as the zero set of any irreducible W-poly in $\mathbb{C}\{y, z\}$ does at $0 \in \mathbb{C}^2$.

In §1.8, as soon as any W-poly $f(y, z)$ of two complex variables is found to be irreducible in $\mathbb{C}\{y, z\}$ by The 2nd Algorithm, in §1.9 the aim is how to use The 3rd Algorithm for computing the standard Puiseux W-poly of the recursive type(or the standard Puiseux expansion) directly which has the same multiplicity sequence at $0 \in \mathbb{C}^2$ as the analytic variety $\{f(y, z) = 0\}$ does at $0 \in \mathbb{C}^2$, by The 1st Algorithm. Thus, the solution of The 3rd Algorithm can be given by Theorem 1.16 of §1.9.

Remark 1.0. The complete proof of three explicit algorithms will be done later in the other sections of this paper.

(a) In order to succeed in the computation of The 1st Algorithm, it is very interesting and important by Definition 1.1 that we can define the new terminology, the standard irreducible(Puiseux) W-polys of two complex variables of the recursive type, which will be shown to be well-defined by Theorem 1.3 of §1.3.

(b) The proofs of Theorem 1.4 and Theorem 1.6 will be done by Theorem 11.2 and Theorem 11.4 of §11, respectively. The proofs of Theorem 1.15 and Theorem 1.16 will be done by Theorem 16.6 and Theorem 17.1, respectively.

(c) To find The 1st Algorithm with proof, we use Theorem A([K2]), instead of the well-known theorem(Theorem B), because Theorem B is not applicable to compute The 1st algorithm in this paper:

Theorem A: Whenever any two irreducible parametrization have the same Puiseux pairs(that is, the same standard Puiseux expansions) by a nonsingular change of the parametrization, then they have the same multiplicity sequences, and conversely.

Theorem B: As far as arbitrary Puiseux expansion of irreducible plane curve singularities is concerned, any two irreducible plane curve singularities have the same topological types if and only if they have the same Puiseux pairs.

Note that Theorem B was proved by K. Brauner[Br], W.Burau[Bu] and O.Zariski[Z1] and that the proof of Theorem A was done by Theorem 8.8([K2]) and Theorem 8.10([K2])), using σ -process only, without using Theorem B.

(d) To find three algorithms with proofs, we have not studied the theory of Newton polygon. Then we did not mention and use the definition of the Newton polygon at all throughout this paper. In preparation for the representation of the 2nd and the 3rd algorithms in §1.6, it is very important to say that Theorem 1.8(The Division Algorithm for the W-polys) of §1.5 can have an important role of the 2nd and the 3rd algorithms in §1.6, which will be shown by Theorem 15.4 of §15 and Theorem 16.6 of §16. \square

§1.1. The definition for the standard irreducible W-polys of two complex variables of the recursive type(the standard Puiseux W-polys of two complex variables of the recursive type)

Throughout this paper, the family and the set have the same meaning. In preparation for the definition of the new terminology in this section, we are going to review the definition of irreducible W-polys of n complex variables at $0 \in \mathbb{C}^n$, as follows:

Let $\mathbb{C}\{z_1, z_2, \dots, z_n\}$ be the ring of either convergent power series or germs of analytic functions at $0 \in \mathbb{C}^n$, and ${}_n\mathcal{O} = {}_n\mathcal{O}_0$ denote the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. As the ring, $\mathbb{C}\{z_1, z_2, \dots, z_n\}$ can be identified with ${}_n\mathcal{O}$. Also, let $\mathbb{C}[z_1, z_2, \dots, z_n]$ be the ring of polynomials of n complex variables.

(i) $f \in \mathbb{C}\{z_1, z_2, \dots, z_{n-1}\}[z_n] = {}_{n-1}\mathcal{O}[z_n]$ is called a W-poly of degree ν in z_n with coefficients in $\mathbb{C}\{z_1, z_2, \dots, z_{n-1}\} = {}_{n-1}\mathcal{O}$ if $f(z_1, z_2, \dots, z_n) = z_n^\nu + a_1 z_n^{\nu-1} + \dots + a_{\nu-1} z_n + a_\nu$ where the $a_j = a_j(z_1, z_2, \dots, z_{n-1})$ are holomorphic functions at $0 \in \mathbb{C}^{n-1}$ and $a_j(0) = 0$ for $1 \leq j \leq \nu$. Note that ${}_{n-1}\mathcal{O}[z_n] \subseteq {}_n\mathcal{O}$.

(ii) A nonunit $f \in {}_n\mathcal{O}$ is said to be reducible in ${}_n\mathcal{O}$ if it can be written as a product $f = g_1 g_2$ where g_1 and g_2 are not unit in ${}_n\mathcal{O}$, and a nonunit $f \in {}_n\mathcal{O}$ that is not reducible in ${}_n\mathcal{O}$ is called irreducible in ${}_n\mathcal{O}$.

Definition 1.1. Let N_0 be the set of nonnegative integers and N_0^k be its k -dimensional copy. Let r be an arbitrary positive integer.

$g_r \in \mathbb{C}\{y, z\}$ is called the standard irreducible(Puiseux) W-poly of the recursive r-type in z if there are sequences $\{X_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$ satisfying the following six conditions:

Six conditions are denoted by The 1st Cond⁽⁰⁾, ..., The 6-th Cond⁽⁰⁾ for notation.

The 1st Cond⁽⁰⁾ Let $\{X_j : j = 1, 2, \dots, r\}$ with $X_j \subset N_0$ be defined as follows:

- (1) (1a) $X_1 = \{n_1, \beta_{1,1}\}$ with $n_1 \geq 2$ and $\beta_{1,1} \geq 1$.
- (1b) $X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with $n_j \geq 2$ where $j = 2, \dots, r$.

If $j \geq 2$, assume that at least one of $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}$ is nonzero.

The 2nd Cond⁽⁰⁾ For each $j = 1, 2, \dots, r$, let $g_j = g_j(y, z)$ be in $\mathbb{C}\{y, z\}$, each of which is defined by the following way:

- (2) (2a) $g_1 = z^{n_1} + y^{\beta_{1,1}}$.
- (2b) $g_j = g_{j-1}^{n_j} + y^{\beta_{j,1}} z^{\beta_{j,2}} g_1^{\beta_{j,3}} \dots g_{j-2}^{\beta_{j,j}}$ with $g_{-1} = y$ and $g_0 = z$, where $j = 2, \dots, r$.

The 3rd Cond⁽⁰⁾ Let $\{\Delta_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r\}$ be a sequence such that each Δ_k is an integer-valued function defined by the following:

- (3) (3a) $\Delta_1(t) = t$ for each $t \in N_0$.
- (3b) $\Delta_j(t_k)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, r$.

The 4-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq r$.

(4) (4a) $\Delta_1(\beta_{1,1}) = \beta_{1,1} > 0$ with $n_1 \geq 2$.

(4b) $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ where $j = 2, \dots, r$.

The 5-th Cond⁽⁰⁾ The following inequalities hold:

(5) (5a) $\gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq r$.

The 6-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq r$.

(6) (6a) $2 \leq n_1 < \beta_{1,1}$.

(6b) $n_j \geq 2$, $\beta_{j,1} > 0$, and $0 \leq \beta_{j,k} < n_{k-1}$ for $2 \leq j \leq r$ and $2 \leq k \leq j$. \square

Remark 1.1.1.

(i) It is clear that $g_r(y, z)$ of Definition 1.1 is a W-poly of degree $n_1 n_2 \cdots n_r$ in z with coefficients in $\mathbb{C}\{y\}$, without using The 5-th Cond⁽⁰⁾.

(ii) It can be proved by Theorem 5.0 that $g_r \in \mathbb{C}\{y, z\}$ of Definition 1.1 is irreducible in $\mathbb{C}\{y, z\}$, without using The 6-th Cond⁽⁰⁾.

(iii) It can be proved by Theorem 5.0 that $g_r(y, z)$ of Definition 1.1 is an irreducible W-poly of degree $n_1 n_2 \cdots n_r$ in z with coefficients in $\mathbb{C}\{y\}$. In preparation for computing The 1st Algorithm, the aim is how to construct the standard Puiseux W-polys from the irreducible W-polys as we have seen in the construction of the standard Puiseux expansions from the Puiseux expansions. \square

§1.2. The new terminology and notations in preparation for studying three families with equivalence relations

In preparation for a good success of the main aim, we define three Families, that is, Family(1), Family(2), Family(3) with equivalence relations, respectively, as follows: Throughout this paper, the family and the set have the same meaning.

Definition 1.2. [I](1) For brevity, let **Family(0)** be the 0-th family, consisting of all the convergent power series $f \in \mathbb{C}\{y, z\}$ such that f is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $0 \in \mathbb{C}^2$.

(2) For convenience of notation, we use the following:

For any two finite sequences $A = \{a_i \in \mathbb{C} : 1 \leq i \leq n\}$ and $B = \{b_i \in \mathbb{C} : 1 \leq i \leq m\}$, it is said that A and B are equivalent if and only if either $a_i = b_i$ for $i = 1, 2, \dots, n = m$ or $A \equiv B$ as sequence. If A and B are not equivalent, we write $A \not\equiv B$ as sequence.

[II](1) **Family(1)** is called the first family, consisting of all the standard Puiseux W-polys $f \in \mathbb{C}\{y, z\}$ of the recursive type with isolated singularity at $0 \in \mathbb{C}^2$, denoted by

(1.2.1) Family(1) = $\{f \text{ is arbitrary standard Puiseux W-poly of the recursive r-type: } f \in \text{Family(0) and } r \text{ are arbitrary positive integers}\}$.

Now, in preparation for finding an equivalence relation on Family(1) by (1c) later, it remains to define arbitrary two elements g_r and ϕ_ρ in Family(1) by (1a) and (1b), as follows:

(1a) $g_r \in \mathbb{C}\{y, z\}$ is called the standard Puiseux W-poly in $\mathbb{C}[y, z]$ of the recursive r-type if $g_r \in \mathbb{C}\{y, z\}$ satisfies the same properties and notations as we have seen in Definition 1.1.

(1b) By the same method as we have seen in either Definition 1.1 or (1a), another element $\phi_\rho \in \text{Family(1)}$ is called the standard Puiseux W-poly $\phi_\rho \in \mathbb{C}\{y, z\}$ of the recursive ρ -type at $(0, 0) \in \mathbb{C}^2$, if there are sequences $\{W_k : k = 1, 2, \dots, \rho\}$ with $W_k \subset N_0$, $\{\phi_k : k = 1, 2, \dots, \rho\}$ with $\phi_k \in \mathbb{C}\{y, z\}$ and $\{\omega_k : N_0^k \rightarrow N_0 \text{ is an integer-valued function for } k = 1, 2, \dots, \rho\}$ satisfying six conditions:

Six conditions for ϕ_ρ are denoted by The 1st Cond⁽⁰⁾, ..., The 6-th Cond⁽⁰⁾.

The 1st Cond⁽⁰⁾ Let $\{W_j : j = 1, 2, \dots, \rho\}$ with $W_j \subset N_0$ be defined as follows:

(1) (1.1) $W_1 = \{\ell_1, \delta_{1,1}\}$ with $\ell_1 \geq 2$ and $\delta_{1,1} \geq 1$.

(1.2) $W_j = \{\ell_j, \delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}\}$ with $\ell_j \geq 2$ where $j = 2, \dots, \rho$.

If $j \geq 2$, then assume that at least one of $\delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}$ is nonzero.

The 2nd Cond⁽⁰⁾ For each $j = 1, 2, \dots, r$, let $\phi_j = \phi_j(y, z)$ be in $\mathbb{C}\{y, z\}$, each of which is defined by the following way:

- (2) (2.1) $\phi_1 = z^{\ell_1} + y^{\delta_{1,1}}$.
 (2.2) $\phi_j = \phi_{j-1}^{\ell_j} + y^{\delta_{j,1}} z^{\delta_{j,2}} \phi_1^{\delta_{j,3}} \dots \phi_{j-2}^{\delta_{j,j}}$ where $j = 2, \dots, \rho$.

The 3rd Cond⁽⁰⁾ Let $\{\omega_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, \rho\}$ be a sequence such that each ω_k is an integer-valued function defined by the following:

- (3) (3.1) $\omega_1(t) = t$ for each $t \in N_0$.
 (3.2) $\omega_j(t_k)_{k=1}^j = t_j \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1} + \ell_{j-1} \omega_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, \rho$.

The 4-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq \rho$.

- (4) (4.1) $\omega_j(\delta_{1,1}) = \delta_{1,1} > 0$ with
 (4.1) $\omega_j(\delta_{j,k})_{k=1}^j > \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq \rho$.

The 5-th Cond⁽⁰⁾ The following inequalities hold:

- (5) (5.1) $\gcd(\ell_j, \omega_j(\delta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq \rho$.

The 6-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq \rho$.

- (6) (6.1) $2 \leq \ell_1 < \delta_{1,1}$.
 (6.2) $\ell_j \geq 2$, $\delta_{j,1} > 0$, and $0 \leq \delta_{j,k} < \ell_{k-1}$ for $2 \leq j \leq \rho$ and $2 \leq k \leq j$.

(1c) For any standard Puiseux W-poly $g_r \in \mathbb{C}\{y, z\}$ of the recursive r-type in (1a) and any standard Puiseux W-poly $\phi_\rho \in \mathbb{C}\{y, z\}$ of the recursive ρ -type in (1b), it is said that g_r and ϕ_ρ are equivalent, denoted by $g_r \equiv \phi_\rho$ in Family(1), if the following are satisfied:

- (1.2.2) $n_j = \ell_j$ for each $j = 1, 2, \dots, r = \rho$, and
 $\beta_{j,k} = \delta_{j,k}$ for each $j = 1, 2, \dots, r$ and for all $k = 1, 2, \dots, j$.

(2) Family(2) is the 2nd family, consisting of all the irreducible curves with the standard Puiseux expansions, denoted by

- (1.2.3) Family(2) = $\{C_r(t) : C_r(t) \text{ is the standard Puiseux expansion of the } r\text{-type for any } r \in \mathbb{N}\}$.

In more detail, we define the standard Puiseux expansion $C_r(t)$ of the r -type by (1.2.4), and also the standard Puiseux expansion $C_s(t)$ of the s -type by (1.2.5), respectively. After then, we will define an equivalence relation for any two standard Puiseux expansions $C_r(t)$ of the r -type and $C_s(t)$ of the s -type by (1.2.6) of (2c).

Note by Definition 8.1 that the parametrization $C(t)$ for arbitrary irreducible plane curve C can be defined by $y(t) = t^n$ and $z(t) = c_1 t^{k_1} + c_2 t^{k_2} + \dots = c_1 t^{k_1} (1 + H(t))$, where $1 < n$, $1 < k_1 < k_2 < \dots$, and the c_i are nonzero complex numbers and $H(t)$ is just the substitution.

(2a) The standard Puiseux expansion $C_r(t)$ of the r -type for the curve C is as follows:

- (1.2.4) $C_r(t) := \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$
 where $2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_r$ and
 $n > d_1 > d_2 > \dots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

(2b) The standard Puiseux expansion $C_s(t)$ of the s -type for the curve C' is as follows:

- (1.2.5) $C_s(t) := \begin{cases} y = t^m, \\ z = t^{\beta_1} + t^{\beta_2} + \dots + t^{\beta_s}, \end{cases}$
 where $2 \leq m < \beta_1 < \beta_2 < \dots < \beta_s$ and
 $m > \bar{d}_1 > \bar{d}_2 > \dots > \bar{d}_s = 1$ with $\bar{d}_i = \gcd(m, \beta_1, \dots, \beta_i)$, $1 \leq i \leq s$.

(2c) Whenever the standard Puiseux expansions $C_r(t)$ of the r -type in (1.2.4) and $C_s(t)$ of the s -type in (1.2.5) are chosen arbitrary, then it is said that $C_r(t)$ and $C_s(t)$ are equivalent if the following conditions are satisfied:

$$(1.2.6) \quad n = m \quad \text{and} \quad \alpha_i = \beta_i \quad \text{for} \quad i = 1, 2, \dots, r = s.$$

(3) **Family(3)** is the 3rd family, consisting of all the multiplicity sequences of irreducible plane curves with isolated singularity under the standard resolution, denoted by

$$(1.2.7) \quad \text{Family}(3) = \{\text{Multiseq}(V(f)) : f \in \text{Family}(0) \text{ and } f \text{ is irreducible in } {}_2\mathcal{O}\}$$

where for any $f \in \text{Family}(0)$, we define $\text{Multiseq}(V(f))$ by the multiplicity sequence of f , and next an equivalence relation for any two multiplicity sequences in $\text{Family}(3)$, as follows:

(3a) For any $f \in \text{Family}(0)$, to define the multiplicity sequence of f , we may assume for notation that $\tau_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ with $\tau = \tau_\xi$ is the standard resolution of the singularity of $V(f)$ as we have used for $f \in \text{Family}(0)$, which is the composition of a finite number ξ of successive blow-ups π_i at the origin in \mathbb{C}^2 . Let c_0 be the multiplicity of this curve germ f at this point. If we blow up once, then we again find at most one singularity. Let c_1 be the multiplicity of the curve of the germ blown up once, c_2 be the multiplicity of the curve of the germ blown up twice, and continue to the standard resolution. The sequence ends with a sequences of ones. The sequences of these multiplicities, $\{c_0, c_1, \dots, c_{\xi-1}\}$, where the last one is not counted, is then the multiplicity sequence.

Then, $\text{Multiseq}(V(f))$ is written as follows:

$$(1.2.8) \quad \text{Multiseq}(V(f)) = \{c_i : i = 0, 1, \dots, \xi - 1\}.$$

(3b) For any f and g in $\text{Family}(0)$, an equivalence relation for any two multiplicity sequences $\text{Multiseq}(V(f))$ and $\text{Multiseq}(V(g))$ in $\text{Family}(3)$ is defined as follows:

$$(1.2.9) \quad f \text{ and } g \text{ in } \text{Family}(0) \text{ have the same multiplicity sequence}$$

$$\iff \text{either } f \stackrel{\text{multiseq}}{\sim} g \text{ or } V(f) \stackrel{\text{multiseq}}{\sim} V(g) \text{ at } 0 \in \mathbb{C}^2$$

$$\iff \text{Multiseq}(V(f)) \equiv \text{Multiseq}(V(g)) \text{ as sequence.} \quad \square$$

§1.3. What does an equivalence relation of any two elements in **Family(1)**(the family of the standard Puiseux W-polys in $\mathbb{C}\{y, z\}$ of the recursive type) mean?

In order to succeed in the computation of The 1st Algorithm for finding a one-to-one correspondence between $\text{Family}(1)$ and $\text{Family}(2)$, the aim is to study the equivalent class of the standard Puiseux W-polys in $\text{Family}(1)$ with respect to the multiplicity sequences.

Theorem 1.3. Assumptions *Let r and ρ be arbitrary positive integers. By the same way as in Definition 1.1, let g_r be the standard irreducible W-poly in z of the recursive r -type, satisfying the same properties and notations as in Definition 1.1. Also, let ϕ_ρ be the standard irreducible W-poly in z of the recursive ρ -type, satisfying the same properties and notations as in Definition 1.2.*

Conclusions *Then, we have the following:*

$$(1.3.1) \quad g_r \text{ and } \phi_\rho \text{ have the same multiplicity sequence.}$$

$$\iff n_j = \ell_j \text{ and } \beta_{j,k} = \delta_{j,k} \text{ for all } j = 1, 2, \dots, r = \rho \text{ and all } k = 1, 2, \dots, j.$$

That is, g_r and ϕ_ρ are equivalent in the sense of Definition 1.2.

Moreover, it can be easily proved by Theorem 7.3 that the following holds:

$$(1.3.2) \quad g_r \text{ and } \phi_\rho \text{ have the same multiplicity sequence}$$

$$\iff g_r \stackrel{\text{divisor}}{\sim} \phi_\rho \text{ under the standard resolutions.} \quad \square$$

Remark 1.3.1. (i) Theorem 1.3 can be proved by Theorem 7.3 and Theorem 10.2 where the new terminology of (1.3.2) is defined by Definition 2.4 and Definition 2.6.

(ii) Without assuming that both $2 \leq n_1 < \beta_{1,1}$ and $2 \leq \ell_1 < \delta_{1,1}$, it can be easily proved that the conclusion of Theorem 1.3 may not be true by the following example:

$$(1.3.1.1) \quad \begin{aligned} g_1 &= z^3 + y^8 \quad \text{and} \\ \phi_2 &= \phi_1^3 + y^2 z^3 \quad \text{with} \quad \phi_1 = z + y^2, \end{aligned}$$

because g_1 and ϕ_2 have the same multiplicity sequence, and also they have the same divisor under two standard resolutions, but the condition in (1.3.1) does not hold. \square

Remark 1.3.2. It will be proved by Theorem 1.3 and Theorem A(Theorem 8.10([K2])) that we can compute one-to-one function F from $\text{Family}(1)$ into $\text{Family}(2)$. It will be proved later that such a function $F: \text{Family}(1) \rightarrow \text{Family}(2)$ must be onto. \square

Chapter II: The rigorous representation of explicit irreducibility algorithms for the Weierstrass polynomials of two complex variables with examples and without proofs and related topics in the Puiseux expansions

§1.4. The 1st Algorithm for computing a one-to-one function between Family(1) and Family(2)(the family of the standard Puiseux expansions) with its examples

In this section, the first half of The 1st Algorithm can be given by Algorithm 1.4.1 for Theorem 1.4 with Example 1.4.1, and also the second half of the 1st Algorithm can be given by Algorithm 1.6.2 for Theorem 1.6 with Example 1.6.3.

§1.4.A. The first half of The 1st Algorithm(Theorem 1.4)

Theorem 1.4(Theorem 11.2:Algorithm for finding a one-to-one function from Family(1) into Family(2)).

Assumptions Let $g_r \in \mathbb{C}\{y, z\}$ be the standard Puiseux W-poly of the recursive r -type in z in Family(1) satisfying six conditions with the same notations as in Definition 1.1.

Conclusions

[I] By explicit algorithm in (1.4.1), we can compute the standard Puiseux expansion for the curve $C(g_r : t)$ such that $\text{Multiseq}(V(g_r)) \equiv \text{Multiseq}(C(g_r : t))$ as sequence:

(Algorithm 1.4.1 for Theorem 1.4)

$$(1.4.1) \quad C(g_r : t) := \begin{cases} y = t^n \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$$

such that $n = n_1 n_2 \dots n_r$ and $\alpha_1 = \beta_{1,1} n_2 \dots n_r$,

$$\alpha_j = \alpha_{j-1} + \widehat{\Delta}_j n_{j+1} n_{j+2} \dots n_r,$$

where $\widehat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} > 0$ for $2 \leq j \leq r$ and $\Delta_1(t) = t$.

[II] Let $\Psi : \text{Family}(1) \rightarrow \text{Family}(2)$ be a function defined by $\Psi(g_r) = C(g_r : t)$ for any g_r in Family(1). Then, Ψ is a one-to-one function from Family(1) into Family(2). \square

Remark 1.4.0. (a) Note that the parametrization in (1.4.1) satisfies the following:

- (a.1) $n < \alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_r$.
- (a.2) $n > d_1 > d_2 > \dots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, where $d_i = n_{i+1} n_{i+2} \dots n_r$ for $i = 1, 2, \dots, r-1$. So, $\alpha_j - \alpha_{j-1} = \widehat{\Delta}_j d_j$.
- (b) Ψ , being given by a one-to-one function from Family(1) into Family(2) by Algorithm 1.4.1 for Theorem 1.4, can be proved to onto map which is computable by Algorithm 1.6.2 for Theorem 1.6. \square

Example 1.4.1 for Theorem 1.4:

Let the polynomial g_3 in $\mathbb{C}[y, z]$ be given as follows:

$$(1.4.2) \quad g_1 = z^3 + y^4, \quad g_2 = g_1^5 + y^{18} z^2, \quad g_3 = g_2^3 + y^{61} z^1.$$

Then, g_3 is the standard Puiseux W-poly of the recursive 3rd type because of the following computations (i), (ii), (iii) and (iv):

- (i) By Definition 1.1 and (1.4.2), $n_1 = 3$, $n_2 = 5$, $n_3 = 3$, $\Delta_1(\beta_{1,1}) = \beta_{1,1} = 4$, $\Delta_2(\beta_{2,1}, \beta_{2,2}) = 62$, and $\Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) = \beta_{3,3} \Delta_2(\beta_{2,1}, \beta_{2,2}) + n_2 \Delta_2(\beta_{3,1}, \beta_{3,2}) = 5 \cdot 187 = 935$.
- (ii) $n_1 = 3 < \beta_{1,1} = 4$, $\Delta_2(\beta_{2,1}, \beta_{2,2}) = 62 > n_2 n_1 \Delta_1(\beta_{1,1}) = 60$, and $\Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) = 935 > n_3 n_2 \Delta_2(\beta_{2,1}, \beta_{2,2}) = 930$.
- (iii) $\gcd(n_1, \beta_{1,1}) = \gcd(3, 4) = 1$, $\gcd(n_2, \Delta_2(\beta_{2,k})_{k=1}^2) = \gcd(5, 62) = 1$ and $\gcd(n_3, \Delta_3(\beta_{3,k})_{k=1}^3) = \gcd(3, 935) = 1$.
- (iv) $n_j \geq 2$, $\beta_{j,1} > 0$ for all $j = 1, 2, 3$. Also, $0 \leq \beta_{j,k} < n_{k-1}$ for $2 \leq j \leq 3$ and $2 \leq k \leq j$.

Note by (i), (ii), (iii) and (iv) and by Theorem 5.0 that g_3 is irreducible in $\mathbb{C}\{y, z\}$.

Now, it is easy to compute by (1.4.1) in Algorithm 1.4.1 for Theorem 1.4 that the standard Puiseux expansion for $C_3(t)$ such that $V(g_r) \equiv C_3(t)$ (multi. seq.) is given by

$$(1.4.3) \quad C_3(t) := \begin{cases} y = t^{45} \\ z = t^{60} + t^{66} + t^{71}. \end{cases}$$

because of the following computations (a) and (b):

- (a) $n = n_1 n_2 n_3 = 45$, $\alpha_1 = \beta_{1,1} n_2 n_3 = 60$, and $\alpha_2 - \alpha_1 = \hat{\Delta}_2 n_3 = 2 \cdot 3 = 6$ implies that $\alpha_2 = 66$ because $\hat{\Delta}_2 = \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_2 n_1 \beta_{1,1} = 2$.
- (b) $\alpha_3 - \alpha_2 = \hat{\Delta}_3 = 5$ implies that $\alpha_3 = 71$ because $\hat{\Delta}_3 = \Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) - n_3 n_2 \Delta_2(\beta_{2,1}, \beta_{2,2}) = 935 - 3 \cdot 5 \cdot 62 = 5$. \square

§1.4.B. The second half of The 1st Algorithm(Theorem 1.6)

Sublemma 1.5.(Corollary 7.6) for Theorem 1.6.

Assumptions Let $A \geq 2$ and $B \geq 2$ be integers with $\gcd(A, B) = 1$. Let p be an integer such that $p > nAB$ for some integer $n \geq 2$.

Conclusions We can compute a unique pair of two integers s_1 and t_1 such that $p = s_1 A + t_1 B$ with $0 \leq s_1 < B$ and $t_1 > A$. \square

Theorem 1.6(Theorem 11.4:Algorithm for finding the unique element of Family(1) corresponding to any given standard Puiseux expansion of Family(2)).

Assumptions Let the standard Puiseux expansion of the r -type for the curve $C_r(t)$ be given by

$$(1.6.1) \quad C_r(t) := \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$$

where $2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_r$ and
 $n > d_1 > d_2 > \dots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

Conclusions To compute the standard Puiseux W -poly g_r of the recursive r -type with $V(g_r) \stackrel{\text{multiseq}}{\sim} C_r(t)$ at $0 \in \mathbb{C}^2$ is to find **explicit algorithm(Algorithm 1.6.2)**, using a finite number $\frac{r(r-1)}{2}$ of Sublemma 1.5(Corollary 7.6), as soon as the standard Puiseux W -poly g_r of the recursive r -type satisfies the same kind of properties and notations as in Definition 1.1, for notation.

(Algorithm 1.6.2 for Theorem 1.6) To compute an algorithm for finding one and only one standard Puiseux W -poly $g_r \in \mathbb{C}\{y\}[z]$ of the recursive r -type such that $V(g_r) \stackrel{\text{multiseq}}{\sim} C_r(t)$ at $0 \in \mathbb{C}^2$, we may assume that the above g_r satisfies the same properties and notations as g_r of Definition 1.1 does.

To find such an algorithm, using Step(1) and Step(2), it suffices to compute the family of sets $\{X_j : 1 \leq j \leq r\}$, satisfying following properties:

$$(1)(1a) \quad X_1 = \{n_1, \beta_{1,1}\} \text{ with } 2 \leq n_1 < \beta_{1,1}.$$

$$(1b) \quad X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\} \text{ with } n_j \geq 2 \text{ for } j = 2, \dots, r, \text{ satisfying the six conditions in Definition 1.1 and the following equations in (1.6.2):}$$

$$(1.6.2)(i) \quad n = n_1 d_1 \text{ and } \alpha_1 = \beta_{1,1} d_1 \text{ with } d_1 = \gcd(n, \alpha_1)$$

$$(ii) \quad d_{j-1} = n_j d_j \text{ and } \alpha_j - \alpha_{j-1} = \hat{\Delta}_j d_j \text{ with } d_j = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1}) \text{ for } 2 \leq j \leq r, \\ \text{where } \hat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}.$$

Note that all $n_i \geq 2$, and so $n > d_1 > \dots > d_r = 1$. It may be assumed by Remark 1.4.0 that the equations in (1.6.2) of Theorem 1.6 and the equations in (1.4.1) are the same.

Step(1) for Algorithm 1.6.2. We can easily compute a finite set $\{(n_j, \hat{\Delta}_j) \in N^2 : j = 1, 2, \dots, r\}$ of unique pairs, each of which satisfies the following:

- (1.1) Let $d_1 = \gcd(n, \alpha_1)$. It is easy to compute a unique pair $(n_1, \hat{\Delta}_1) \in N^2$ such that $n = n_1 d_1$ and $\alpha_1 = \hat{\Delta}_1 d_1$ with $\gcd(n_1, \hat{\Delta}_1) = 1$. Note that $\hat{\Delta}_1 > n_1 \geq 2$ because $n > d_1$, and write $\hat{\Delta}_1 = \beta_{1,1}$ for notation.
- (1.2) Let $d_j = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1})$ for $2 \leq j \leq r$. It is easy to compute a unique pair $(n_j, \hat{\Delta}_j) \in N^2$ such that $d_{j-1} = n_j d_j$ and $\alpha_j - \alpha_{j-1} = \hat{\Delta}_j d_j$ with $\gcd(d_j, \hat{\Delta}_j) = 1$. Note that $d_j = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j)$.

Step(2) for Algorithm 1.6.2. Let $\{(n_j, \hat{\Delta}_j) \in N^2 : j = 1, 2, \dots, r\}$ be already given by Step(1) where $\gcd(n_j, \hat{\Delta}_j) = 1$ for $1 \leq j \leq r$.

- (2.1) With $\{n_1, \hat{\Delta}_1\}$ in (1.1) of Step (1), let $\Delta_1 : N_0 \rightarrow N_0$ be a function defined by

$$\Delta_1(t) = t.$$

We can compute a solution $\beta_{1,1} = \hat{\Delta}_1$ such that $\Delta_1(\beta_{1,1}) = \beta_{1,1} > n_1 \geq 2$.

- (2.2) With $(n_2, \hat{\Delta}_2)$ in (1.2) of Step (1), let $\Delta_2 : N_0^2 \rightarrow N_0$ be a function defined by

$$\Delta_2(t_1, t_2) = t_2 \beta_{1,1} + n_1 t_1 \quad \text{with} \quad \beta_{1,1} = \hat{\Delta}_1.$$

Given $p_2 = n_2 n_1 \beta_{1,1} + \hat{\Delta}_2$, we can compute a unique pair (a_2, b_2) in N^2 such that $a_2 \beta_{1,1} + b_2 n_1 = p_2$ with $b_2 > \beta_{1,1}$ and $0 \leq a_2 < n_1$ (by Sublemma 1.5, because $p_2 > 2n_1 \beta_{1,1}$ and $\gcd(n_1, \beta_{1,1}) = 1$).

Write $\beta_{2,1} = a_2$ and $\beta_{2,2} = b_2$ with $0 \leq a_2 < n_1$ and $b_2 > \beta_{1,1}$, and then $p_2 = \Delta_2(\beta_{2,1}, \beta_{2,2})$.

- (2.3) With $\{n_j, \hat{\Delta}_j\}$ in (1.2) of Step (1), suppose we have proved that the following are true:

For each $j = 2, 3, \dots, \ell$ with $\ell < r$, assume that $d_j > 1$, and then use the induction assumption on the positive integer j . Given $p_j = \hat{\Delta}_j + n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $j = 2, 3, \dots, \ell$, we may assume by a finite number $(j-1)$ of use of Sublemma 1.5 that we can compute a unique sequence $X_j = \{\beta_{j,k} : k = 1, 2, \dots, j\} \subset N_0$ such that $\Delta_j(\beta_{j,k})_{k=1}^j = p_j$ satisfying the six conditions in Definition 1.1.

- (2.3.0) Then, a computational algorithm for $\{\beta_{\ell+1,k} : k = 1, 2, \dots, \ell+1\} \subset N_0$ such that $\Delta_{\ell+1}(\beta_{\ell+1,k})_{k=1}^{\ell+1} = p_{\ell+1}$ with the six conditions in Definition 1.1 can be easily represented by (i), (ii) and (iii).

We use Sublemma 1.5 ℓ -times, as follows:

With $\{n_{\ell+1}, \hat{\Delta}_{\ell+1}\} \in N^2$ in (1, 2) of Step (1), let $\Delta_{\ell+1} : N_0^{\ell+1} \rightarrow N_0$ be a function defined by

$$\Delta_{\ell+1}(t_k)_{k=1}^{\ell+1} = t_{\ell+1} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} + n_{\ell} \Delta_{\ell}(t_k)_{k=1}^{\ell}.$$

(i) Given $p_{\ell+1} = n_{\ell+1} n_{\ell} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} + \hat{\Delta}_{\ell+1}$, we can compute a unique pair $(a_{\ell+1}, b_{\ell+1})$ in N^2 such that $p_{\ell+1} = a_{\ell+1} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} + b_{\ell+1} n_{\ell}$ with $b_{\ell+1} > \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell}$ and $0 \leq a_{\ell+1} = \beta_{\ell+1,\ell+1} < n_{\ell}$ (by Sublemma 1.5 once, because $p_{\ell+1} > 2n_{\ell} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell}$ and $\gcd(n_{\ell}, \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell}) = 1$).

(ii) Since $b_{\ell+1} > \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} \geq 2n_{\ell} n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1}$, then by induction on the positive integer $\ell < s$, and by a finite number $(\ell-1)$ of use of Sublemma 1.5, we can compute a unique sequence $(\beta_{\ell+1,k})_{k=1}^{\ell} \subset N_0^{\ell}$ such that

$\Delta_{\ell}(\beta_{\ell+1,k})_{k=1}^{\ell} = b_{\ell+1}$ with $\beta_{\ell+1,1} > 0$ and $0 \leq \beta_{\ell+1,k} < n_{k-1}$ for $2 \leq k \leq \ell$.

(iii) By (i) and (ii), by a finite number ℓ of use of Sublemma 1.5, we can compute a unique sequence $\{\beta_{\ell+1,k} : k = 1, 2, \dots, \ell+1\} \subset N_0$ such that $p_{\ell+1} = a_{\ell+1} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} + b_{\ell+1} n_{\ell} = \beta_{\ell+1,\ell+1} \Delta_{\ell}(\beta_{\ell,k})_{k=1}^{\ell} + n_{\ell} \Delta_{\ell}(\beta_{\ell+1,k})_{k=1}^{\ell} = \Delta_{\ell+1}(\beta_{\ell+1,k})_{k=1}^{\ell+1}$ satisfying the six conditions in Definition 1.1. \square

Example 1.6.3 for Theorem 1.6(See page 517 of [Bri-Kn]). Let the parametrization $C_4(t)$ for the Puiseux expansion be given by an example in page 517 of [Bri-Kn].

$$(1.6.3) \quad C_4(t) := \begin{cases} y = t^{100} \\ z = t^{250} + t^{375} + t^{410} + t^{417}. \end{cases}$$

This is the standard Puiseux expansion because of the computations (i) and (ii):

- (i) $n < \alpha_1 < \dots < \alpha_4$ where $n = 100$, $\alpha_1 = 250$, $\alpha_2 = 375$, $\alpha_3 = 410$ and $\alpha_4 = 417$.
- (ii) $n = 100 > d_1 = 50 > d_2 = 25 > d_3 = 5 > d_4 = 1$ where $d_1 = \gcd(n, \alpha_1)$, $d_2 = \gcd(d_1, \alpha_2 - \alpha_1)$, $d_3 = \gcd(d_2, \alpha_3 - \alpha_2)$ and $d_4 = \gcd(d_3, \alpha_4 - \alpha_3)$.

Now, the problem is how to find a one and only one $g_4 \in \text{Family}(1)$ such that $V(g_4) \stackrel{\text{multiseq}}{\sim} C_4(t)$ at the origin in \mathbb{C}^2 . For the solution of the above problem, by a finite number $\frac{r(r-1)}{2} = \frac{4 \cdot 3}{2}$ of use of Sublemma 1.5, it suffices to follow (**Algorithm 1.6.2 for Theorem 1.6**).

After the following computation is done, the above polynomial g_4 of the recursive 4-th type is as follows:

$$(1.6.4) \quad g_1 = z^2 + y^5, \quad g_2 = g_1^2 + y^{10}z, \quad g_3 = g_2^5 + y^{58}g_1 \quad \text{and} \quad g_4 = g_3^5 + y^{300}zg_1g_2.$$

Step(1) for Algorithm 1.6.2. We can compute a finite set $\{(n_j, \hat{\Delta}_j) \in N^2 : j = 1, 2, 3, 4\}$ of pairs, each of which satisfies the following: Recall that $d_j = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1})$ for $1 \leq j \leq 4$ where $d_0 = n$ and $\alpha_0 = 0$.

- (1) Let $n = d_1 n_1$ and $\alpha_1 = d_1 \hat{\Delta}_1$. Then, $d_1 = 50$, $n_1 = 2$ and $\gamma_{11} = \hat{\Delta}_1 = 5$.
- (2) Let $d_1 = d_2 n_2$ and $\alpha_2 - \alpha_1 = d_2 \hat{\Delta}_2$. Then, $d_2 = 25$, $n_2 = 2$ and $\hat{\Delta}_2 = 5$.
- (3) Let $d_2 = d_3 n_3$ and $\alpha_3 - \alpha_2 = d_3 \hat{\Delta}_3$. Then, $d_3 = 5$, $n_3 = 5$ and $\hat{\Delta}_3 = 7$.
- (4) Let $d_3 = d_4 n_4$ and $\alpha_4 - \alpha_3 = d_4 \hat{\Delta}_4 = 7$.

Step(2) for Algorithm 1.6.2. Let $\{(d_j, \hat{\Delta}_j) \in N^2 : j = 1, 2, 3, 4\}$ of pairs be given by Step (1) where $\gcd(n_j, \hat{\Delta}_j) = 1$ for $1 \leq j \leq 4$.

By the Euclidean algorithm in Sublemma 1.5(Corollary 7.6), for each $j = 2, 3, 4$, we compute a finite unique sequence $\{\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with six conditions in Definition 1.1 for the standard Puiseux W-poly in Family(1), as follows:

- (i) To compute g_1 , $n_1 = 2$ and $\beta_{1,1} = 5$. So, $g_1 = z^2 + y^5$.
- (ii) To compute g_2 , we use Sublemma 1.5 once. Note that $\Delta_2(\beta_{2,1}, \beta_{2,2}) = \beta_{1,1}\beta_{2,2} + n_1\beta_{2,1} = 5\beta_{2,2} + 2\beta_{2,1}$ where $\Delta_2(\beta_{2,1}, \beta_{2,2}) = p_2 = n_2 n_1 \beta_{1,1} + \hat{\Delta}_2 = 2 \cdot 2 \cdot 5 + 5 = 25$. Since $\Delta_2(\beta_{2,1}, \beta_{2,2}) > n_2 n_1 \beta_{1,1}$, we can compute a unique solution $\{\beta_{2,1}, \beta_{2,2}\} \subset N_0$ such that $\Delta_2(\beta_{2,1}, \beta_{2,2}) = 25$ with $\beta_{2,2} < n_1 = 2$. By Sublemma 1.5, $\beta_{2,1} = 10$ and $\beta_{2,2} = 1$. So, $g_2 = g_1^2 + y^{10}z^1$.
- (iii) To compute g_3 , we use Sublemma 1.5 twice. Note that $\Delta_3(\beta_{3,k})_{k=1}^3 = \beta_{3,3}\Delta_2(\beta_{2,1}, \beta_{2,2}) + n_2\Delta_2(\beta_{3,1}, \beta_{3,2}) = 25\beta_{3,3} + 2\Delta_2(\beta_{3,1}, \beta_{3,2}) > n_3 n_2 \Delta_2(\beta_{2,1}, \beta_{2,2})$ where $\Delta_3(\beta_{3,k})_{k=1}^3 = p_3 = n_3 n_2 \Delta_2(\beta_{2,1}, \beta_{2,2}) + \hat{\Delta}_3 = 5 \cdot 2 \cdot 25 + 7 = 257$.
- (iii-a) To compute $\beta_{3,3}$, since $\gcd(n_2, \Delta_2(\beta_{2,1}, \beta_{2,2})) = 1$, by Sublemma 1.5 we can compute a unique pair $(a_3, b_3) = (1, 116)$ such that $25a_3 + 2b_3 = 257$ where $0 \leq 1 = a_3 = \beta_{3,3} < n_2$ and $b_3 = 116 > \Delta_2(\beta_{2,1}, \beta_{2,2})$. Then, $\beta_{3,3} = 1$ and $\Delta_2(\beta_{3,1}, \beta_{3,2}) = b_3 = 116$.
- (iii-b) Since $116 = \Delta_2(\beta_{3,1}, \beta_{3,2}) = 2\beta_{3,1} + 5\beta_{3,2} > \Delta_2(\beta_{2,1}, \beta_{2,2})$ by (iii-a), then by Sublemma 1.5 we can compute a unique solution $\{\beta_{3,1}, \beta_{3,2}\} \subset N_0$ such that $\Delta_2(\beta_{3,1}, \beta_{3,2}) = 116$ with $\beta_{3,2} = 0 < n_1 = 2$ and $\beta_{3,1} = 58$. So, by (iii-a) and (iii-b), $\beta_{3,1} = 58$, $\beta_{3,2} = 0 < n_1$ and $\beta_{3,3} = 1 < n_2$. So, $g_3 = g_2^5 + y^{58}g_1^1$.
- (iv) To compute g_4 , we use Sublemma 1.5 three times. It is clear that $\Delta_4(\beta_{4,k})_{k=1}^4 = \beta_{4,4}\Delta_3(\beta_{3,k})_{k=1}^3 + n_3\Delta_3(\beta_{4,k})_{k=1}^3 = 257\beta_{4,4} + 5\Delta_3(\beta_{4,1}, \beta_{4,2}, \beta_{4,3}) > n_4 n_3 \Delta_3(\beta_{3,k})_{k=1}^3$ where $\Delta_4(\beta_{4,k})_{k=1}^4 = p_4 = n_4 n_3 \Delta_3(\beta_{3,k})_{k=1}^3 + \hat{\Delta}_4 = 5 \cdot 5 \cdot 257 + 7$.
- (iv-a) To compute $\beta_{4,4}$, since $\gcd(n_3, \Delta_3(\beta_{3,k})_{k=1}^3) = 1$, by Sublemma 1.5 once we can compute a unique pair $(a_4, b_4) = (1, 1235)$ such that $257a_4 + 5b_4 = 5 \cdot 5 \cdot 257 + 7$ where $0 \leq 1 = a_4 = \beta_{4,4} < n_3$ and $b_4 = 1235 > \Delta_3(\beta_{3,k})_{k=1}^3$.
- (iv-b) To compute a unique solution $\{\beta_{4,k}; k = 1, 2, 3\}$ such that $b_4 = \Delta_3(\beta_{4,k})_{k=1}^3 > \Delta_3(\beta_{3,k})_{k=1}^3$ with $\beta_{4,1} > 0$, $\beta_{4,2} < n_1$ and $\beta_{4,3} < n_2$, then using Sublemma 1.5 twice and the same method as we have used in (iii), we can compute $\beta_{4,1} = 300$, $\beta_{4,2} = 1 < n_1$ and $\beta_{4,3} = 1 < n_2$. So, by (iv-a) and (iv-b), $\beta_{4,1} = 300$, $\beta_{4,2} = 1$, $\beta_{4,3} = 1 < n_2$ and $\beta_{4,4} = 1$.

Thus, by (i), (ii), (iii) and (iv), the above polynomial $f(y, z) = g_4$ of the recursive type is as follows:

$$f = g_4 = g_3^5 + y^{300} z^1 g_1^1 g_2^1 \text{ where } g_1 = z^2 + y^5, g_2 = g_1^2 + y^{10} z^1 \text{ and } g_3 = g_2^5 + y^{58} g_1^1.$$

Therefore, f is the unique standard Puiseux W-poly of the recursive type because of The 6-th Cond⁽⁰⁾. \square

§1.5. The division algorithm for the W-polys in preparation for the computations of The 2nd Algorithm and The 3rd Algorithm

As in Definition 15.0 with Notation 15.0.1, the Weierstrass preparation theorem and the Weierstrass division theorem can be written by The WPT and The WDT respectively, for brevity of notation. In order to succeed in the computations of the 2nd and the 3rd algorithms in §1.6, in this section it is very important to say without any other proofs that The Division Algorithm for the W-polys(Theorem 1.8(Theorem 15.4) with two sublemmas) can have an important role of the 2nd and the 3rd algorithms in §1.6, which will be shown in §16, later.

Theorem 1.7(Theorem 15.2: The WDT for the W-polys).

Assumptions Let $h \in {}_{n-1}\mathcal{O}[z_n]$ be a W-poly of degree $\nu > 0$ in z_n . Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a W-poly of degree $\mu \geq \nu$ in z_n , and ℓ be a positive integer with $\ell\nu \leq \mu < (\ell + 1)\nu$.

Conclusions

(1) Then, f can be written uniquely in the form

$$(1.7.1) \quad f = \sum_{i=0}^{\ell} r_i h^i \quad \text{with} \quad h^0 = 1,$$

where if $\mu \geq \ell\nu$ then for $i = 0, 1, \dots, \ell - 1$, each $r_i \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n with $r_i(0, \dots, 0, z_n)$ identically zero, and if $\mu = \ell\nu$ then r_ℓ is equal to one and if $\mu > \ell\nu$ then $r_\ell \in {}_{n-1}\mathcal{O}[z_n]$ is a W-poly of degree $\mu - \ell\nu < \nu$ in z_n .

(2) In addition, suppose $h \in {}_{n-1}\mathcal{O}[z_n]$ has a multiplicity $\nu > 0$ at $0 \in \mathbb{C}^n$ and $f \in {}_{n-1}\mathcal{O}[z_n]$ has a multiplicity $\mu \geq \nu$ at $0 \in \mathbb{C}^n$. Then, the above representation $f = \sum_{i=0}^{\ell} r_i h^i$ of (1.7.1) satisfies the property such that for $i = 0, 1, \dots, \ell - 1$, each r_i has a multiplicity $\geq \mu - i\nu$ at $0 \in \mathbb{C}^n$ and such that if $\mu = \ell\nu$ then r_ℓ has a zero multiplicity at $0 \in \mathbb{C}^n$. \square

Theorem 1.8(Theorem 15.4: The Division Algorithm for the W-polys).

Assumptions Let $f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be a W-poly of degree $n \geq 2$ in z where for $0 \leq i \leq n - 2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Assume that f may not be irreducible in ${}_2\mathcal{O}_0$, and note that a_{n-1} is identically zero for convenience. Write $n = \prod_{k=1}^{\ell} n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

Conclusions We can compute a unique W-poly in z , $f_1 \in \mathbb{C}\{y\}[z]$, satisfying the following notations and properties: Let $f_{-1} = y$ and $f_0 = z$.

$$(1.8.1) \quad \begin{aligned} f_1 &= f_0^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i} f_0^i \quad \text{and} \\ f &= f_1^{d_2} + \sum_{i=0}^{d_2-2} S_{2,i} f_1^i, \end{aligned}$$

such that (i) $n = d_2 n_1$ with $n = d_1$,
(ii) $f_1 = f_1(y, z) \in \mathbb{C}\{y\}[z]$ is a W-poly of degree n_1 in z ,
(iii) $f \in \mathbb{C}\{y\}[z, f_1] \subseteq \mathbb{C}\{y, z\}[f_1]$ is a W-poly of degree d_2 in f_1 ,

considering $f_{-1} = y, f_0 = z, f_1$ as independent complex (3)-variables at the origin in \mathbb{C}^3 , with two properties (1) and (2):

- (1)(1a) Let i be fixed with $0 \leq i \leq n_1 - 2$. If exists, then $R_{1,i} = R_{1,i}(y)$ is a nonunit in $\mathbb{C}\{y\}$.
- (1b) Let i be fixed with $0 \leq i \leq d_2 - 2$. Then $S_{2,i} = S_{2,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< n_1$ in z and $S_{2,i}(0, z) = 0$.
- (2)(2a) Let i be fixed with $0 \leq i \leq n_1 - 2$. For any nonzero monomial y^{δ_1} in $R_{1,i} = R_{1,i}(y) \in \mathbb{C}\{y\}$, $\delta_1 > 0$.
- (2b) Let i be fixed with $0 \leq i \leq d_2 - 2$. For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $S_{2,i} = S_{2,i}(y, z) \in \mathbb{C}\{y\}[z]$, $\delta_1 > 0$ and $\delta_2 < n_1$. \square

Remark 1.8.1. (a) Note that Theorem 15.4 is a generalization of Theorem 1.8. It will be proved by Sublemma 15.4.α and Sublemma 15.5 of §15 that Theorem 15.4 is true. So, it can be easily proved by Sublemma 1.9(Sublemma 15.4.α) and Sublemma 1.10(Sublemma 15.5) that Theorem 1.8 is true.

(b) It is most interesting and important in this paper that explicit algorithm for finding a construction of (1.8.1) in the conclusion of Theorem 1.8 can be completely computed from an equation in (1.9.1) of Sublemma 1.9 and an equation in (1.10.1) of Sublemma 1.10, using an equation in (1.7.1) of Theorem 1.7. \square

Sublemma 1.9 for Theorem 1.8(Sublemma 15.4.α for Theorem 15.4).

Assumptions Suppose that the same properties and notations as in the assumption of Theorem 1.8 hold.

Conclusions We show that $h_{1,1}$ and f can be constructed as follows:

$$(1.9.1) \quad \begin{cases} h_{1,1} &= f_0^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(1)} f_0^i, \\ f &= h_{1,1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} h_{1,1}^i, \end{cases}$$

where $h_{1,1} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree n_1 in z and $n = n_1 d_2$, satisfying the following facts, Fact(A), Fact(B), Fact(C), Fact(D) and Fact(E).

Fact(A) For each $i = 0, 1, \dots, n_1 - 2$, $R_{1,i}^{(1)} = R_{1,i}^{(1)}(y) \in \mathbb{C}\{y\}$ with $R_{1,i}^{(1)}(0) = 0$, if exists.

Fact(B) For each $i = 0, 1, \dots, n_1 - 2$, and for any nonzero monomial y^{δ_1} in $R_{1,i}^{(1)} \in \mathbb{C}\{y\}$, $\delta_1 > 0$.

Fact(C) For each $i = 0, 1, \dots, d_2 - 1$, $T_{2,i} = T_{2,i}(y, z) = \sum a_{p,q} y^p z^q$ with a nonzero constant $a_{p,q}$ such that $p > 0$ and $q < n_1$ and that $T_{2,i}(0, z) = 0$.

Moreover, considering $y, z, f_1 = h_{1,1}$ as independent complex (3)-variables at the origin in \mathbb{C}^3 , then, $T_{2,i} \in \mathbb{C}\{y\}[z] \subseteq \mathbb{C}\{y, z\}$ satisfies two facts Fact(D) and Fact(E).

Fact(D) For each $i = 0, 1, \dots, d_2 - 1$, and for any nonzero monomial $\prod_{t=1}^2 f_t^{\gamma_t}$ in $T_{2,i}$, $\gamma_1 > 0$ and $\gamma_2 < n_1$.

Fact(E) In particular, if $i = d_2 - 1$ for $T_{2,i}$ of Fact(D), then $\gamma_2 \leq n_1 - 2$. \square

Sublemma 1.10 for Theorem 1.8(Sublemma 15.5 for Theorem 15.4).

Assumptions Suppose that the same properties and notations as in the assumption of Theorem 1.8 hold. By the same way as we have seen in (1.9.1) of the conclusion of Sublemma 1.9, we may assume that $(h_{1,1}, f)$ can be constructed as follows:

$$(1.10.1) \quad \begin{cases} h_{1,1} &= f_0^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(1)} f_0^i, \\ f &= h_{1,1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} h_{1,1}^i, \end{cases}$$

satisfying the facts, denoted by Fact(A), Fact(B), Fact(C), Fact(D) and Fact(E). For brevity of notation, let $h_1 = h_{1,1}$, $R_i^{(1)} = R_{1,i}^{(1)}$ for $0 \leq i \leq n_1 - 2$ and $T_i^{(1)} = T_{2,i}^{(1)} = T_{2,i}$ for $0 \leq i \leq d_2 - 1$, respectively.

Conclusions Then, (f_1, f) for f in the conclusion of Theorem 1.8 can be constructed as follows:

Case[I]: If $T_{d_2-1}^{(1)}$ in (h_1, f) is zero, let $f_1 = h_1$, $R_{1,i} = R_i^{(1)}$ for $0 \leq i \leq n_1 - 2$ and $S_{2,i} = T_i^{(1)}$ for $0 \leq i \leq d_2 - 2$, respectively. Then, the construction of (f_1, f) has been already finished.

Case[II]: If $T_{d_2-1}^{(1)}$ is not zero, for finding such a construction of (f_1, f) , it suffices to follow two steps, Step(1) and Step(2).

Step(1) for Case[II] Then, there is a sequence of pairs, $H = \{(h_p, f) : p = 1, 2, \dots\}$, each pair of which can be constructed with five properties, called Property(1), Property(2), Property(3), Property(4) and Property(5), as follows:

$$(1.10.2)(1.10.2.1) \quad \begin{cases} h_1 &= f_0^{n_1} + \sum_{i=0}^{n_1-2} R_i^{(1)} f_0^i & \text{with } R_i^{(1)} = R_{1,i}^{(1)} \text{ in (1.10.1),} \\ f &= h_1^{d_2} + \sum_{i=0}^{d_2-1} T_i^{(1)} h_1^i & \text{with } T_i^{(1)} = T_{2,i}^{(1)} \text{ in (1.10.1),} \end{cases}$$

$$(1.10.2.2) \quad \begin{cases} h_2 &= h_1 + \frac{1}{d_2} T_{d_2-1}^{(1)} = f_0^{n_1} + \sum_{i=0}^{n_1-2} R_i^{(2)} f_0^i, \\ f &= h_2^{d_2} + \sum_{i=0}^{d_2-1} T_i^{(2)} h_2^i, \end{cases}$$

$$(1.10.2.3) \quad \begin{cases} h_3 &= h_2 + \frac{1}{d_2} T_{d_2-1}^{(2)} = f_0^{n_1} + \sum_{i=0}^{n_1-2} R_i^{(3)} f_0^i, \\ f &= h_3^{d_2} + \sum_{i=0}^{d_2-1} T_i^{(3)} h_3^i, \end{cases}$$

...

satisfying the following properties and notations:

Property(1) Let p be fixed with $p \geq 1$. For each $i = 0, 1, \dots, d_2 - 2$, $R_i^{(p+1)} = R_i^{(p+1)}(y) \in \mathbb{C}\{y\}$ with $R_i^{(p+1)}(0) = 0$, if exists.

Property(2) Let p be fixed with $p \geq 1$. For each $i = 0, 1, \dots, d_2 - 1$, $T_i^{(p+1)} = T_i^{(p+1)}(y, z) = \sum a_{\alpha, \beta}^{(p+1)} y^\alpha z^\beta$ with a nonzero constant $a_{\alpha, \beta}^{(p+1)}$ such that $\alpha > 0$ and $0 \leq \beta < n_1$ and that $T_i^{(p+1)}(0, z) = 0$.

Consider $f_{-1} = y, f_0 = z$ as independent complex (2)-variables at the origin in \mathbb{C}^2 .

Property(3) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_1 - 2$. For any nonzero monomial $f_{-1}^{\delta_1}$ in $R_i^{(p+1)} = R_i^{(p+1)}(y) \in \mathbb{C}\{y\}$, $\delta_1 > 0$.

Property(4) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_2 - 1$. For any nonzero monomial $f_{-1}^{\delta_1} f_0^{\delta_2}$ in $T_i^{(p+1)} = T_i^{(p+1)}(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}, f_0\}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

Property(5) In particular, if $i = d_2 - 1$ for $T_i^{(p+1)}$ of Property(4), then $\delta_2 \leq n_1 - 2$.

Step(2) for Case[II] By Step(1) for Case[II], there is a pair $(h_{\nu+1}, f) \in H$ which satisfies the following property:

Property(6) There is an integer $\nu \leq \frac{n_1+1}{2}$ such that $T_{d_2-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu$ and $T_{d_2-1}^{(\nu+1)} = T_{d_2-1}^{(\nu+2)} = \dots = 0$. That is, $(h_\nu, f) \neq (f_1, f)$ and $(h_{\nu+1}, f) = (f_1, f)$ for an integer $\nu \leq \frac{n_1+1}{2}$. \square

Remark 1.10.1. (a) It is clear by Sublemma 1.9 that Property(1), Property(2), Property(3), Property(4) and Property(5) for (h_1, f) in (1.10.2.1) are equivalent to Fact(A), Fact(C), Fact(B), Fact(D) and Fact(E) for $(h_{1,1}, f)$ in (1.9.1), respectively.

(b) It is clear by Sublemma 15.4.α that (h_1, f) of Sublemma 1.9 was already constructed with five properties. \square

§1.6. Two fundamental lemmas for the representation of the local defining equations of irreducible plane curve singularities

In order to succeed in the computations of the 2nd and the 3rd algorithms in §1.7, §1.8 and §1.9, it is very important to say without any other proofs that we can write two fundamental lemmas for the representation of the local defining equations of irreducible plane curve singularities, that is, Lemma 1.11 and Lemma 1.12 in this section.

Lemma 1.11(The fundamental lemma for the representation of the local defining equations of irreducible plane curve singularities).

Assumptions Let $f = f(y, z) = b_n z^n + b_0 y^{\beta_0} + \sum_{i=1}^{n-1} b_i y^{\beta_i} z^i$ be in $\mathbb{C}\{y, z\}$ where for $0 \leq i \leq n$, each $b_i = b_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ if exists and the β_i are positive integers. Let m be the multiplicity of f at $0 \in \mathbb{C}^2$ with $n \geq 2$ and $\beta_0 \geq 2$. Let $d = \gcd(n, \beta_0)$, and write $n = n_1 d$ and $\beta_0 = \beta_{1,0,1} d$ with $\gcd(n_1, \beta_{1,0,1}) = 1$. Note that d may be equal to n .

In particular, if $b_i(y, z) = b_i(y)$ for all i and $b_n = 1$ then $f(y, z)$ is called a W -poly in z .

Conclusions

Fact[I]: If f is irreducible in ${}_2\mathcal{O}_0$ then b_n and b_0 are units in $\mathbb{C}\{y, z\}$, and $m = n$ or β_0 . So, f must satisfy the following necessary condition:

$$(1.11.1) \quad \frac{\beta_i}{n-i} \geq \frac{\beta_0}{n} \quad \text{for } 0 \leq i \leq n-1.$$

For f of (1.11.1), it suffices to consider two cases:

Case(A) $\gcd(n, \beta_0) = 1$, and Case(B) $\gcd(n, \beta_0) > 1$.

Case(A) Let $\gcd(n, \beta_0) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (1.11.1) holds. In this case, $f \in \text{the type}[1]$ in the sense of Definition 2.5.

Case(B) Let $\gcd(n, \beta_0) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, then $f \in \text{the type}[\ell]$ with $\ell \geq 1$ in the sense of Definition 2.5.

Fact[II]: If f is irreducible in ${}_2\mathcal{O}_0$, f can be represented as follows:

$$(1.11.2) \quad \begin{aligned} g_1 &= z^{n_1} + \xi y^{k_1} \quad \text{with} \quad k_1 = \beta_{1,0,1}, \\ f &= A \cdot g_1^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d, \end{aligned}$$

where the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} d$, satisfying the following properties :

- (i) A and ξ are the unique nonzero complex numbers such that $A = b_n(0, 0) \neq 0$, $dA\xi = b_{n-n_1}(0, 0) \neq 0$, and $\binom{d}{i} A \xi^i = b_{n-in_1}(0, 0)$ for $1 \leq i \leq d$.
- (ii) $\frac{\beta_i}{n-i} \geq \frac{\beta_0}{n} = \frac{\beta_{1,0,1}}{n_1}$ for $0 \leq i \leq n-1$. \square

The proof of Theorem 1.11 will be done by Theorem 3.2, Theorem 3.4 and Theorem 3.6.

Remark 1.11.0. (a) Let $d = \gcd(n, \beta_0) = 1$. If f is irreducible in ${}_2\mathcal{O}_0$, $f \stackrel{\text{resol}}{\sim} z^n + y^{\beta_0}$ at $0 \in \mathbb{C}^2$ in the sense of Definition 2.4.

(b) If f is defined by $f = (z^2 + y^3)^2 + y^2 z^4$ satisfying an equation in (1.11.2), note that f is not irreducible in $\mathbb{C}\{y, z\}$, satisfying an equation in (1.11.1). \square

Lemma 1.11.1 Assumptions Let $V(f) = \{(y, z) : f(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 with isolated singularity at the origin, which is written in the form,

$$(1.11.3) \quad \begin{aligned} g_1 &= z^{n_1} + \xi y^{k_1} \quad \text{with} \quad k_1 = \beta_{1,0,1}, \\ f &= A \cdot g_1^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d, \end{aligned}$$

where A and ξ are nonzero complex numbers, and the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + k_1 \beta > n_1 k_1 d$ and $2 \leq n_1 < k_1$ and $\gcd(n_1, k_1) = 1$. Note that f may not be irreducible in $\mathbb{C}\{y, z\}$.

Conclusions We may assume that g_1 of (1.11.3) can be identified with either g_1 of (3.6.1) in Theorem 3.6 or g_1 of (5.4.0) in Sublemma 5.4. Following the same properties and notations as in either Theorem 3.6 or Sublemma 5.4, let $\tau_m = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(g_1)$. Since $V(g_1)$ has the singular point at the origin, along $v = 0$ $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings and $(f \circ \tau_m)_{\text{total}}$ can be rewritten in the following form: Note that $2 \leq j \leq r$.

$$(1.11.4) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{k_1} u^b), \\ (g_1 \circ \tau_m)_{\text{proper}} &= (1 + \xi u), \\ (f \circ \tau_m)_{\text{total}} &= (f \circ \tau_m)(v, u) = v^{e_m} u^{\rho_m} (f \circ \tau_m)_{\text{proper}} \quad \text{with} \quad g_j = f, \\ (f \circ \tau_m)_{\text{proper}} &= A(1 + \xi u)^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^{n_1 \alpha + k_1 \beta - n_1 k_1 d} u^{a\alpha + b\beta - bn_1 d}, \end{aligned}$$

where

- (i) a and b are some nonnegative integers such that $ak_1 - bn_1 = 1$,
- (ii) $e_m = n_1 k_1 d$ and $\rho_m = bn_1 d$ and $\rho_{\alpha, \beta} = a\alpha + b\beta - bn_1 d \geq 0$,
- (iii) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind,
- (iv) $V^{(m)}(g_1) \cap (\cup_{i=1}^m E_i) = V^{(m)}(g_1) \cap E_m = \{(v, 1 + \xi u) = (0, 0)\}$.

because note by Theorem 3.6 or Sublemma 5.4 that we can use the same τ_m for the composition of the first finite number m of successive blow-ups in preparation for finding the standard resolution of the singular point $(0, 0)$ of $V(f)$ if exists, as a reduced variety.

Moreover, it is very interesting that $(f \circ \tau_m)_{\text{proper}}$, as an element in $\mathbb{C}\{v, 1 + \xi u\}$ with $(v, 1 + \xi u) = (0, 0)$ and $y = v$, has the same properties and notations at $(y, z) = (0, 0)$ as $f(y, z)$ does at $(y, z) = (0, 0)$ in Lemma 1.11 in the sense of Definition 2.6. \square

Lemma 1.12(The fundamental algorithm for finding an irreducibility criterion of any W-poly in $f \in \mathbb{C}\{y\}[z]$ with $f \in \text{type}[1]$ in the sense of Definition 2.5 which has the same multiplicity sequence as the standard Puiseux expansion of the 1-type($y = t^n$ and $z = t^\alpha$) does and its generalizations).

Assumptions Let $f = f(y, z) = a_n z^n + a_{n-2} y^{\alpha_{n-2}} z^{n-2} + \dots + a_1 y^{\alpha_1} z + a_0 y^{\alpha_0}$ be in $\mathbb{C}\{y, z\}$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y, z)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. In addition, assume that we have the following:
(1.12.0) $2 \leq n \leq \alpha_0$.

Conclusions

Fact: If f is irreducible in ${}_2\mathcal{O}_0$, f must satisfy the necessary condition:

$$(1.12.1) \quad \frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n} \quad \text{for } 0 \leq i \leq n-2.$$

Also, $d_2 = \gcd(n, \alpha_0) < n$ because f is irreducible in ${}_2\mathcal{O}_0$ and a_{n-1} is identically zero.

For f of (1.12.1), it suffices to consider two cases:

Case(A) $\gcd(n, \alpha_0) = 1$, and Case(B) $1 < \gcd(n, \alpha_0) < n$.

Case(A) The necessary and sufficient condition for $f(y, z)$ to be irreducible in ${}_2\mathcal{O}_0$ with $f \in \text{type}[1]$ in the sense of Definition 2.5 is as follows:

$$(1.12.2) \quad \gcd(n, \alpha_0) = 1 \quad \text{and} \quad \frac{\alpha_i}{n-i} > \frac{\alpha_0}{n} \quad \text{for } 0 \leq i \leq n-2.$$

In this case, $f \stackrel{\text{multiseq}}{\sim} z^n + \xi y^{\alpha_0} = g_1$ with $\xi = a_0(0)$. Equivalently, f can be rewritten as follows:

$$(1.12.2^*) \quad f = z^n + \xi y^{\alpha_0} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } n\alpha + \alpha_0\beta > n\alpha_0 \quad \text{and} \quad \gcd(n, \alpha_0) = 1.$$

Moreover, $V(f) \stackrel{\text{resol}}{\sim} V(g_1)$ at the origin in \mathbb{C}^2 in the sense of Definition 2.4.

Case(B) Let $1 < \gcd(n, \alpha_0) < n$ with a_{n-1} zero. If f is irreducible in ${}_2\mathcal{O}_0$, then $f \in \text{type}[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5. \square

Lemma 1.12.1(The fundamental algorithm for finding an irreducibility criterion of any W-poly in $f \in \mathbb{C}\{y\}[z]$ with $f \in \text{type}[2]$ in the sense of Definition 2.5 which has the same multiplicity sequence as the standard Puiseux expansion of the 2-type($y = t^n$ and $z = t^\alpha + t^\beta$) does and its generalizations).

Assumptions Let $f = f(y, z) = z^n + a_{n-2} y^{\alpha_{n-2}} z^{n-2} + \dots + a_1 y^{\alpha_1} z + a_0 y^{\alpha_0}$ be a W-poly of degree n in z where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in $\mathbb{C}\{y\}$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. By the same way as we have seen in (1.12.0) of Lemma 1.12, assume that we have the same additional inequality:

$$(1.12.0) \quad 2 \leq n \leq \alpha_0.$$

Conclusions

Fact[I]: If f is irreducible in ${}_2\mathcal{O}_0$, f can be represented as follows:

$$(1.12.3) \quad g_1 = z^{n_1} + \xi_1 y^{k_1} \quad \text{with } k_1 = \alpha_{1,0,1} = \beta_{1,0,1},$$

$$f = g_1^{d_2} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } \alpha > 0 \quad \text{and} \quad \beta \leq n-2,$$

where $\xi_1 = \frac{1}{d_2} a_{n-n_1}(0)$, $\alpha_{1,0,1} = \alpha_{n-n_1}$ and the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + \alpha_{1,0,1}\beta > n_1\alpha_{1,0,1}d_2$, with the following:

- (i) $\xi_1 = \frac{1}{d_2} a_{n-n_1}(0)$ is the unique nonzero complex number such that $\xi_1^{d_2} = a_n(0)$ and $\binom{d_2}{i} \xi_1^i = a_{n-in_1}(0)$ for $1 \leq i \leq d_2$.
- (ii) $\frac{\alpha_{n-in_1}}{in_1} = \frac{\alpha_{1,0,1}}{n_1}$ for $1 \leq i \leq d_2$.
- (iii) Either $\frac{\alpha_j}{n-j} > \frac{\alpha_{1,0,1}}{n_1} = \frac{\alpha_0}{n}$ or $n_1\alpha_j + \alpha_{1,0,1}j > n_1\alpha_{1,0,1}d_2$ for any $j \neq in_1$, if exists.

Fact[II]: If f is irreducible in ${}_2\mathcal{O}_0$, f of (1.12.3) can be rewritten in the form

$$(1.12.4) \quad f = g_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} g_1^i,$$

satisfying the following:

(a) For $i = 0, 1, \dots, d_2-1$ and for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $T_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$. Without assuming irreducibility of f in $\mathbb{C}\{y, z\}$, note by Sublemma 1.9 of Theorem 1.8 that a coefficient T_{2,d_2-1} of $g_1^{d_2-1}$ may not be identically zero.

(b) Let i be fixed with $0 \leq i \leq d_2 - 1$. For brevity of notation, let $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ be integer-valued functions, each of which is defined respectively, as follows:

$$(1.12.5) \quad \begin{aligned} \theta_1(t) &= t \text{ for each } t \in \mathbb{N}_0, \\ \theta_2(t_1, t_2) &= t_2 \theta_1(k_1) + n_1 \theta_1(t_1) = t_2 k_1 + n_1 t_1 \text{ for each } (t_1, t_2) \in \mathbb{N}_0^2. \end{aligned}$$

By (1.11.2) of Lemma 1.11, for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $T_{2,i} = T_{2,i}(y, z) \in \mathbb{C}\{y\}[z]$, $\theta_2(\delta_1, \delta_2) > n_1 k_1 (d_2 - i)$.

(c) Following the same properties and notations as we have used in (1.11.4) of Lemma 1.11.1, let $\tau_m = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(g_1)$. With the representation in (1.11.4), $(f \circ \tau_m)_{\text{total}}$ with $(f \circ \tau_m)_{\text{proper}}$ can be rewritten as follows:

$$(1.12.6) \quad \begin{aligned} (f \circ \tau_m)_{\text{total}} &= (f \circ \tau_m)(v, u) = v^{\epsilon_m} u^{\rho_m} (f \circ \tau_m)_{\text{proper}}, \\ (f \circ \tau_m)_{\text{proper}} &= (1 + \xi u)^{d_2} + \sum_{i=0}^{d_2-1} \varepsilon_i v^{\theta_2(\beta_{2,i,1}, \beta_{2,i,2}) + n_1 k_1 i - n_1 k_1 d_2} (1 + \xi u)^i, \end{aligned}$$

satisfying the property (v) with the properties (i), (ii), (iii), (iv) in (1.11.4), as follows:

(v) If $T_{2,i}$ is nonzero for $i = 0, 1, \dots, d_2 - 1$, then there is a unique nonzero monomial $C_{2,i} y^{\beta_{2,i,1}} z^{\beta_{2,i,2}}$ in $T_{2,i}$ with a constant $C_{2,i} = \varepsilon_i(0, 0)$ such that $\theta_2(\beta_{2,i,1}, \beta_{2,i,2}) = \min\{\theta_2(\gamma_1, \gamma_2)\}$ for any nonzero monomial $y^{\gamma_1} z^{\gamma_2}$ in $T_{2,i}$ where $\varepsilon_i = \varepsilon_i(v, 1 + \xi u)$ is a unit in $\mathbb{C}\{v, 1 + \xi u\}$ for $i = 0, 1, \dots, d_2 - 1$ if exists.

(d) For all $i = 0, 1, \dots, d_2 - 1$, the following holds:

$$(1.12.7) \quad \begin{aligned} \gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) &\geq 1 \quad \text{and} \\ \frac{\theta_2(\beta_{2,i,1}, \beta_{2,i,2})}{d_2 - i} &\geq \frac{\theta_2(\beta_{2,0,1}, \beta_{2,0,2})}{d_2} > n_1 k_1. \end{aligned}$$

Then, either $\gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) = 1$ or $1 < \gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) \leq d_2$.

(1d-1) Suppose $\gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) = 1$. Then f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5 if and only if the inequality in (1.12.6) holds and g_1 is irreducible in ${}_2\mathcal{O}_0$ with $g_1 \in$ the type [1] in the sense of Definition 2.5.

(1d-2) Suppose $1 < \gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) \leq d_2$ in (1.12.6). To find an irreducible criterion of any W -poly in $f \in \mathbb{C}\{y\}[z]$ with $f \in$ the type [2] in the sense of Definition 2.5, it remains to solve two subcases respectively:

Subcase(i) of (1d-2) Let $\gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) = d_2$ in (1.12.6). Then, f is either irreducible or not in ${}_2\mathcal{O}_0$. If f is irreducible in ${}_2\mathcal{O}_0$ then $f \in$ the type $[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5.

In this case, we can find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5, following the method as in (b) of Remark 1.12.1.1 for Lemma 1.12.1.

Subcase(ii) of (1d-2) Let $1 < \theta_2(\beta_{2,0,1}, \beta_{2,0,2}) < d_2$ in (1.12.6). Then, f is either irreducible or not in ${}_2\mathcal{O}_0$. If f is irreducible in ${}_2\mathcal{O}_0$ then $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5. \square

Remark 1.12.1.1 for Lemma 1.12.1. (a) Assuming that a coefficient T_{2,d_2-1} of $g_1^{d_2-1}$ is zero, then it is clear that f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5 if and only if $\gcd(d_2, \theta_2(\beta_{2,0,1}, \beta_{2,0,2})) = 1$ and (1.12.7) holds.

(b) Assuming that a coefficient T_{2,d_2-1} of $g_1^{d_2-1}$ is nonzero, by Theorem 1.8 and Lemma 1.12.1 we will find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5. First, using the same kind of properties and notations as in (1.10.2) in Sublemma 1.10 of Theorem 1.8 and Lemma 1.12.1, it suffices to consider the following: Note that $h_{1,1} = g_1$ with $T_{2,i}^{(1)} = T_{2,i}$ as we have seen in Sublemma 1.10.

$$(1.12.8) \quad \begin{cases} h_{1,\nu_1+1} &= z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(\nu_1+1)} z^i \text{ with } h_{1,\nu_1+1} \stackrel{\text{multiseq}}{\sim} g_1 \\ f &= h_{1,\nu_1+1}^{d_2} + \sum_{i=0}^{d_2-2} T_{2,i}^{(\nu_1+1)} h_{1,\nu_1+1}^i \text{ with } T_{2,d_2-1}^{(\nu_1+1)} = 0, \end{cases}$$

for an integer $\nu_1 \geq 0$ where $R_{1,i}^{(\nu_1+1)} \in \mathbb{C}\{y\}$ for $0 \leq i \leq n_1 - 2$, and $T_{2,i}^{(\nu_1+1)} \in \mathbb{C}\{y\}[z]$ for $0 \leq i \leq d_2 - 2$, satisfying the following:

(i) For $0 \leq i \leq n_1 - 2$, each $R_{1,i}^{(\nu_1+1)} = b_i y^{\beta_{1,i,1}^{(\nu_1+1)}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\beta_{1,i,1}^{(\nu_1+1)}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.

(ii) For any nonzero monomial $y^{\gamma_1} z^{\gamma_2}$ in $T_{2,i}^{(\nu_1+1)}$,

$$(1.12.9) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 < n_1.$$

(iii) Using the same kind of properties and notations as in Lemma 1.12.1, if $T_{2,i}^{(\nu_1+1)} \neq 0$, let $C_{2,i}^{(\nu_1+1)} \Pi_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(\nu_1+1)}}$ be a unique nonzero monomial with a constant $C_{2,i}^{(\nu_1+1)}$ in $T_{2,i}^{(\nu_1+1)}$ such that $\theta_2(\beta_{2,i,k}^{(\nu_1+1)})_{k=1}^2 = \min\{\theta_2(\gamma_k)_{k=1}^2\}$ for any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p)}$ where $f_{-1} = y$ and $f_0 = z$.

Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5 if and only if the following inequalities in (1.12.10) hold:

$$(1.12.10) \quad \begin{aligned} \gcd(n_1, \beta_{1,0,1}^{(\nu_1+1)}) &= 1 \quad \text{and} \quad \frac{\theta_1(\beta_{1,i,1}^{(\nu_1+1)})}{n_1 - i} \geq \frac{\theta_1(\beta_{1,0,1}^{(\nu_1+1)})}{n_1} \quad \text{for } 0 \leq i \leq n_1 - 2, \\ \gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) &= 1 \quad \text{and} \\ \frac{\theta_2(\beta_{2,i,k}^{(\nu_1+1)})_{k=1}^2}{d_2 - i} &\geq \frac{\theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2}{d_2} > n_1 \alpha_{1,0,1} \quad \text{for } 0 \leq i \leq d_2 - 2. \end{aligned}$$

Also, if f satisfies (1.12.10), $h_{1,\nu_1+1} \stackrel{\text{multiseq}}{\sim} g_1$ and $f \stackrel{\text{multiseq}}{\sim} g_1^{d_2} + y^{\beta_{2,0,1}^{(\nu_1+1)}} z^{\beta_{2,0,2}^{(\nu_1+1)}}$. \square

Lemma 1.12.α. Assumptions Let $f = f(y, z) = z^n + a_{n-2} y^{\alpha_{n-2}} z^{n-2} + \dots + a_1 y^{\alpha_1} z + a_0 y^{\alpha_0}$ be a W -poly of degree n in z where for $0 \leq i \leq n - 2$, each $a_i = a_i(y)$ is a unit in $\mathbb{C}\{y\}$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. In addition, assume that we have the following:
(1.12.α.0) $n > \alpha_0 \geq 2$.

Conclusions

Case(A) Let $\gcd(n, \alpha_0) = 1$ with a_{n-1} zero. The necessary and sufficient condition for $f(y, z)$ to be irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [1] in the sense of Definition 2.5 is as follows:

$$(1.12.11) \quad \frac{\alpha_i}{n - i} \geq \frac{\alpha_0}{n} \quad \text{for } 0 \leq i \leq n - 2.$$

In this case, $f \stackrel{\text{multiseq}}{\sim} z^n + \xi y^{\alpha_0} = g_1$. Moreover, $V(f) \stackrel{\text{resol}}{\sim} V(g_1)$ at $0 \in \mathbb{C}^2$ in the sense of Definition 2.4.

Case(B) Let $1 < \gcd(n, \alpha_0) < \alpha_0$ with a_{n-1} zero. If f is irreducible in ${}_2\mathcal{O}_0$, then $f \in$ the type $[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5.

Case(C) Let $\gcd(n, \alpha_0) = \alpha_0$ with a_{n-1} zero. If f is irreducible in ${}_2\mathcal{O}_0$, then $f \in \text{the type}[\ell]$ with $\ell \geq 1$ in the sense of Definition 2.5. \square

Remark 1.12.α.1. Let $f(y, z)$ of Lemma 1.12.α be defined by $(z^2 + y)^4 + y^4 z$. Then, $\gcd(n, \alpha_0) = \alpha_0 = 4 > 1$, $f(y, z)$ is irreducible in ${}_2\mathcal{O}_0$, and $f \in \text{the type}[1]$ in the sense of Definition 2.5. \square

§1.7. Irreducibility criterion of W-polys of two complex variables(A generalized representation of irreducible W-polys of two complex variables)

In this section, in order to find irreducibility criterion for germs of analytic functions of two complex variables, without loss of generality, it suffices to find the necessary and sufficient condition for $f(y, z)$ of all the W-polys of two complex variables to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{type}[\ell]$ in the sense of Definition 2.5 in terms of Theorem 1.13, using Theorem 15.2(The WDT for the W-polys) and Theorem 15.4(The Division Algorithm for the W-polys) and Theorem 12.0. In §1.8 and §1.9, as an application of this theorem, it will be found without proof that we can write The 2nd Algorithm and The 3rd Algorithm.

Theorem 1.13(How to find a generalized representation of irreducible W-polys of two complex variables(Irreducibility criterion of W-polys of two complex variables)).

Assumptions Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W-poly of degree $n \geq 2$ in z . Without loss of generality, we may assume that f satisfies the following form:

$$(Eq.1) \quad f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i,$$

where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ for $0 \leq i \leq n-2$, if exists, and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. Write $n = \prod_{k=1}^r n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

In addition, assume that we have the following:

$$(1.13.0) \quad 2 \leq n \leq \alpha_0.$$

Conclusions The necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{type}[\ell]$ in the sense of Definition 2.5(Theorem 12.0) is as follows:

By Theorem 15.4(The Division Algorithm for W-polys) for each $k = 1, 2, \dots, \ell$, f_k and f can be written in the form

$$(1.13.1) \quad \begin{cases} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \\ f_\ell &= f_{\ell-1}^{n_\ell} + \sum_{i=0}^{n_\ell-2} R_{\ell,i} f_{\ell-1}^i \end{cases} \quad \text{with } f = f_\ell$$

where, considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} with $f_{-1} = y$ and $f_0 = z$,

- (i) $n = \prod_{k=1}^\ell n_k$ with $n_k \geq 2$ for $1 \leq k \leq \ell$;
 - (ii) for each fixed k and for each i with $0 \leq i \leq n_k - 2$, $R_{k,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}$;
 - (iii) for each $k = 1, 2, \dots, \ell - 1$, $f_k = f_k(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}]$;
 - (iv) $f = f(y, z, f_1, \dots, f_{\ell-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{\ell-2}\}[f_{\ell-1}]$ with $f = f_\ell$;
- satisfying a finite number of conditions, each of which is represented respectively, as follows:

(1) Condition[A] for $f_1(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5:

$R_{1,i} \in \mathbb{C}\{y\}$ satisfies the properties (1a), (1b) and (1c) for each $i = 0, 1, \dots, n_1 - 2$, and then $f_1 = f_1(y, z) \in \mathbb{C}\{y, z\}$ satisfies the properties (1d).

(1a) For $0 \leq i \leq n_1 - 2$, each $R_{1,i} = b_i y^{\alpha_{1,i,1}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.

(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers.

(1c) $\theta_1(\alpha_{1,i,1}) > (n_1 - i)$ for all $i = 0, 1, \dots, n_1 - 2$, where n_1 is the multiplicity of f_1 at $0 \in \mathbb{C}^2$ with $n_1 \geq 2$.

(1d) For all $i = 0, 1, \dots, n_1 - 2$,

$$(1.13.2) \quad \gcd(n_1, \alpha_{1,0,1}) = 1 \quad \text{and} \quad \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}.$$

(2) Condition[A] for $f_2(y, z) \in \text{the type}[2]$ in the sense of Definition 2.5:

$R_{2,i} \in \mathbb{C}\{y\}[z]$ satisfies the properties (2a), (2b) and (2c) for each $i = 0, 1, \dots, n_2 - 2$, and then $f_2 = f_2(y, z, f_1) \in \mathbb{C}\{y, z\}[f_1]$ satisfies the properties (2d).

(2a) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(2b) Let \mathbb{N}_0^2 be two-dimensional cartesian product of \mathbb{N}_0 . For given integers $n_1, \alpha_{1,0,1}$ and a function θ_1 in $\text{Cond}[A]$ of the 1st type, define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by

$$(1.13.3) \quad \theta_2(t_1, t_2) = t_2 \theta_1(\alpha_{1,0,1}) + n_1 \theta_1(t_1) = t_2 \alpha_{1,0,1} + n_1 t_1 \quad \text{for each } (t_1, t_2) \in \mathbb{N}_0^2.$$

Then, for any two nonzero monomials $y^{\alpha_1} z^{\alpha_2}$ and $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed,

$$(1.13.3-1) \quad \theta_2(\alpha_1, \alpha_2) = \theta_2(\delta_1, \delta_2) \text{ if and only if } \alpha_1 = \delta_1 \text{ and } \alpha_2 = \delta_2.$$

So, there exists a unique nonzero monomial $A_{2,i} y^{\alpha_{2,i,1}} z^{\alpha_{2,i,2}}$ in $R_{2,i}$
with a nonzero constant $A_{2,i}$ such that $\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2}) = \min\{\theta_2(\delta_1, \delta_2)\}$
for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed.

(2c) For all $i = 0, 1, \dots, n_2 - 2$,

$$(1.13.4) \quad \theta_2(\alpha_{2,i,k})_{k=1}^2 > (n_2 - i) n_1 \alpha_{1,0,1}.$$

(2d) For all $i = 0, 1, \dots, n_2 - 2$,

$$(1.13.5) \quad \gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) = 1 \quad \text{and} \quad \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} \geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2}.$$

(3) Condition[A] for $f_m(y, z) \in \text{the type}[m]$ in the sense of Definition 2.5:

For each fixed m with $3 \leq m \leq \ell - 1$, $R_{m,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}$ satisfies the properties (3a), (3b) and (3c) for each $i = 0, 1, \dots, n_m - 2$, and then $f_m = f_m(y, z, f_1, \dots, f_{m-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}[f_{m-1}]$ satisfies the properties (3d).

(3a) For any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with $f_{-1} = y$ and $f_0 = z$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $k = 2, 3, \dots, m$.

(3b) By induction assumption on the integer $(m - 1) \leq \ell - 1$, there exists a sequence $\{f_3, f_4, \dots, f_{m-1}\}$, each of which satisfies the same kind of properties and notations as we have seen in Condition[A] for $f_2(y, z) \in \text{the type}[2]$ in the sense of Definition 2.5. Then inductively, define $\theta_m : \mathbb{N}_0^m \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^m is its m -dimensional cartesian product by

$$(1.13.6) \quad \theta_m(t_k)_{k=1}^m = t_m \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1} + n_{m-1} \theta_{m-1}(t_k)_{k=1}^{m-1} \quad \text{for each } (t_k)_{k=1}^m \in \mathbb{N}_0^m,$$

where recall by induction assumption that for a fixed i , $A_{m-1,i} \Pi_{k=1}^{m-1} f_{k-2}^{\alpha_{m-1,i,k}}$ is a unique nonzero monomial in $R_{m-1,i}$ with a constant $A_{m-1,i}$ such that

$$(1.13.7) \quad \theta_{m-1}(\alpha_{m-1,i,k})_{k=1}^{m-1} = \min\{\theta_{m-1}(\delta_k)_{k=1}^{m-1}\},$$

for any nonzero monomial $\Pi_{k=1}^{m-1} f_{k-2}^{\delta_k}$ in $R_{m-1,i}$.

Then, for any two nonzero monomials $\Pi_{k=1}^m f_{k-2}^{\alpha_k}$ and $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with i fixed,

$$(1.13.7-1) \quad \theta_m(\alpha_k)_{k=1}^m = \theta_m(\delta_k)_{k=1}^m \text{ if and only if } \alpha_k = \delta_k \text{ for } k = 1, 2, \dots, m.$$

So, there exists a unique nonzero-monomial $A_{m,i} \Pi_{k=1}^m f_{k-2}^{\alpha_{m,i,k}}$ in $R_{m,i}$

with a constant $A_{m,i}$ such that $\theta_m(\alpha_{m,i,k})_{k=1}^m = \min\{\theta_m(\delta_k)_{k=1}^m\}$

for any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$.

(3c) For all $i = 0, 1, \dots, n_m - 2$,

$$(1.13.8) \quad \theta_m(\alpha_{m,i,k})_{k=1}^m > (n_m - i)n_{m-1}\theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1}.$$

(3d) For all $i = 0, 1, \dots, n_m - 2$,

$$(1.13.9) \quad \begin{aligned} \gcd(n_m, \theta_m(\alpha_{m,0,k})_{k=1}^m) &= 1 \quad \text{and} \\ \frac{\theta_m(\alpha_{m,i,k})_{k=1}^m}{n_m - i} &\geq \frac{\theta_m(\alpha_{m,0,k})_{k=1}^m}{n_m}. \quad \square \end{aligned}$$

Remark 1.13.1. (1) There is nothing to prove for Theorem 1.13, because the sufficiency of the condition in Theorem 1.13 can be proved by Theorem 16.5, and the necessity of the condition in Theorem 1.13 can be proved by Theorem 16.6, too.

(2) The converse of Theorem 16.5 can be represented by Theorem 16.6. Moreover, we can compute irreducibility criterion of W-polys defining plane curve singularities at the origin in \mathbb{C}^2 in the process of the proof of Theorem 16 together with Proposition 16.7 and Proposition 16.8 completely and rigorously, using the Euclidean algorithm and Theorem 15.4(The Division Algorithm for the W-polys). \square

Remark 1.13.2. Consider the sequence $S = \{f_k : 1 \leq k \leq \ell\}$ with $f_\ell = f$ where $f_k = f_k(y, z, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-1}\}$, which have the same properties and notations as we have seen in (1.13.1) of the conclusion of Theorem 1.13. If $f \in \mathbb{C}\{y\}[z]$ is irreducible in $\mathbb{C}\{y, z\}$, then $f = f_\ell(y, z, f_1, \dots, f_{\ell-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{\ell-2}\}[f_{\ell-1}]$ is an irreducible W -poly of degree n_ℓ in $f_{\ell-1}$ with coefficients in $\mathbb{C}\{y, z, f_1, \dots, f_{\ell-2}\}$ and with multiplicity n_ℓ at the origin in \mathbb{C}^ℓ . \square

Corollary 1.13.3. Assumptions Under the same assumption and conclusion as in Theorem 1.13, note that f_k is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $(0, 0)$ in \mathbb{C}^2 for $k \geq 1$. In particular, for each $k = 1, 2, \dots, \ell$, let $V(H_k) = \{(y, z) : H_k(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is defined as follows:

$$(1.13.3.1) \quad \begin{aligned} (i) \quad & H_1 = z^{n_1} + y^{\alpha_{1,0,1}} \quad \text{with } n_1 \geq 2 \text{ and } \alpha_{1,0,1} \geq 2. \\ (ii) \quad & H_2 = H_1^{n_2} + y^{\alpha_{2,0,1}} z^{\alpha_{2,0,2}}. \\ & \dots \dots \dots \\ (\ell) \quad & H_\ell = H_{\ell-1}^{n_\ell} + y^{\alpha_{\ell,0,1}} z^{\alpha_{\ell,0,2}} H_1^{\alpha_{\ell,0,3}} \dots H_{\ell-2}^{\alpha_{\ell,0,\ell}}. \end{aligned}$$

Conclusions

Fact[I]: Then, $f_k \stackrel{\text{multiseq}}{\sim} H_k$ for each $k = 1, 2, \dots, \ell$.

Fact[II]: Then, $f_k \stackrel{\text{resol}}{\sim} H_k$ for each $k = 1, 2, \dots, \ell$. \square

Remark for Corollary 1.13.3. (I) Note by Theorem 5.0 that H_{j+1} is irreducible in $\mathbb{C}\{y, z\}$ with $H_{j+1} \in$ the type[j+1] in the sense of Definition of 2.5 $\iff \gcd(n_1, \alpha_{1,0,1}) = 1$, $\gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) = 1, \dots, \gcd(n_{j+1}, \theta_{j+1}(\alpha_{j+1,0,k})_{k=1}^{j+1}) = 1$.

(II) Note that f_{j+1} is irreducible in $\mathbb{C}\{y, z\}$ with $f_{j+1} \in$ the type[j+1] in the sense of Definition of 2.5 \iff the following holds:

$$(1) \quad \gcd(n_1, \alpha_{1,0,1}) = 1 \text{ and } \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1} \text{ for } 0 \leq i \leq n_1 - 2.$$

$$(2) \quad \gcd(n_2, \theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})) = 1 \text{ and } \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} \geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2} \text{ for } 0 \leq i \leq n_2 - 2.$$

$\dots \dots \dots$

$$(j+1) \quad \gcd(n_{j+1}, \theta_{j+1}(\alpha_{j+1,0,k})_{k=1}^{j+1}) = 1 \text{ and } \frac{\theta_{j+1}(\alpha_{j+1,i,k})_{k=1}^{j+1}}{n_{j+1} - i} \geq \frac{\theta_{j+1}(\alpha_{j+1,0,k})_{k=1}^{j+1}}{n_{j+1}} \text{ for } 0 \leq i \leq n_{j+1} - 2. \quad \square$$

Theorem 1.14(A generalized representation of irreducible W-polys of two complex variables(Irreducibility criterion of W-polys of two complex variables)).

Assumptions Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W-poly of degree $n \geq 2$ in z . Without loss of generality, we may assume that f satisfies the following form:

$$(Eq.1) \quad f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i,$$

where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ for $0 \leq i \leq n-2$, if exists, and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. Write $n = \prod_{k=1}^r n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

In preparation for finding the irreducibility criterion for $f(y, z)$ of all the W-polys of two complex variables, it suffices to consider three cases, respectively:

$$(1.14.0) \quad \begin{array}{ll} \text{Case}(\alpha) & 2 \leq n < \alpha_0, \\ \text{Case}(\beta) & 2 \leq \alpha_0 < n \text{ with } \gcd(n, \alpha_0) < \alpha_0, \\ \text{Case}(\gamma) & n = p\alpha_0 \text{ for an integer } p > 0. \end{array}$$

Conclusions (I) Let Case(α) hold. If $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$, note that $2 \leq n < \alpha_0$ if and only if $2 \leq n < \alpha_0$ with $\gcd(n, \alpha_0) < n$.

Then, the necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{type}[\ell]$ in the sense of Definition 2.5(Theorem 12.0) was already given by the condition in the conclusion in Theorem 1.13.

(II) Let Case(β) hold. Then, the necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{type}[\ell]$ in the sense of Definition 2.5 is the same as the given condition by the same method as we have used in the conclusion in Theorem 1.13.

(III) Let Case(γ) hold. Then, the necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{type}[\ell-1]$ in the sense of Definition 2.5 is as follows:

By Theorem 15.4(The Division Algorithm for W-polys) for each $k = 1, 2, \dots, \ell-1$, f_k and f can be written in the form

$$(1.14.1) \quad \begin{cases} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \\ f &= f_{\ell-1}^{n_\ell} + \sum_{i=0}^{n_\ell-2} R_{\ell,i} f_{\ell-1}^i \end{cases}$$

where, considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} with $f_{-1} = y$ and $f_0 = z$,

- (i) $n = \prod_{k=1}^\ell n_k$ with $n_k \geq 2$ for $1 \leq k \leq \ell$;
 - (ii) for each fixed k and for each i with $0 \leq i \leq n_k - 2$, $R_{k,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}$;
 - (iii) for each $k = 1, 2, \dots, \ell-1$, $f_k = f_k(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}]$;
 - (iv) $f = f(y, z, f_1, \dots, f_{\ell-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{\ell-2}\}[f_{\ell-1}]$ with $f = f_\ell$;
- satisfying a finite number of conditions, each of which is represented respectively, as follows:

(1) Condition[A] for $f_1(y, z) \in \text{the type}[0]$ in the sense of Definition 2.5:

$R_{1,i} \in \mathbb{C}\{y\}$ satisfies the properties (1a), (1b) and (1c) for each $i = 0, 1, \dots, n_1 - 2$, and then $f_1 = f_1(y, z) \in \mathbb{C}\{y, z\}$ satisfies the properties (1d).

(1a) For $0 \leq i \leq n_1 - 2$, each $R_{1,i} = b_i y^{\alpha_{1,i,1}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.

(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers.

(1c) $\theta_1(\alpha_{1,i,1}) > (\alpha_{1,0,1} - i)$ for all $i = 0, 1, \dots, n_1 - 2$, where $\alpha_{1,0,1} = 1$ is the multiplicity of f_1 at $0 \in \mathbb{C}^2$ with $n_1 \geq 2$.

(1d) For all $i = 0, 1, \dots, n_1 - 2$,

$$(1.14.2) \quad \gcd(n_1, \alpha_{1,0,1}) = 1 \quad \text{and} \\ \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}.$$

(2) Condition[A] for $f_2(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5:

$R_{2,i} \in \mathbb{C}\{y\}[z]$ satisfies the properties (2a), (2b) and (2c) for each $i = 0, 1, \dots, n_2 - 2$, and then $f_2 = f_2(y, z, f_1) \in \mathbb{C}\{y, z\}[f_1]$ satisfies the properties (2d).

(2a) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(2b) Let \mathbb{N}_0^2 be two-dimensional cartesian product of \mathbb{N}_0 . For given integers $n_1, \alpha_{1,0,1}$ and a function θ_1 in $\text{Cond}[A]$ of the 1st type, define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by

$$(1.14.3) \quad \theta_2(t_1, t_2) = t_2 \theta_1(\alpha_{1,0,1}) + n_1 \theta_1(t_1) = t_2 \alpha_{1,0,1} + n_1 t_1 \quad \text{for each } (t_1, t_2) \in \mathbb{N}_0^2.$$

Then, for any two nonzero monomials $y^{\alpha_1} z^{\alpha_2}$ and $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed,

$$(1.14.3-1) \quad \theta_2(\alpha_1, \alpha_2) = \theta_2(\delta_1, \delta_2) \text{ if and only if } \alpha_1 = \delta_1 \text{ and } \alpha_2 = \delta_2. \\ \text{So, there exists a unique nonzero monomial } A_{2,i} y^{\alpha_{2,i,1}} z^{\alpha_{2,i,2}} \text{ in } R_{2,i} \\ \text{with a nonzero constant } A_{2,i} \text{ such that } \theta_2(\alpha_{2,i,1}, \alpha_{2,i,2}) = \min\{\theta_2(\delta_1, \delta_2)\} \\ \text{for any nonzero monomial } y^{\delta_1} z^{\delta_2} \text{ in } R_{2,i} \text{ with } i \text{ fixed.}$$

(2c) For all $i = 0, 1, \dots, n_2 - 2$,

$$(1.14.4) \quad \theta_2(\alpha_{2,i,k})_{k=1}^2 > (n_2 - i) n_1 \alpha_{1,0,1}.$$

(2d) For all $i = 0, 1, \dots, n_2 - 2$,

$$(1.14.5) \quad \gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) = 1 \quad \text{and} \\ \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} \geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2}.$$

(3) Condition[A] for $f_m(y, z) \in \text{the type}[m-1]$ in the sense of Definition 2.5:

For each fixed m with $3 \leq m \leq \ell - 1$, $R_{m,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}$ satisfies the properties (3a), (3b) and (3c) for each $i = 0, 1, \dots, n_m - 2$, and then $f_m = f_m(y, z, f_1, \dots, f_{m-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}[f_{m-1}]$ satisfies the properties (3d).

(3a) For any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with $f_{-1} = y$ and $f_0 = z$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $k = 2, 3, \dots, m$.

(3b) By induction assumption on the integer $(m-1) \leq \ell - 1$, there exists a sequence $\{f_3, f_4, \dots, f_{m-1}\}$, each of which satisfies the same kind of properties and notations as we have seen in Condition[A] for $f_2(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5. Then inductively, define $\theta_m : \mathbb{N}_0^m \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^m is its m -dimensional cartesian product by

$$(1.14.6) \quad \theta_m(t_k)_{k=1}^m = t_m \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1} + n_{m-1} \theta_{m-1}(t_k)_{k=1}^{m-1} \quad \text{for each } (t_k)_{k=1}^m \in \mathbb{N}_0^m,$$

where recall by induction assumption that for a fixed i , $A_{m-1,i} \Pi_{k=1}^{m-1} f_{k-2}^{\alpha_{m-1,i,k}}$ is a unique nonzero monomial in $R_{m-1,i}$ with a constant $A_{m-1,i}$ such that

$$(1.14.7) \quad \theta_{m-1}(\alpha_{m-1,i,k})_{k=1}^{m-1} = \min\{\theta_{m-1}(\delta_k)_{k=1}^{m-1}\},$$

for any nonzero monomial $\Pi_{k=1}^{m-1} f_{k-2}^{\delta_k}$ in $R_{m-1,i}$.

Then, for any two nonzero monomials $\Pi_{k=1}^m f_{k-2}^{\alpha_k}$ and $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with i fixed,

$$(1.14.7-1) \quad \theta_m(\alpha_k)_{k=1}^m = \theta_m(\delta_k)_{k=1}^m \text{ if and only if } \alpha_k = \delta_k \text{ for } k = 1, 2, \dots, m.$$

So, there exists a unique nonzero-monomial $A_{m,i} \Pi_{k=1}^m f_{k-2}^{\alpha_{m,i,k}}$ in $R_{m,i}$

with a constant $A_{m,i}$ such that $\theta_m(\alpha_{m,i,k})_{k=1}^m = \min\{\theta_m(\delta_k)_{k=1}^m\}$

for any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$.

(3c) For all $i = 0, 1, \dots, n_m - 2$,

$$(1.14.8) \quad \theta_m(\alpha_{m,i,k})_{k=1}^m > (n_m - i)n_{m-1}\theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1}.$$

(3d) For all $i = 0, 1, \dots, n_m - 2$,

$$(1.14.9) \quad \begin{aligned} \gcd(n_m, \theta_m(\alpha_{m,0,k})_{k=1}^m) &= 1 \quad \text{and} \\ \frac{\theta_m(\alpha_{m,i,k})_{k=1}^m}{n_m - i} &\geq \frac{\theta_m(\alpha_{m,0,k})_{k=1}^m}{n_m}. \quad \square \end{aligned}$$

Corollary 1.14.1. Assumptions Suppose that the same assumption as in Theorem 1.14 holds. In addition, assume that we have the following:

$$(1.14.0) \quad n = p\alpha_0 \text{ for an integer } p > 0.$$

In particular, for each $k = 1, 2, \dots, \ell$, let $V(H_k) = \{(y, z) : H_k(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is defined as follows:

$$(1.14.1.1) \quad \begin{aligned} (i) \quad &H_1 = z^{n_1} + y^{\alpha_{1,0,1}} \quad \text{with } n_1 \geq 2 \text{ and } \alpha_{1,0,1} \geq 1. \\ (ii) \quad &H_2 = H_1^{n_2} + y^{\alpha_{2,0,1}} z^{\alpha_{2,0,2}}. \\ &\dots\dots\dots \\ (\ell) \quad &H_\ell = H_{\ell-1}^{n_\ell} + y^{\alpha_{\ell,0,1}} z^{\alpha_{\ell,0,2}} H_1^{\alpha_{\ell,0,3}} \dots H_{\ell-2}^{\alpha_{\ell,0,\ell}}. \end{aligned}$$

Conclusions

Fact[I]: Then, $f_k \stackrel{\text{multiseq}}{\sim} H_k$ for each $k = 1, 2, \dots, \ell$.

Fact[II]: Then, $f_k \stackrel{\text{resol}}{\sim} H_k$ for each $k = 1, 2, \dots, \ell$. \square

§1.8. The 2nd Algorithm for computing completely irreducible W-polys from all the W-polys of two complex variables.

Noting that the statements of Theorem 1.15 and Corollary 1.15.1 are different, observe by either Theorem 1.15 or Corollary 1.15.1 that The 2nd Algorithm can be solved as follows:

Assumptions Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W-poly of degree $n \geq 2$ in z , satisfying the same properties and notations as in the assumptions of Theorem 1.15(Corollary 1.15.1).

Conclusions

(i) The aim of Theorem 1.15 is to prove the following:

We can compute explicitly when f is irreducible in $\mathbb{C}\{y, z\}$. If f is irreducible in $\mathbb{C}\{y, z\}$, we can find $H_\ell \in \text{Family}(1)$ such that $f \stackrel{\text{multiseq}}{\sim} H_\ell$ for any irreducible W-poly $f \in \mathbb{C}\{y\}[z] \iff$ by the induction method on the positive integer ℓ we must follow the computations over all the k-steps, $k = 1, 2, \dots, \ell$, in Theorem 1.15. Note by Definition 1.2 that $H_\ell \in \mathbb{C}\{y\}[z]$ is called the standard irreducible(Puiseux) W-poly of the recursive ℓ -type in z .

(ii) The aim of Corollary 1.15.1 is to prove the following:

We can compute explicitly when f is irreducible in $\mathbb{C}\{y, z\}$. If f is irreducible in $\mathbb{C}\{y, z\}$, we can find a generalized representation of any irreducible W-poly $f \in \mathbb{C}\{y\}[z]$ with $f \stackrel{\text{multiseq}}{\sim} H_\ell$ in the sense of Theorem 1.13 \iff by the induction method on the positive integer ℓ we must follow the computations over all the k-steps, $k = 1, 2, \dots, \ell$, in Corollary 1.15.1.

Theorem 1.15(The 2nd Algorithm: Explicit algorithm for finding completely irreducible W-polys from all the W-polys of two complex variables).

Assumptions Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W-poly of degree $n \geq 2$ in z . Without loss of generality, we may assume that f satisfies the following form:

$$(Eq.1) \quad f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i,$$

where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ for $0 \leq i \leq n-2$, if exists, and the α_i are positive integers. Note that a_{n-1} is identically zero. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $d_2 = \gcd(n, \alpha_0)$. Write $n = \prod_{k=1}^r n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

In addition, assume that we have the following:

$$(1.15.0) \quad 2 \leq n \leq \alpha_0.$$

Conclusions If f is irreducible in ${}_2\mathcal{O}_0$ with isolated singularity at $0 \in \mathbb{C}^2$, then $f \in$ the type $[\ell]$ for some $\ell \leq r$ in the sense of Definition 2.5. By the induction method on the positive integer r , the aim is to compute an elementary algorithm for finding irreducible W-polys from all the W-polys in $\mathbb{C}\{y\}[z]$, using q iterations of the following steps with $q \leq \ell$: Observe that the statement on the 3rd step may be omitted if necessary, to simplify the statements for this theorem by the induction method.

The 1st step: To find the irreducibility algorithm for $f \in$ the type [1] in the sense of Definition 2.5.

With equations in (Eq.2), the aim in this step is how to compute the necessary and sufficient condition for $f \in$ the type [1] in the sense of Definition 2.5.

If f is irreducible in ${}_2\mathcal{O}_0$, then f of (Eq.1) must satisfy the following necessary condition:

$$(Eq.2) \quad \frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n} \quad \text{and} \quad 1 \leq \gcd(n, \alpha_0) < n \quad \text{for} \quad 0 \leq i \leq n-2.$$

If f satisfies (Eq.2), it suffices to consider the following two cases for the 1st step:

Case(A) $\gcd(n, \alpha_0) = 1$ and Case(B) $1 < \gcd(n, \alpha_0) < n$.

Case(A) of The 1st step: Let $\gcd(n, \alpha_0) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0 \iff$ (Eq.2) holds. In this case, $f \in$ the type [1] in the sense of Definition 2.5.

Remark for Case(A) of The 1st step. The equation in (Eq.1) itself is a generalized representation of f in the sense of Theorem 1.13, since $a_{n-1} = 0$.

Case(B) of The 1st step: Let $1 < \gcd(n, \alpha_0) < n$. Then, take the 2nd step. \square

Remark for The 1st step: To find $H_1 \in$ Family(1) such that $f \stackrel{\text{multiseq}}{\sim} H_1$ for any $f \in$ the type [1] in the sense of Definition 2.5. In Case(A) of The 1st step, $f \in$

the type [1] in the sense of Definition 2.5 if and only if $f \stackrel{\text{multiseq}}{\sim} H_1$ where $H_1 = z^n + y^{\alpha_0}$ with $\gcd(n, \alpha_0) = 1$. \square

The 2nd step: To find the irreducibility algorithm for $f \in$ the type [2] in the sense of Definition 2.5.

With the equations, which are defined by (Eq.3), (Eq.4) and (Eq.5) in the 2nd step later, the aim in this step is how to compute the necessary and sufficient condition for $f \in$ the type [2] in the sense of Definition 2.5.

For the irreducibility algorithm for the 2nd step, it suffices to consider Case(B) for the 1st step only. Let $1 < \gcd(n, \alpha_0) < n$.

Then, note that $f \in$ the type $[\ell]$ for some $\ell \geq 2$ in the sense of Definition 2.5.

Let $d_2 = \gcd(n, \alpha_0)$, and then write $n = n_1 d_2$ and $\alpha_0 = \alpha_{1,0,1} d_2$. To solve the above problem, if f is irreducible in ${}_2\mathcal{O}_0$, it is easy to compute by (1.12.3) of Lemma 1.12.1 that f of (Eq.1) must satisfy the following necessary condition:

$$(Eq.3) \quad (a) \quad g_1 = z^{n_1} + \xi_1 y^{\alpha_{1,0,1}},$$

$$(b) \quad f = g_1^{d_2} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } n_1 \alpha + \alpha_{1,0,1} \beta > n_1 \alpha_{1,0,1} d_2, \alpha > 0, \beta \leq n - 2,$$

where $\xi_1 = \frac{1}{d_2} a_{n-n_1}(0)$, $\alpha_{1,0,1} = \alpha_{n-n_1}$ and the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers $\alpha > 0$ and $\beta \leq n - 2$ such that $n_1 \alpha + \alpha_{1,0,1} \beta > n_1 \alpha_{1,0,1} d_2$.

Then, g_1 must satisfy the following necessary condition:

$$(Eq.3.1) \quad 1 < d_2 < n \quad \text{and} \quad \xi_1 = \frac{1}{d_2} a_{n-n_1}(0) \neq 0.$$

Note that g_1 is irreducible in ${}_2\mathcal{O}_0$, since $\gcd(n_1, \alpha_{1,0,1}) = 1$.

Apply the WDT(Theorem 1.7) to $f(y, z)$ with a divisor $g_1(y, z)$. Whether or not f is irreducible in ${}_2\mathcal{O}_0$, by either (1.12.4) of Lemma 1.12.1 or (1.9.1) of Sublemma 1.9, (g_1, f) can be written in the form

$$(Eq.4)(Eq.4.1) \quad \begin{cases} g_1 &= z^{n_1} + \xi_1 y^{\sigma_1} \\ f &= g_1^{d_2} + \sum_{i=1}^{d_2-1} T_{2,i}^{(1)} g_1^i, \end{cases} \quad \text{with } \sigma_1 = \alpha_{1,0,1} \quad \text{and} \quad \xi_1 = \frac{1}{d_2} a_{n-n_1}(0),$$

where $T_{2,i}^{(1)} \in \mathbb{C}\{y\}[z]$ for $0 \leq i \leq d_2 - 1$, satisfying the following (i) and (ii):

(i) For any nonzero monomial $y^{\gamma_1} z^{\gamma_2}$ in $T_{2,i}^{(1)}$,

$$(Eq.4.2) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 < n_1.$$

(ii) Define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by $\theta_2(t_k)_{k=1}^2 = t_2 \alpha_{1,0,1} + n_1 t_1$. Then, θ_2 is one-to-one.

If $T_{2,i}^{(1)} \neq 0$, let $C_{2,i}^{(1)} \prod_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(1)}}$ be a unique nonzero monomial with a constant $C_{2,i}^{(1)}$ in $T_{2,i}^{(1)}$ such that $\theta_2(\beta_{2,i,k}^{(1)})_{k=1}^2 = \min\{\theta_2(\gamma_k)_{k=1}^2\}$ for any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(1)}$. Note by Lemma 1.12.1 that the construction of $T_{2,i}^{(1)}$ is trivial where $f_{-1} = y$ and $f_0 = z$.

If f is irreducible in ${}_2\mathcal{O}_0$, then f must satisfy the following necessary condition:

$$(Eq.4.3) \quad \frac{\theta_2(\beta_{2,i,k}^{(1)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2}{d_2} > n_1 \alpha_{1,0,1} \quad \text{for } 0 \leq i \leq d_2 - 1.$$

If f satisfies (Eq.4.3), in order to find an irreducibility algorithm for f , it suffices to consider the following two cases for the 2nd step:

Case(A) $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 1$ and Case(B) $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$.

Case(A) of the 2nd step: Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ \iff (Eq.4.3) holds.

In this case, $f \in$ the type [2] in the sense of Definition 2.5. So $f \stackrel{\text{multiseq}}{\sim} H_1^{d_2} + \prod_{k=1}^2 H_{k-2}^{\sigma_{2,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$ and $\sigma_{2,k} = \beta_{2,0,k}^{(1)}$ for $1 \leq k \leq 2$.

Remark for Case(A) of the 2nd step. Note that (g_1, f) in (Eq.4.1) may not be a generalized representation of f in the sense of Theorem 1.13, since $T_{2,d_2-1}^{(1)}$ may not be zero. To find a generalized representation of f in the sense of Theorem 1.13, until we get the same result in the conclusion of Sublemma 1.10(Sublemma 15.5), it suffices to apply either Sublemma 1.10(Sublemma 15.5) or the same kind of a fine sequence of pairs in (Eq.5) to (g_1, f) .

Case(B) of the 2nd step: Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$. To find a necessary condition for f to be irreducible in ${}_2\mathcal{O}_0$, we may assume that f satisfies (Eq.2), (Eq.3) and (Eq.4).

Then, it suffices to consider the following two subcases of Case(B) for the 2nd step:

Subcase(B1) $T_{2,d_2-1}^{(1)} \neq 0$, and Subcase(B2) $T_{2,d_2-1}^{(1)} = 0$.

Subcase(B1) of the 2nd step: Assume that $T_{2,d_2-1}^{(1)} \neq 0$. To solve the subcase, first of all, we must eliminate $T_{2,d_2-1}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Sublemma 1.10 for Theorem 1.8 (Sublemma 15.5 for Theorem 15.4), we can compute a unique finite sequence of pairs $\{(h_{1,p}, f) : 1 \leq p \leq \nu_1 + 1\}$ in (Eq.5) for a unique integer $\nu_1 \leq \frac{n_1+1}{2}$, each pair of which can be written in the form

$$(Eq.5)(Eq.5.1)(Eq.5.1.1) \quad \begin{cases} h_{1,1} &= z^{n_1} + \xi_1 y^{\alpha_{1,0,1}} = z^{n_1} + R_{1,0}^{(1)} \text{ with } h_{1,1} = g_1 \\ f &= h_{1,1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(1)} h_{1,1}^i, \end{cases}$$

and for $1 \leq p \leq \nu_1 - 1$,

$$(Eq.5.1.2) \quad \begin{cases} h_{1,p+1} &= h_{1,p} + \frac{1}{d_2} T_{2,d_2-1}^{(p)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p+1)} z^i \\ f &= h_{1,p+1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{1,p+1}^i \text{ with } T_{2,d_2-1}^{(p+1)} \neq 0, \end{cases}$$

and

$$(Eq.5.1.3) \quad \begin{cases} h_{1,\nu_1+1} &= h_{1,\nu_1} + \frac{1}{d_2} T_{2,d_2-1}^{(\nu_1)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(\nu_1+1)} z^i \\ f &= h_{1,\nu_1+1}^{d_2} + \sum_{i=0}^{d_2-2} T_{2,i}^{(\nu_1+1)} h_{1,\nu_1+1}^i \text{ with } T_{2,d_2-1}^{(\nu_1+1)} = 0, \end{cases}$$

where $R_{1,i}^{(p)} \in \mathbb{C}\{y\}$ for $1 \leq p \leq \nu_1 + 1$ and $0 \leq i \leq n_1 - 2$; and $T_{2,i}^{(p)} \in \mathbb{C}\{y\}[z]$ for $1 \leq p \leq \nu_1 + 1$ and $0 \leq i \leq d_2 - 1$, satisfying the following properties:

- (i) For any nonzero monomial y^δ in $R_{1,i}^{(p)}$, $\delta > 0$.
- (ii) For any nonzero monomial $y^{\gamma_1} z^{\gamma_2}$ in $T_{2,i}^{(p)}$,

$$(Eq.5.2) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 < n_1.$$

(iii) If $T_{2,i}^{(p)} \neq 0$, let $C_{2,i}^{(p)} \prod_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(p)}}$ be a unique nonzero monomial with a constant $C_{2,i}^{(p)}$ in $T_{2,i}^{(p)}$ such that $\theta_2(\beta_{2,i,k}^{(p)})_{k=1}^2 = \min\{\theta_2(\gamma_k)_{k=1}^2\}$ for any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p)}$ where $f_{-1} = y$ and $f_0 = z$.

Remark. It is easy to compute $C_{2,i}^{(p)} \prod_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(p)}}$ by an elementary way because $T_{2,i}^{(p)} \in \mathbb{C}\{y\}[z]$ is a polynomial of finite degree $< n_1$ in z and θ_2 is one to one.

If f is irreducible in ${}_2\mathcal{O}_0$, then f satisfy the following necessary condition:

$$(Eq.5.3)(Eq.5.3.1) \quad \frac{\theta_2(\beta_{2,i,k}^{(p)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(p)})_{k=1}^2}{d_2} > n_1 \alpha_{1,0,1} \quad \text{for } 1 \leq p \leq \nu_1, 0 \leq i \leq d_2 - 1.$$

$$(Eq.5.3.2) \quad \frac{\theta_2(\beta_{2,i,k}^{(\nu_1+1)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2}{d_2} > n_1 \alpha_{1,0,1} \quad \text{for } 0 \leq i \leq d_2 - 2.$$

Remark. If f satisfies (Eq.5.3), $h_{1,p} \stackrel{\text{multiseq}}{\sim} h_{1,1} = g_1$ for all $p \geq 1$.

Assuming that f satisfies (Eq.5.3), to find an irreducibility algorithm for Subcase(B1), it suffices to consider the following two subcases, Subcase(B1-a) and Subcase(B1-b):

Subcase(B1-a) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) = 1$.

Subcase(B1-b) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$.

Subcase(B1-a) of the 2nd step

(a) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.5.3.2) holds. Thus, $f \in$ the type [2] in the sense of Definition 2.5.

(b) Assume that $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) = 1$ for some $p \leq \nu_1 + 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if an inequality in (Eq.5.3.1) holds without mentioning any inequality in (Eq.5.3.2).

Remark for Subcase(B1-a) of the 2nd step. $(f_1, f) = (h_{1,\nu_1+1}, f)$ is a generalized representation of f in the sense of Theorem 1.13 since $T_{2,d_2-1}^{(\nu_1+1)} = 0$.

Subcase(B1-b) of the 2nd step Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$, noting that $T_{2,d_2-1}^{(\nu_1+1)} = 0$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the next step.

Subcase(B2) of the 2nd step It was assumed by this subcase that $T_{2,d_2-1}^{(1)} = 0$, noting that $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the next step. \square

Remark for The 2nd step: To find $H_2 \in \text{Family}(1)$ such that $f \stackrel{\text{multiseq}}{\sim} H_2$ for any $f \in \text{the type [2]}$ in the sense of Definition 2.5. In this case, $f \in \text{the type [2]}$ in the sense of Definition 2.5, and so $f \stackrel{\text{multiseq}}{\sim} H_2 = H_1^{d_2} + \Pi_{k=1}^2 H_{k-2}^{\sigma_{2,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$ and $\sigma_{2,k} = \beta_{2,0,k}^{(\nu_1+1)}$ for $1 \leq k \leq 2$ and for some positive integer $\nu_1 + 1$. \square

The 3rd step: To find the irreducibility algorithm for $f \in \text{the type [3]}$ in the sense of Definition 2.5.

With three equations (Eq.6), (Eq.7) and (Eq.8) in the 3rd step, the aim in this step is to find the necessary and sufficient condition for $f \in \text{the type [3]}$ in the sense of Definition 2.5.

For the proof of this step, recall the defining equation (h_{1,ν_1+1}, f) of (Eq.5) as we have seen in Subcase(B1) of the 2nd step:

$$(Eq.5) \quad \begin{cases} h_{1,\nu_1+1} &= z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(\nu_1+1)} z^i \\ f &= h_{1,\nu_1+1}^{d_2} + \sum_{i=0}^{d_2-2} T_{2,i}^{(\nu_1+1)} h_{1,\nu_1+1}^i \text{ with } T_{2,d_2-1}^{(\nu_1+1)} = 0. \end{cases}$$

By either Subcase(B1-b) or Subcase(B2) of the 2nd step, it may be assumed that $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$ and $T_{2,d_2-1}^{(\nu_1+1)} = 0$ for some positive integer $\nu_1 + 1$.

Remark. If $T_{2,d_2-1}^{(1)} = 0$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$ from $(h_{1,1}, f)$ of (Eq.5.1), then $(h_{1,1}, f) = (g_1, f)$ can be viewed as (h_{1,ν_1+1}, f) of (Eq.5) with $\nu_1 = 0$.

For brevity of notation, $h_{1,\nu_1+1}, T_{2,i}^{(\nu_1+1)}, C_{2,i}^{(\nu_1+1)}$ and $\beta_{2,i,k}^{(\nu_1+1)}$ can be replaced by $f_1, T_{2,i}, C_{2,i}$ and $\beta_{2,i,k}$ for $0 \leq i \leq d_2 - 2$ and $1 \leq k \leq 2$, respectively because $g_1 = h_{1,1} \stackrel{\text{multiseq}}{\sim} h_{1,p}$ for all $p \geq 1$ and $T_{2,d_2-1}^{(\nu_1+1)} = 0$. Recall that $f_1 = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i} z^i$ from (h_{1,ν_1+1}, f) where $R_{1,i}^{(\nu_1+1)}$ is defined to be $R_{1,i}$ for each i .

Let $d_3 = \gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2)$, and then write $d_2 = n_2 d_3$. To solve the above problem, we need to construct g_2 as follows:

$$(Eq.6) \quad g_2 = f_1^{n_2} + \xi_2 y^{\sigma_{2,1}} z^{\sigma_{2,2}} \in \mathbb{C}\{y\}[z, f_1],$$

where $\xi_2 = \frac{1}{d_2} C_{2,d_2-n_2}$ and $\sigma_{2,k} = \beta_{2,d_2-n_2,k}$ for $1 \leq k \leq 2$.

Then, g_2 must satisfy the following necessary condition:

$$(Eq.6.1) \quad 1 < d_3 < d_2, \quad \xi_2 \neq 0, \quad \theta_2(\sigma_{2,k})_{k=1}^2 > n_2 n_1 \beta_{1,0,1},$$

and $\gcd(n_2, \theta_2(\sigma_{2,k})_{k=1}^2) = 1$.

If g_2 of (Eq.6) satisfies inequalities in (Eq.6.1), then $g_2(y, z)$ is irreducible in ${}_2\mathcal{O}_0$ with $g_2 \in \text{the type [2]}$ in the sense of Definition 2.5 because $g_2(y, z, f_1)$ can be viewed as an element of ${}_2\mathcal{O}_0$.

Apply the WDT to $f(y, z)$ with a divisor $g_2(y, z)$. Then, (g_2, f) can be written in the form

$$(Eq.7)(Eq.7.1) \quad \begin{cases} g_2 &= f_1^{n_2} + \xi_2 y^{\sigma_{2,1}} z^{\sigma_{2,2}} \\ f &= g_2^{d_3} + \sum_{i=0}^{d_3-1} T_{3,i}^{(1)} g_2^i, \end{cases}$$

where $T_{3,i}^{(1)} \in \mathbb{C}\{y\}[z, f_1]$ for $0 \leq i \leq d_3 - 1$, satisfying the following property:

(i) For any nonzero monomial $\Pi_{k=1}^3 f_{k-2}^{\gamma_k}$ in $T_{3,i}^{(1)}$,

$$(Eq.7.2) \quad \gamma_1 > 0, \quad \gamma_2 < n_1 \quad \text{and} \quad \gamma_3 < n_2.$$

(ii) Define $\theta_3 : \mathbb{N}_0^{(3)} \rightarrow \mathbb{N}_0$ by $\theta_3(t_k)_{k=1}^3 = t_3 \theta_2(\sigma_{2,k})_{k=1}^2 + n_2 \theta_2(t_k)_{k=1}^2$, and then θ_3 is one-to-one.

If $T_{3,i}^{(1)} \neq 0$, let $C_{3,i}^{(1)} \Pi_{k=1}^3 f_{k-2}^{\beta_{3,i,k}^{(1)}}$ be a unique nonzero monomial with a constant $C_{3,i}^{(1)}$ in $T_{3,i}^{(1)}$ such that $\theta_3(\beta_{3,i,k}^{(1)})_{k=1}^3 = \min\{\theta_3(\gamma_k)_{k=1}^3\}$ for any nonzero monomial $\Pi_{k=1}^3 f_{k-2}^{\gamma_k}$ in $T_{3,i}^{(1)}$.

Remark. The above construction of $T_{3,i}^{(1)}$ is as follows: Note that $g_2(y, z)$ is a polynomial in z of degree $\Pi_{k=1}^2 n_k$, and for $0 \leq i \leq d_3 - 1$, $T_{3,i}^{(1)} \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree $< \Pi_{k=1}^2 n_k$, and $f_1(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree n_1 . Again, apply the WDT to each $T_{3,i}^{(1)}$ with a divisor $f_1(y, z)$. Then $T_{3,i}^{(1)}$ can be written just as above. \square

Since f is irreducible in ${}_2\mathcal{O}_0$, f must satisfy the following necessary condition:

$$(Eq.7.3) \quad \frac{\theta_3(\beta_{3,i,k}^{(1)})_{k=1}^3}{d_3 - i} \geq \frac{\theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3}{d_3} > n_2 \theta_2(\sigma_{2,k})_{k=1}^2. \quad \text{for } 0 \leq i \leq d_3 - 1.$$

Recall that $\sigma_{2,k} = \beta_{2,d_2-n_2,k}$ for $1 \leq k \leq 2$ from the construction of g_2 .

Since f satisfies (Eq.7.3), then to find an irreducibility algorithm for f , it suffices to consider the following two subcases Case(A) and Case(B):

Case(A) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) = 1$, and Case(B) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$.

Case(A) of the 3rd step Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.7.2) holds. In this case, if f is irreducible in ${}_2\mathcal{O}_0$, then $f \in$ the type [3] in the sense of Definition 2.5 and also $f \stackrel{\text{multiseq}}{\sim} H_2^{d_3} + \Pi_{k=1}^3 H_{k-2}^{\sigma_{3,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\beta_{1,0,1}}$ and $H_2 = H_1^{n_2} + y^{\sigma_{2,1}} z^{\sigma_{2,2}}$ and $\sigma_{3,k} = \beta_{3,0,k}^{(1)}$ for $1 \leq k \leq 3$.

Remark for Case(A) of the 3rd step. Let $f_1 = h_{1,\nu_1+1}$ in (Eq.5), and (g_2, f) in (Eq.7). Then, (f_1, g_2, f) may not be a generalized representation of f in the sense of Theorem 16.5, since $T_{3,d_3-1}^{(1)}$ may not be zero. To find a generalized representation of f in the sense of Theorem 16.5, until we get the same kind of result in the conclusion of Sublemma 15.5, it suffices to apply either Sublemma 15.5 or the same kind of a fine sequence of pairs in (Eq.8) to (g_2, f) .

Case(B) of the 3rd step: Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$. To find a necessary condition for f to be irreducible in ${}_2\mathcal{O}_0$, we may assume that f satisfies (Eq.7) with (Eq.2), (Eq.3), \dots , (Eq.6).

Then, it suffices to consider the following two subcases:

Subcase(B1) $T_{3,d_3-1}^{(1)} \neq 0$, and Subcase(B2) $T_{3,d_3-1}^{(1)} = 0$.

Subcase(B1) of the 3rd step: Let $T_{3,d_3-1}^{(1)} \neq 0$. To solve the case, first of all, we must eliminate $T_{3,d_3-1}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Theorem 1.8(Theorem 15.4), we can compute a unique finite sequence of pairs, $\{(h_{2,p}, f) : 1 \leq p \leq \nu_2 + 1\}$ in (Eq.8) for a unique integer $\nu_2 \leq \frac{n_2+1}{2}$, each pair of which can be written in the form

$$(Eq.8)(Eq.8.1)(Eq.8.1.1) \quad \begin{cases} h_{2,1} &= f_1^{n_2} + \xi_2 y^{\sigma_{2,1}} z^{\sigma_{2,2}} = f_1^{n_2} + R_{2,0}^{(1)} \text{ with } h_{2,1} = g_2 \\ f &= h_{2,1}^{d_3} + \sum_{i=0}^{d_3-1} T_{3,i}^{(1)} h_{2,1}^i, \end{cases}$$

and for $1 \leq p \leq \nu_2 - 1$,

$$(Eq.8.1.2) \quad \begin{cases} h_{2,p+1} &= h_{2p} + \frac{1}{d_3} T_{3,d_3-1}^{(p)} = f_1^{n_2} + \sum_{i=0}^{n_2-2} R_{2i}^{(p+1)} f_1^i \\ f &= h_{2,p+1}^{d_3} + \sum_{i=0}^{d_3-1} T_{3,i}^{(p+1)} h_{2,p+1}^i \text{ with } T_{3,d_3-1}^{(p+1)} \neq 0, \end{cases}$$

and

$$(Eq.8.1.3) \quad \begin{cases} h_{2,\nu_2+1} &= h_{2,\nu_2} + \frac{1}{d_3} T_{3,d_3-1}^{(\nu_2)} = f_1^{n_2} + \sum_{i=0}^{n_2-2} R_{2,i}^{(\nu_2+1)} f_1^i \\ f &= h_{2,\nu_2+1}^{d_3} + \sum_{i=0}^{d_3-2} T_{3,i}^{(\nu_2+1)} h_{2,\nu_2+1}^i \text{ with } T_{3,d_3-1}^{(\nu_2+1)} = 0, \end{cases}$$

where $R_{2,i}^{(p)} \in \mathbb{C}\{y\}[z]$ for $1 \leq p \leq \nu_2 + 1$ and $0 \leq i \leq n_2 - 2$; and $T_{3,i}^{(p)} \in \mathbb{C}\{y\}[z, f_1]$ for $1 \leq p \leq \nu_2 + 1$ and $0 \leq i \leq d_3 - 1$, satisfying the following properties:

- (i) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}^{(p)}$, $\delta_1 > 0$ and $\delta_2 < n_1$.
- (ii) For any nonzero monomial $\Pi_{k=1}^3 f_{k-2}^{\gamma_k}$ in $T_{3,i}^{(p)}$,

$$(Eq.8.2) \quad \gamma_1 > 0, \quad \gamma_2 < n_1 \quad \text{and} \quad \gamma_3 < n_2.$$

- (iii) If $T_{3,i}^{(p)} \neq 0$, let $C_{3,i}^{(p)} \Pi_{k=1}^3 f_{k-2}^{\beta_{3,i,k}^{(p)}}$ be a unique nonzero monomial with a constant $C_{3,i}^{(p)}$ in $T_{3,i}^{(p)}$ such that $\theta_3(\beta_{3,i,k}^{(p)})_{k=1}^3 = \min\{\theta_3(\gamma_k)_1^3\}$ for any nonzero monomial $\Pi_{k=1}^3 f_{k-2}^{\gamma_k}$ in $T_{3,i}^{(p)}$ where $f_{-1} = y$ and $f_0 = z$.

Remark. It is possible to compute $C_{3,i}^{(p)} \Pi_{k=1}^3 f_{k-2}^{\beta_{3,i,k}^{(p)}}$ by an elementary way because $T_{3,i}^{(p)} \in \mathbb{C}\{y\}[z, f_1]$ in z and f_1 , recalling that for any nonzero monomial $\Pi_{k=1}^3 f_{k-2}^{\gamma_k}$ in $T_{3,i}^{(p)}$, $\gamma_2 < n_1$ and $\gamma_3 < n_2$ and θ_3 is one-to-one.

Now, Since f is irreducible in ${}_2\mathcal{O}_0$, then f must satisfy the following necessary condition:

$$(Eq.8.3)(Eq.8.3.1) \quad \frac{\theta_3(\beta_{3,i,k}^{(p)})_{k=1}^3}{d_3 - i} \geq \frac{\theta_3(\beta_{3,0,k}^{(p)})_{k=1}^3}{d_3} > n_2 \theta_2(\sigma_{2,k})_{k=1}^2 \quad \text{for } 1 \leq p \leq \nu_2, 0 \leq i \leq d_3 - 1.$$

$$(Eq.8.3.2) \quad \frac{\theta_3(\beta_{3,i,k}^{(\nu_2+1)})_{k=1}^3}{d_3 - i} \geq \frac{\theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3}{d_3} > n_2 \theta_2(\sigma_{2,k})_{k=1}^2 \quad \text{for } 0 \leq i \leq d_3 - 2.$$

Remark. Since f satisfies (Eq.8.3), then $h_{2,p} \stackrel{\text{multiseq}}{\sim} h_{2,1} = g_2$ for all $p \geq 1$.

Since f satisfies (Eq.8.3), to find an irreducibility algorithm for Subcase(B1), then there exist the following two possibilities, Subcase(B1-a) and Subcase(B1-b):

Subcase(B1-a) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$ and $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) = 1$.

Subcase(B1-b) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) > 1$.

Subcase(B1-a) of the 3rd step

(a) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$ and $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.8.3.2) holds. Thus, $f \in$ the type [3] in the sense of Definition 2.5.

(b) Assume that $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(p+1)})_{k=1}^3) = 1$ for some $p \leq \nu_2 + 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if an inequality in (Eq.8.3.1) holds without mentioning any inequality in (Eq.8.3.2).

Remark for Subcase(B1-a) of the 3rd step. $(f_1, f_2, f) = (f_1, h_{2,\nu_2+1}, f)$ is a generalized representation of f in the sense of Theorem 1.13, since $T_{2,d_2-1}^{(\nu_2+1)} = 0$.

Subcase(B1-b) of the 3rd step Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) > 1$, noting that $T_{3,d_3-1}^{(\nu_2+1)} = 0$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the next step.

Subcase(B2) of the 3rd step It was assumed by this subcase that $T_{3,d_3-1}^{(1)} = 0$, noting that $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the next step. \square

Remark for The 3rd step: To find $H_3 \in \text{Family}(1)$ such that $f \stackrel{\text{multiseq}}{\sim} H_3$ for any $f \in \text{the type [3]}$ in the sense of Definition 2.5. Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.7.2) holds. In this case, if f is irreducible in ${}_2\mathcal{O}_0$, then $f \in \text{the type [3]}$ in the sense of Definition 2.5 and also $f \stackrel{\text{multiseq}}{\sim} H_2^{d_3} + \Pi_{k=1}^3 H_{k-2}^{\sigma_{3,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\beta_{1,0,1}}$ and $H_2 = H_1^{n_2} + y^{\sigma_{2,1}} z^{\sigma_{2,2}}$ and $\sigma_{3,k} = \beta_{3,0,k}^{(\nu+1)}$ for $1 \leq k \leq 3$ and for some positive integer $\nu_2 + 1$. \square

The general case will be proved by induction. Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W -poly of degree $n \geq 2$ in z . Suppose we have shown for any integer $q \geq 3$ that for each (j)-th step, $1 \leq j \leq q-1$, the irreducibility algorithm for $f \in \text{the type [j]}$ has been found, by using the same kind of properties and notations as we have seen in the proof of finding the irreducibility algorithm for the (j)-th step, $1 \leq j \leq q-1$. Now, we may assume by induction on q that the irreducibility algorithm for the (q-1)-th step can be solvable for $q \geq 3$, and so if f is irreducible in ${}_2\mathcal{O}_0$, we may assume without loss of generality that if $f \in \text{the type [k]}$ for an integer $k \geq q$ in the sense of Definition 2.5 then $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1,0,k}^{(\nu_{q-2}+1)})_{k=1}^{q-1}) > 1$ and $T_{q-1,d_{q-1}-1}^{(\nu_{q-2}+1)} = 0$ for some positive integer $\nu_{q-2} + 1$. Then, it remains to show that the generalized irreducibility algorithm for $f \in \text{the type [q]}$ can be written as follows.

The q-th step: To find the irreducibility algorithm for $f \in \text{the type [q]}$ in the sense of Definition 2.5.

With three equations, which is defined by (Eq.(3q-3)), (Eq.(3q-2)) and (Eq.(3q-1)) in the q -th step later, the aim in this step is how to compute the necessary and sufficient condition for $f \in \text{the type [q]}$ in the sense of Definition 2.5.

Suppose by induction on the positive integer (q-1) that f is irreducible in ${}_2\mathcal{O}_0$ and let f satisfy a finite number (3q-4) of conditions, which have been represented by (Eq.1), (Eq.2), (Eq.3), ..., (Eq.(3q-6)), (Eq.(3q-5)), (Eq.(3q-4)).

For the proof of this step, recall the defining equation $(h_{q-2,\nu_{q-2}+1}, f)$ of (Eq.(3q-4)) as we have already seen in Subcase(B1) of the (q-1)-th step: (Eq.(3q-4))

$$\begin{cases} h_{q-2,\nu_{q-2}+1} &= f_{q-3}^{n_{q-2}} + \sum_{i=0}^{n_{q-2}-2} R_{q-2,i}^{(\nu_{q-2}+1)} f_{q-3}^i \\ f &= h_{q-2,\nu_{q-2}+1}^{d_{q-1}} + \sum_{i=0}^{d_{q-1}-2} T_{q-1,i}^{(\nu_{q-2}+1)} h_{q-2,\nu_{q-2}+1}^i \text{ with } T_{q-1,d_{q-1}-1}^{(\nu_{q-2}+1)} = 0. \end{cases}$$

By either Subcase(B1-b) or Subcase(B2) of the (q-1)-th step, it may be assumed that $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1,0,k}^{(\nu_{q-2}+1)})_{k=1}^{q-1}) > 1$ and $T_{(q-1),d_{q-1}-1}^{(\nu_{q-2}+1)} = 0$ for some positive integer $\nu_{q-2} + 1$.

Remark. If $T_{q-1,d_{q-1}-1}^{(1)} = 0$ and $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1,0,k}^{(1)})_{k=1}^{q-1}) > 1$ for Subcase(B2), then $(h_{q-2,1}, f) = (g_{q-2}, f)$ can be viewed as $(h_{q-2,\nu+1}, f)$ with $\nu_{q-2} = 0$.

For brevity of notation, $h_{q-2,\nu_{q-2}+1}$, $T_{q-1,i}^{(\nu_{q-2}+1)}$, $C_{q-1,i}^{(\nu_{q-2}+1)}$ and $\beta_{q-1,i,k}^{(\nu_{q-2}+1)}$ can be replaced by f_{q-2} , $T_{q-1,i}$, $C_{q-1,i}$ and $\beta_{q-1,i,k}$ for $0 \leq i \leq d_{q-1} - 2$ and $1 \leq k \leq q-1$, respectively because $g_{q-2} = h_{q-2,1} \stackrel{\text{multiseq}}{\sim} h_{q-2,p}$ for all $p \geq 1$ and $T_{q-1,d_{q-1}-1}^{(\nu_{q-2}+1)} = 0$. Recall that $f_{q-2} = f_{q-3}^{n_{q-2}} + \sum_{i=0}^{n_{q-2}-2} R_{q-2,i}^{(\nu_{q-2}+1)} f_{q-3}^i$ from $(h_{q-2,\nu_{q-2}+1}, f)$ where $R_{q-2,i}^{(\nu_{q-2}+1)}$ is defined to be $R_{q-2,i}$ for each i .

Let $d_q = \gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1,0,k}^{q-1})_{k=1}^{q-1})$ and then write $d_{q-1} = n_{q-1}d_q$. To solve the above problem, we need to construct g_{q-1} as follows:

$$\text{(Eq.(3q-3))} \quad g_{q-1} = f_{q-2}^{n_{q-1}} + \xi_{q-1} \Pi_{k=1}^{q-1} f_{k-2}^{\sigma_{q-1,k}} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{q-2}],$$

where $\xi_{q-1} = \frac{1}{d_{q-1}} C_{q-1,d_{q-1}-n_{q-1}}$ and $\sigma_{q-1,k} = \beta_{q-1,d_{q-1}-n_{q-1},k}$ for $1 \leq k \leq q-1$.

Then, g_{q-1} must satisfy the following necessary condition:

$$(Eq.(3q-3).1) \quad 1 < d_q < d_{q-1}, \quad \xi_{q-1} \neq 0, \quad \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} > n_{q-1}n_{q-2}\theta_{q-2}(\alpha_{q-2,0,k})_{k=1}^{q-2},$$

and $\gcd(n_{q-1}, \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1}) = 1$.

If g_{q-1} of (Eq.(3q-3)) satisfies inequalities in (Eq.(3q-3).1), then $g_{q-1}(y, z)$ is irreducible in ${}_{q-1}\mathcal{O}_0$ with $g_{q-1} \in \text{the type}[q-1]$ because $g_{q-1}(y, z, f_1, \dots, f_{q-2})$ can be viewed as an element of ${}_q\mathcal{O}_0$.

Apply the WDT to $f(y, z)$ with a divisor $g_{q-1}(y, z)$. Then, (g_{q-1}, f) can be written in the form

$$(Eq.(3q-2))(Eq.(3q-2).1) \quad \begin{cases} g_{q-1} &= f_{q-2}^{n_{q-1}} + \xi_{q-1} \Pi_{k=1}^q f_{k-2}^{\sigma_{q-1,k}} \\ f &= g_{q-1}^{d_q} + \sum_{i=0}^{d_q-1} T_{q,i}^{(1)} g_{q-1}^i, \end{cases}$$

where $T_{q,i}^{(1)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{q-2}]$ for $0 \leq i \leq d_q - 1$, satisfying the following property:

(i) For any nonzero monomial $\Pi_{k=1}^q f_{k-2}^{\gamma_k}$ in $T_{q,i}^{(1)}$,

$$(Eq.(3q-2).2) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_k < n_{k-1} \quad \text{for} \quad 2 \leq k \leq q.$$

(ii) Define $\theta_q : \mathbb{N}_0^{(q)} \rightarrow \mathbb{N}_0$ by $\theta_q(t_k)_{k=1}^q = t_q \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} + n_{q-1} \theta_{q-1}(t_k)_{k=1}^{q-1}$, and then θ_q is one-to-one.

If $T_{q,i}^{(1)} \neq 0$, let $C_{q,i}^{(1)} \Pi_{k=1}^q f_{k-2}^{\beta_{q,i,k}^{(1)}}$ be a unique nonzero monomial with a constant $C_{q,i}^{(1)}$ in $T_{q,i}^{(1)}$ such that $\theta_q(\beta_{q,i,k}^{(1)})_{k=1}^q = \min\{\theta_q(\gamma_k)_{k=1}^q\}$ for any nonzero monomial $\Pi_{k=1}^q f_{k-2}^{\gamma_k}$ in $T_{q,i}^{(1)}$.

Remark. The above construction of $T_{q,i}^{(1)}$ is as follows: Note that $g_{q-1}(y, z)$ is a polynomial of degree $\Pi_{k=1}^{q-1} n_k$ in z , and for $0 \leq i \leq d_q - 1$, $T_{q,i}^{(1)} \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{k=1}^{q-1} n_k$ in z , and $f_{q-2}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $\Pi_{k=1}^{q-2} n_k$ in z . Again, apply the WDT to each $T_{q,i}^{(1)}$ with a divisor $f_{q-2}(y, z)$. Then $T_{q,i}^{(1)}$ can be written just as above.

Since f is irreducible in ${}_2\mathcal{O}_0$, f must satisfy the following necessary condition:

$$(Eq.(3q-2).3) \quad \frac{\theta_q(\beta_{q,i,k}^{(1)})_{k=1}^q}{d_q - i} \geq \frac{\theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q}{d_q} > n_{q-1} \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} \quad \text{for } 0 \leq i \leq d_q - 1.$$

Recall that $\sigma_{q-1,k} = \beta_{q-1,d_{q-1}-n_{q-1},k}$ for $1 \leq k \leq q-1$ from the construction of g_{q-1} .

If f satisfies (Eq.(3q-2).3), then to find an irreducibility algorithm for f , it suffices to consider the following two cases:

Case(A) $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) = 1$, and Case(B) $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) > 1$.

Case(A) of the q-th step Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.(3q-2).3) holds. In this case, if f is irreducible in ${}_2\mathcal{O}_0$, then $f \in \text{the type}[q]$ and also $f \stackrel{\text{multiseq}}{\sim} H_{q-1}^{d_q} + \Pi_{k=1}^q H_{k-2}^{\sigma_{q,k}} where $H_{-1} = y, H_0 = z, H_1 = z^{n_1} + y^{\beta_{1,0,1}}, H_2 = H_1^{n_2} + y^{\sigma_{2,1}} z^{\sigma_{2,2}}, H_j = H_{j-1}^{n_j} + \Pi_{k=1}^j H_{k-2}^{\sigma_{j,k}}$ for $3 \leq j \leq q-2, H_{q-1} = H_{q-2}^{d_{q-1}} + \Pi_{k=1}^{q-1} H_{k-2}^{\sigma_{q-1,k}}$, noting that for each fixed $j = 1, 2, \dots, q$, $\sigma_{j,k} = \beta_{j,0,k}^{(1)}$ for $1 \leq k \leq j$.$

Remark for Case(A) of the q-th step. Let (g_{q-1}, f) be in (Eq.(3q-2)). By Theorem 1.13, $(f_1, \dots, f_{q-2}, g_{q-1}, f)$ may not be a generalized representation of f , since $T_{q-1,d_{q-1}-1}^{(\nu_{q-1}+1)}$ may not be zero. To find a generalized representation of f in the sense of Theorem 1.13, until we get the same kind of result in the conclusion of Sublemma 15.5, it suffices to apply either Sublemma 15.5 or the same kind of a fine sequence of pairs in (Eq.(3q-1)) to (g_{q-1}, f)

Case(B) of the q-th step Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) > 1$. To find a necessary condition for f to be irreducible in ${}_2\mathcal{O}_0$, then we may assume that f satisfies (Eq.2), (Eq.3), \dots , (Eq.3q-2).

Then, it suffices to consider the following two subcases:

Subcase(B1) $T_{q,d_q-1}^{(1)} \neq 0$, and Subcase(B2) $T_{q,d_q-1}^{(1)} = 0$.

Subcase(B1) of the q-th step Let $T_{q,d_q-1}^{(1)} \neq 0$. To solve the case, first of all, we must eliminate $T_{q,d_q-1}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Theorem 1.8(Theorem 15.4), we can compute a unique finite sequence of pairs, $\{(h_{q-1,p}, f) : 1 \leq p \leq \nu+1\}$ in (Eq.(3q-1)) for a unique integer $\nu_{q-1} \leq \frac{n_{q-1}+1}{2}$, each pair of which can be written in the form

$$\begin{cases} h_{q-1,1} &= f_{q-2}^{n_{q-1}} + \xi_{q-1} \prod_{k=1}^{q-1} f_{k-2}^{\sigma_{q-1,k}} = f_{q-2}^{n_{q-1}} + R_{q-1,0}^{(1)} \text{ with } h_{q-1,1} = g_{q-1} \\ f &= h_{q-1,1}^{d_q} + \sum_{i=0}^{d_q-1} T_{q,i}^{(1)} h_{q-1,1}^i, \end{cases}$$

and for $1 \leq p \leq \nu_{q-1} - 1$,

$$\text{(Eq.(3q-1).1.2)} \quad \begin{cases} h_{q-1,p+1} &= h_{q-1,p} + \frac{1}{d_q} T_{q,d_q-1}^{(p)} = f_{q-2}^{n_{q-1}} + \sum_{i=0}^{n_{q-1}-2} R_{q-1,i}^{(p+1)} f_{q-2}^i \\ f &= h_{q-1,p+1}^{d_q} + \sum_{i=0}^{d_q-1} T_{q,i}^{(p+1)} h_{q-1,p+1}^i \text{ with } T_{q,d_q-1}^{(p+1)} \neq 0, \end{cases}$$

and

$$\text{(Eq.(3q-1).1.3)} \quad \begin{cases} h_{q-1,\nu_{q-1}+1} &= h_{q-1,\nu_{q-1}} + \frac{1}{d_q} T_{q,d_q-1}^{(\nu_{q-1})} = f_{q-2}^{n_{q-1}} + \sum_{i=0}^{n_{q-1}-2} R_{q-1,i}^{(\nu_{q-1}+1)} f_{q-2}^i \\ f &= h_{q-1,\nu_{q-1}+1}^{d_q} + \sum_{i=0}^{d_q-2} T_{q,i}^{(\nu_{q-1}+1)} h_{q-1,\nu_{q-1}+1}^i \text{ with } T_{q,d_q-1}^{(\nu_{q-1}+1)} = 0, \end{cases}$$

where $\sigma_{q-1,k} = \beta_{q-1,d_{q-1}-n_{q-1},k}$ for $1 \leq k \leq q-1$, $R_{q-1,i}^{(p)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{q-3}]$ for $1 \leq p \leq \nu_{q-1}+1$ and $0 \leq i \leq n_{q-1}-2$; and $T_{\nu_{q-1},i}^{(p)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{q-2}]$ for $1 \leq p \leq \nu+1$ and $0 \leq i \leq d_q-1$, satisfying the following properties:

- (i) For any nonzero monomial $\prod_{k=1}^{q-1} f_{k-2}^{\delta_k}$ in $R_{q-1,i}^{(p)}$,
 $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $2 \leq k \leq q-1$.
- (ii) For any nonzero monomial $\prod_{k=1}^q f_{k-2}^{\gamma_k}$ in $T_{q,i}^{(p)}$,

$$\text{(Eq.(3q-1).2)} \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_q < n_{q-1} \quad \text{for } 2 \leq k \leq q.$$

- (iii) Define a function $\theta_q : \mathbb{N}_0^q \rightarrow \mathbb{N}_0$ by $\theta_q(t_k)_{k=1}^q = t_q \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} + n_{q-1} \theta_{q-1}(t_k)_{k=1}^{q-1}$, and then θ_q is one-to-one. If $T_{q,i}^{(p)} \neq 0$, let $C_{q,i}^{(p)} \prod_{k=1}^q f_{k-2}^{\beta_{q,i,k}^{(p)}}$ be a unique nonzero monomial with a constant $C_{q,i}^{(p)}$ in $T_{q,i}^{(p)}$ such that $\theta_q(\beta_{q,i,k}^{(p)})_{k=1}^q = \min\{\theta_q(\gamma_k)_{k=1}^q\}$ for any nonzero monomial $\prod_{k=1}^q f_{k-2}^{\gamma_k}$ in $T_{q,i}^{(p)}$ where $f_{-1} = y$ and $f_0 = z$.

Remark. It is possible to compute $C_{q,i}^{(p)} \prod_{k=1}^q f_{k-2}^{\beta_{q,i,k}^{(p)}}$ by an elementary way because $T_{q,i}^{(p)} \in \mathbb{C}\{y\}[z, f_1, f_2, \dots, f_{q-2}]$ in z, f_1, \dots, f_{q-2} , recalling that for any nonzero monomial $\prod_{k=1}^q f_{k-2}^{\gamma_k}$ in $T_{q,i}^{(p)}$, $\gamma_1 > 0$ and $\gamma_k < n_{k-1}$ for $2 \leq k \leq q$, θ_q is one-to-one.

Since f is irreducible in ${}_2\mathcal{O}_0$, then f must satisfy the following necessary condition:

$$((\text{Eq.(3q-1).3})(\text{Eq.(3q-1).3.1}))$$

$$\frac{\theta_q(\beta_{q,i,k}^{(p)})_{k=1}^q}{d_q - i} \geq \frac{\theta_q(\beta_{q,0,k}^{(p)})_{k=1}^q}{d_q} > n_{q-1} \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} \text{ for } 1 \leq p \leq \nu_{q-1}, 0 \leq i \leq d_q - 1,$$

$$((\text{Eq.(3q-1).3.2}))$$

$$\frac{\theta_q(\beta_{q,i,k}^{(\nu_{q-1}+1)})_{k=1}^q}{d_q - i} \geq \frac{\theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q}{d_q} > n_{q-1} \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} \text{ for } 0 \leq i \leq d_q - 2.$$

Remark. Since f satisfies (Eq.(3q-1).3), $h_{q-1,p} \stackrel{\text{multiseq}}{\sim} h_{q-1,1} = g_{q-1}$ for all $p \geq 1$.

Since f satisfies (Eq.(3q-1).3), to find an irreducibility algorithm for Subcase(B1), then there exist the following two possibilities, Subcase(B1-a) and Subcase(B1-b):

Subcase(B1-a) Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) > 1$ and $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) = 1$.

Subcase(B1-b) Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) > 1$.

Subcase(B1-a) of the q-th step

(a) Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) > 1$ and $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.(3q-1).3.2) holds. Thus, $f \in$ the type $[q]$ in the sense of Definition 2.5.

(b) Assume that $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(p)})_{k=1}^q) = 1$ for some $p \leq \nu_{q-1} + 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if an inequality in (Eq.(3q-1).3.1) holds without mentioning any inequality in (Eq.(3q-1).3.2).

Remark for Subcase(B1-a) of the q-th step. $(f_1, \dots, f_{q-2}, f_{q-1}, f)$ with $f_{q-1} = h_{q-1, \nu_{q-1}+1}$ is a generalized representation of irreducible W-poly of the recursive q-type for f in the sense of Theorem 1.13, since $T_{q-1, d_{q-1}-1}^{(\nu_{q-1}+1)} = 0$.

Subcase(B1-b) of the q-th step Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) > 1$, noting that $T_{q, d_{q-1}}^{(\nu_{q-1}+1)} = 0$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the $(q+1)$ -th step.

Subcase(B2) of the q-th step It was assumed by this subcase that $T_{q, d_{q-1}}^{(1)} = 0$, noting that $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(1)})_{k=1}^q) > 1$. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the $(q+1)$ -th step. \square

Remark for the q-th step: To find $H_q \in$ Family(1) such that $f \stackrel{\text{multiseq}}{\sim} H_q$ for any $f \in$ the type $[q]$ in the sense of Definition 2.5. Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.(3q-1).2) holds. In this case, if f is irreducible in ${}_2\mathcal{O}_0$, then $f \in$ the type $[q]$ and also $f \stackrel{\text{multiseq}}{\sim} H_{q-1}^{d_q} + \Pi_{k=1}^q H_{k-2}^{\sigma_{q,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\beta_{1,0,1}}$, and for $3 \leq j \leq q$, $H_{j-1} = H_{j-2}^{d_{j-1}} + \Pi_{k=1}^{j-1} H_{k-2}^{\sigma_{j-1,k}} \in \text{Family}[1]$, noting that for each fixed $j = 1, 2, \dots, q-1$ and for $1 \leq k \leq j$, $\sigma_{j,k} = \beta_{j, d_j - n_j, k}$, and for $1 \leq k \leq q-1$ $\sigma_{q,k} = \beta_{q,0,k}^{(\nu_{q-1}+1)}$ with some positive integer $\nu_{q-1} + 1$, because of the equations (Eq.(3j-3), (Eq.(3j-2) and (Eq.(3j-1) with $2 \leq j \leq q$ in the conclusion of Theorem 1.16. Note that $d_{j+1} = \gcd(d_j, \theta_j(\sigma_{j,k})_{k=1}^j)$ with $d_j = n_j d_{j+1}$ for $j \geq 2$. \square

Corollary 1.15.1 for Theorem 1.15.

Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 1.15 hold.

Conclusions If f is irreducible in ${}_2\mathcal{O}_0$ with isolated singularity at $0 \in \mathbb{C}^2$, then $f \in$ the type $[\ell]$ for some $\ell \leq r$ in the sense of Definition 2.5. By the induction method on the positive integer r , the aim is to compute an elementary algorithm for finding irreducible W-polys from all the W-polys in $\mathbb{C}\{y\}[z]$, using q iterations of the following steps with $q \leq \ell$: Observe that the statement on the 3rd step may be omitted if necessary, to simplify the statements for this theorem by the induction method.

Following the same properties and notations as in the conclusions of Theorem 1.15, to find $f \in$ the type $[\ell]$ for some $\ell \leq r$ in the sense of Definition 2.5, there is nothing to prove by Theorem 1.15 that the following are true:

The 1st step: To find the irreducibility algorithm for $f \in$ the type $[1]$ in the sense of Definition 2.5.

If f satisfies (Eq.2) of Theorem 1.15, it suffices to consider the following case for the coefficient a_{n-1} of z^{n-1} .

Case(I) $a_{n-1} = 0$. There are two subcases only.

Subcase(I-1) $\gcd(n, \alpha_0) = 1$ and Subcase(I-2) $1 < \gcd(n, \alpha_0) < n$.

Subcase(I-1) Let $\gcd(n, \alpha_0) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [1] \iff the inequality in (Eq.2) of Theorem 1.15 holds.

Subcase(I-2) Let $1 < \gcd(n, \alpha_0) < n$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq 2$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, take the next step.

The 2nd step: To find the irreducibility algorithm for $f \in$ the type [2] in the sense of Definition 2.5.

With three equations, which is defined by (Eq.3), (Eq.4) and (Eq.5) in the 2nd step of Theorem 1.15 later, the aim in this step is how to compute the necessary and sufficient condition for $f \in$ the type [2] in the sense of Definition 2.5, **without need of computing any inequality in (Eq.5.3.1) of Theorem 1.15.**

For the irreducibility algorithm for the 2nd step, it suffices to consider the following subcase only for the 1st step:

Subcase(I-2) Let $1 < \gcd(n, \alpha_0) < n$ and $a_{n-1} = 0$.

If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq 2$ in the sense of Definition 2.5. To find the irreducibility algorithm for $f \in$ the type [2], using the inequalities in (Eq.3) with (Eq.3.1) and (Eq.4) with (Eq.4.1) and (Eq.4.2) of Theorem 1.15 it suffices to consider two cases:

Case(I) $T_{2,d_2-1}^{(1)} = 0$, and **Case(II)** $T_{2,d_2-1}^{(1)} \neq 0$.

Case(I) Let $T_{2,d_2-1}^{(1)} = 0$. There are two subcases only.

Subcase(I-1) $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 1$ and **Subcase(I-2)** $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$.

Subcase(I-1) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] \iff the inequality in Eq(4.3) of Theorem 1.15 holds.

Subcase(I-2) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq 3$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the next step. To compute the inequality in (Eq.4.3) may not be needed, if necessary.

Case(II) Let $T_{2,d_2-1}^{(1)} \neq 0$. To solve the subcase, first of all, we must eliminate $T_{2,d_2-1}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Sublemma 1.10 for Theorem 1.8 (Sublemma 15.5 for Theorem 15.4), we can compute a unique finite sequence of pairs $\{(h_{1,p}, f) : 1 \leq p \leq \nu_1 + 1\}$ in (Eq.5) of Theorem 1.15 for a unique integer $\nu_1 \leq \frac{n_1+1}{2}$, which satisfies the following:

(i) $T_{2,d_2-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu_1$ and $T_{2,d_2-1}^{(\nu_1+1)} = 0$.

(ii) If f satisfies (Eq.5.1) and (Eq.5.2) of Theorem 1.15, without computing any inequality in (Eq.5.3.1) of Theorem 1.15, to find an irreducibility algorithm for f , it suffices to consider the following two subcases:

Subcase(II-1) $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) = 1$ and **Subcase(II-2)** $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$.

Subcase(II-1) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] \iff (Eq.5.3.2) of Theorem 1.15 holds.

Subcase(II-2) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq 3$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the next step. To compute the inequality in Eq(5.3.1) of Theorem 1.15 may not be needed, if necessary.

The 3rd step: To find the irreducibility algorithm for $f \in$ the type [3] in the sense of Definition 2.5.

With three equations, which is defined by (Eq.6), (Eq.7) and (Eq.8) of Theorem 1.15 in the 3rd step later, the aim in this step is how to compute the necessary and sufficient condition for $f \in$ the type [3] in the sense of Definition 2.5, **without need of computing any inequality in (Eq.8.3.1) of Theorem 1.15.**

For the irreducibility algorithm for the 3rd step, it suffices to consider the following subcase only for the 2nd step:

Subcase(II-2) Let $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(\nu_1+1)})_{k=1}^2) > 1$ and $T_{2,d_2-1}^{(\nu_1+1)} = 0$ for some integer $\nu_1 \geq 0$.

If f is irreducible in ${}_2\mathcal{O}_0$ for Subcase(II-2), note that $f \in$ the type $[\ell]$ for some $\ell \geq 3$ in the sense of Definition 2.5. To find the irreducibility algorithm for $f \in$ the type $[3]$, using the inequalities in (Eq.6) with (Eq.6.1) and (Eq.7) with (Eq.7.1) and (Eq.7.2) of Theorem 1.15 it suffices to consider two new cases:

Case(I) $T_{3,d_3-1}^{(1)} = 0$, and Case(II) $T_{3,d_3-1}^{(1)} \neq 0$.

Case(I) Let $T_{3,d_3-1}^{(1)} = 0$. There are two subcases only.

Subcase(I-1) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) = 1$ and Subcase(I-2) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$.

Subcase(I-1) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type $[3] \iff$ (Eq.7.3) of Theorem 1.15 holds.

Subcase(I-2) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(1)})_{k=1}^3) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq 4$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the 4-th step. To compute the inequality in (Eq.7.3) of Theorem 1.15 may not be needed, if necessary.

Case(II) Let $T_{3,d_3-1}^{(1)} \neq 0$. To solve the subcase, first of all, we must eliminate $T_{3,d_3-1}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Sublemma 1.10 for Theorem 1.8 (Sublemma 15.5 for Theorem 15.4), we can compute a unique finite sequence of pairs $\{(h_{2,p}, f) : 1 \leq p \leq \nu_2 + 1\}$ in (Eq.7) of Theorem 1.15 for a unique integer $\nu_2 \leq \frac{n_3+1}{2}$, which satisfies the following:

(i) $T_{3,d_3-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu_2$ and $T_{2,d_3-1}^{(\nu_2+1)} = 0$.

(ii) If f satisfies (Eq.8.1) and (Eq.8.2) of Theorem 1.15, without computing any inequality in (Eq.8.3.1) of Theorem 1.15, to find an irreducibility algorithm for f , it suffices to consider the following two subcases:

Subcase(II-1) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) = 1$ and Subcase(II-2) $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) > 1$.

Subcase(II-1) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type $[3] \iff$ (Eq.8.3.2) of Theorem 1.15 holds.

Subcase(II-2) Let $\gcd(d_3, \theta_3(\beta_{3,0,k}^{(\nu_2+1)})_{k=1}^3) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$ for Subcase(II-2), note that $f \in$ the type $[\ell]$ for some $\ell \geq 4$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the next step. To compute the inequality in Eq(8.3.1) of Theorem 1.15 may not be needed, if necessary.

The general case will be proved by induction. Let $f \in \mathbb{C}\{y\}[z]$ be an arbitrary W -poly of degree $n \geq 2$ in z . Suppose we have shown for any integer $q \geq 3$ that for each (j)-th step, $1 \leq j \leq q-1$, the irreducibility algorithm for $f \in$ the type $[j]$ has been found, by using the same kind of properties and notations as we have seen in the proof of finding the irreducibility algorithm for the (j)-th step, $1 \leq j \leq q-1$. Now, we may assume by induction on q that the irreducibility algorithm for the (q-1)-th step can be solvable for $q \geq 3$, and so if f is irreducible in ${}_2\mathcal{O}_0$, we may assume without loss of generality that if $f \in$ the type $[k]$ for an integer $k \geq q$ in the sense of Definition 2.5 then $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1,0,k}^{(\nu_{q-2}+1)})_{k=1}^{q-1}) > 1$ and $T_{q-1,d_{q-1}-1}^{(\nu_{q-2}+1)} = 0$ for some positive integer $\nu_{q-2} + 1$. Then, it remains to show that the generalized irreducibility algorithm for $f \in$ the type $[q]$ can be written as follows.

The q-th step: To find the irreducibility algorithm for $f \in$ the type $[q]$ in the sense of Definition 2.5.

With three equations, which is defined by (Eq.(3q-3)), (Eq.(3q-2)) and (Eq.(3q-1)) in the q-th step of Theorem 1.15 later, the aim in this step is how to compute the necessary and sufficient condition for $f \in$ the type $[q]$ in the sense of Definition 2.5, **without need of computing any inequality in (Eq.(3q-1)).3.1) of Theorem 1.15.**

Suppose by induction on the positive integer $(q-1)$ that f is irreducible in ${}_2\mathcal{O}_0$ and let f satisfy a finite number $(3q-4)$ of conditions, which have been represented by (Eq.1), (Eq.2), (Eq.3), ..., (Eq.(3q-6)), (Eq.(3q-5)), (Eq.(3q-4)) of Theorem 1.15.

For the proof of this step, recall the defining equation $(h_{q-2, \nu_{q-2}+1}, f)$ of (Eq.(3q-4)) of Theorem 1.15 as we have already seen in Subcase(B1) of the $(q-1)$ -th step of Theorem 1.15:

$$(Eq.(3q-4)) \quad \begin{cases} h_{q-2, \nu_{q-2}+1} &= f_{q-3}^{n_{q-2}} + \sum_{i=0}^{n_{q-2}-2} R_{q-2, i}^{(\nu_{q-2}+1)} f_{q-3}^i \\ f &= h_{q-2, \nu_{q-2}+1}^{d_{q-1}} + \sum_{i=0}^{d_{q-1}-2} T_{q-1, i}^{(\nu_{q-2}+1)} h_{q-2, \nu_{q-2}+1}^i \end{cases}$$

of Theorem 1.15, noting that $T_{q-1, d_{q-1}-1}^{(\nu_{q-2}+1)} = 0$.

By either Subcase(I-2) or Subcase(II-2) of the $(q-1)$ -th step, it may be assumed that $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1, 0, k}^{(\nu_{q-2}+1)})_{k=1}^{q-1}) > 1$ and $T_{(q-1), d_{(q-1)}-1}^{(\nu_{q-2}+1)} = 0$ for some positive integer $\nu_{q-2}+1$.

Remark. If $T_{q-1, d_{q-1}-1}^{(1)} = 0$ and $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1, 0, k}^{(1)})_{k=1}^{q-1}) > 1$ for Subcase(I-2), then $(h_{q-2, 1}, f) = (g_{q-2}, f)$ can be viewed as $(h_{q-2, \nu_{q-2}+1}, f)$ with $\nu_{q-2} = 0$. \square

For the irreducibility algorithm for the q -th step, it suffices to consider the following subcase only for the $(q-1)$ -th step:

Subcase(II-2) Let $\gcd(d_{q-1}, \theta_{q-1}(\beta_{q-1, 0, k}^{(\nu_{q-2}+1)})_{k=1}^{q-1}) > 1$ for some integer $\nu_{q-2} \geq 0$.

If f is irreducible in ${}_2\mathcal{O}_0$ for Subcase(II-2), note that $f \in$ the type $[\ell]$ for some $\ell \geq 3$ in the sense of Definition 2.5. To find the irreducibility algorithm for $f \in$ the type $[q]$ in the sense of Definition 2.5, using the inequalities in (Eq.3q-3) with (Eq.(3q-3).1) and (Eq.3q-2) with (Eq.(3q-2).1) and (Eq.(3q-2).2) of Theorem 1.15, it suffices to consider two new cases:

Case(I) $T_{q, d_{q-1}}^{(1)} = 0$, and Case(II) $T_{q, d_{q-1}}^{(1)} \neq 0$.

Case(I) Let $T_{q, d_{q-1}}^{(1)} = 0$. There are two subcases only.

Subcase(I-1) $\gcd(d_q, \theta_q(\beta_{q, 0, k}^{(1)})_{k=1}^q) = 1$ and Subcase(I-2) $\gcd(d_q, \theta_q(\beta_{q, 0, k}^{(1)})_{k=1}^q) > 1$.

Subcase(I-1) Let $\gcd(d_q, \theta_q(\beta_{q, 0, k}^{(1)})_{k=1}^q) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type $[q]$ in the sense of Definition 2.5 \iff the inequality in (Eq.(3q-2).3) of Theorem 1.15 holds.

Subcase(I-2) Let $\gcd(d_q, \theta_q(\beta_{q, 0, k}^{(1)})_{k=1}^q) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq q+1$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the $(q+1)$ -th step. To compute the inequality in Eq.(3q-2).3) of Theorem 1.15 may not be needed, if necessary.

Case(II) Let $T_{q, d_{q-1}}^{(1)} \neq 0$. To solve the subcase, first of all, we must eliminate $T_{q, d_{q-1}}^{(1)}$ whether or not f is irreducible in ${}_2\mathcal{O}_0$. To do it, by Sublemma 1.10 for Theorem 1.8 (Sublemma 15.5 for Theorem 15.4, we can compute a unique finite sequence of pairs $\{(h_{q-1, p}, f) : 1 \leq p \leq \nu_{q-1} + 1\}$ in (Eq.3q-2) of Theorem 1.15 for a unique integer $\nu_{q-1} \leq \frac{n_q+1}{2}$, which satisfies the following:

(i) $T_{q, d_{q-1}}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu_{q-1}$ and $T_{2, d_{q-1}}^{(\nu_{q-1}+1)} = 0$.

(ii) If f satisfies (Eq.(3q-1).1) and (Eq.(3q-1).2) of Theorem 1.15, without computing any inequality in (Eq.(3q-1).3.1) of Theorem 1.15, to find an irreducibility algorithm for f , it suffices to consider the following two subcases:

Subcase(II-1) $\gcd(d_q, \theta_q(\beta_{q-1, 0, k}^{(\nu_{q-1}+1)})_{k=1}^{q-1}) = 1$; Subcase(II-2) $\gcd(d_q, \theta_q(\beta_{q-1, 0, k}^{(\nu_{q-1}+1)})_{k=1}^{q-1}) > 1$.

Subcase(II-1) Let $\gcd(d_q, \theta_q(\beta_{q-1, 0, k}^{(\nu_{q-1}+1)})_{k=1}^{q-1}) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type $[q]$ in the sense of Definition 2.5 \iff (Eq.(3q-1).3.2) of Theorem 1.15 holds.

Subcase(II-2) Let $\gcd(d_q, \theta_q(\beta_{q-1, 0, k}^{(\nu_{q-1}+1)})_{k=1}^{q-1}) > 1$. If f is irreducible in ${}_2\mathcal{O}_0$, note that $f \in$ the type $[\ell]$ for some $\ell \geq q+1$ in the sense of Definition 2.5. To find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$ in this case, take the $(q+1)$ -th step. (Eq.(3q-1).3.1) of Theorem 1.15 may not be needed if necessary. \square

Remark 1.15.2. Note that the proof of Theorem 1.13 can be done by Theorem 16.5 and Theorem 16.6. Using the same method as we have used in the process of the proof of Theorem 16.6 together with Proposition 16.7 and Proposition 16.8, there is nothing to prove for Theorem 1.15 with Corollary 1.15.1(The 2nd Algorithm). \square

§ 1.9. The 3rd Algorithm for computing the corresponding standard Puiseux expansion from any irreducible W-poly of two complex variables with respect to the multiplicity sequences

In preparation for finding the representation of the desired algorithm, we prefer to use the properties and notations in Definition 1.2, Theorem 1.4 and Theorem 1.6. By Definition 1.2, Family(1) is the first family, denoted by $\text{Family}(1) = \{f \in \text{Family}(0) : f \text{ is arbitrary standard Puiseux polynomial of the recursive } r\text{-type and } r \text{ are any positive integers}\}$ and Family(2) is the 2nd family, denoted by $\text{Family}(2) = \{C_r(t) : C_r(t) \text{ is the standard Puiseux expansion of the } r\text{-type for any } r \in \mathbb{N}\}$.

Then, we use the following notations in (i) and (ii).

(i) For any f and g in $\text{Family}(0)$, an equivalence relation for any two multiplicity sequences $\text{Multiseq}(V(f))$ and $\text{Multiseq}(V(g))$ in $\text{Family}(3)$ is defined as follows:

$$(*) \quad \begin{aligned} & \text{Multiseq}(V(f)) \text{ and } \text{Multiseq}(V(g)) \text{ are equivalent} \\ \iff & f \stackrel{\text{multiseq}}{\sim} g \text{ at the origin in } \mathbb{C}^2 \end{aligned}$$

(ii) For any f in $\text{Family}(0)$ and any $C_r(t)$ in $\text{Family}(2)$, an equivalence relation for any two multiplicity sequences $\text{Multiseq}(V(f))$ and $\text{Multiseq}(C_r(t))$ in $\text{Family}(3)$ is defined as follows:

$$(**) \quad \begin{aligned} & \text{Multiseq}(V(f)) \text{ and } \text{Multiseq}(C_r(t)) \text{ are equivalent} \\ \iff & f \stackrel{\text{multiseq}}{\sim} C_r \text{ at the origin in } \mathbb{C}^2 \quad \square \end{aligned}$$

Theorem 1.16(The 3rd Algorithm: Explicit algorithm for finding the corresponding standard Puiseux expansion from any given irreducible W-poly of two complex variables).

Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 1.15 hold. In addition, let f of (Eq.1) of Theorem 1.15 be irreducible in ${}_2\mathcal{O}_0$ with isolated singularity at $0 \in \mathbb{C}^2$. Write $n = \prod_{k=1}^r n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers. Then $f \in$ the type $[\ell]$ for some $1 \leq \ell \leq r$ in the sense of Definition 2.5.

Conclusions Following the same properties and notations as in the assumptions and the conclusions of Theorem 1.15, to find explicit algorithm for computing the corresponding standard Puiseux expansion from any irreducible W-poly $f \in \mathbb{C}\{y\}[z]$, it suffices to solve The 1st Problem by The 1st Conclusion and The 2nd Problem by the proof of The 2nd Conclusion respectively, because Conclusions of this theorem can be divided by The 1st Conclusion and The 2nd Conclusion, as follows:

The 1st Problem The problem is to find explicit algorithm for computing the standard Puiseux polynomial of the recursive r -type $H_q \in \text{Family}[q]$ for any $f \in$ the type $[q]$ in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_q$ at $0 \in \mathbb{C}^2$ for each $q = 1, 2, \dots, r$, as an application of Theorem 1.15.

The 2nd Problem The problem is to find Explicit algorithm for computing standard Puiseux expansion $C_q \in \text{Family}[2]$ for any standard Puiseux polynomial of the recursive r -type $H_q \in \text{Family}[1]$ such that $H_q \stackrel{\text{multiseq}}{\sim} C_q$ at $0 \in \mathbb{C}^2$, as an application of Theorem 1.4.

[I] The 1st Conclusion(A solution of The 1st problem)

Let $f \in$ the type $[\ell]$ for some $\ell \leq r$ in the sense of Definition 2.5, and follow the same properties and notations as in the assumptions and the conclusions of Theorem 1.15 for a

solution of The 1st problem. By the induction method on the set of the positive integers, the aim is to solve the 1st problem, using q iterations of the following steps with $q \leq \ell$.

The 1st step: To find $H_1 \in \text{Family}[1]$ for any $f \in$ the type $[1]$ in the sense of

Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_1$ at $0 \in \mathbb{C}^2$

For the aim in this step, by Remark for The 1st step in the conclusion of Theorem 1.15 we can find $H_1 \in \text{Family}[1]$ for any $f \in$ the type $[1]$ in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_1$ where $H_1 = z^n + y^{\alpha_0} \in \text{Family}[1]$ with $\gcd(n, \alpha_0) = 1$. \square

The 2nd step: To find $H_2 \in \text{Family}[1]$ for any $f \in$ the type $[2]$ in the sense of

Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_2$ at $0 \in \mathbb{C}^2$

For the aim in this step, by Remark for The 2nd step in the conclusion of Theorem 1.15, we can find $H_2 \in \text{Family}[1]$ for any $f \in$ the type $[2]$ in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_2 = H_1^{d_2} + \Pi_{k=1}^2 H_{k-2}^{\sigma_{2,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\alpha_{1,0,1}} \in \text{Family}[1]$ and $\sigma_{2,k} = \beta_{2,0,k}^{(\nu_1+1)}$ for $1 \leq k \leq 2$ and for some integer $\nu_1 + 1 > 0$, because of the equations, (Eq.3), (Eq.4) and (Eq.5) in the conclusion of Theorem 1.15. \square

The q -th step: To find $H_q \in \text{Family}[1]$ for any $f \in$ the type $[q]$ in the sense of

Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_q$ at $0 \in \mathbb{C}^2$

For the aim in this step, by Remark for The q -th step in the conclusion of Theorem 1.15, we can find $H_q \in \text{Family}[1]$ for any $f \in$ the type $[q]$ in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_q = H_{q-1}^{d_q} + \Pi_{k=1}^q H_{k-2}^{\sigma_{q,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$, and for $3 \leq j \leq q$, $H_{j-1} = H_{j-2}^{d_{j-1}} + \Pi_{k=1}^{j-1} H_{k-2}^{\sigma_{j-1,k}} \in \text{Family}[1]$, noting that for each fixed $j = 1, 2, \dots, q-1$ and for $1 \leq k \leq j$, $\sigma_{j,k} = \beta_{j,d_j-n_j,k}$, and for $1 \leq k \leq q-1$ $\sigma_{q,k} = \beta_{q,0,k}^{(\nu_{q-1}+1)}$ with some positive integer $\nu_{q-1} + 1$, because of the equations, (Eq.(3j-3)), (Eq.(3j-2)) and (Eq.(3j-1)) with $2 \leq j \leq q$ in the conclusion of Theorem 1.15. Note that $d_{j+1} = \gcd(d_j, \theta_j(\sigma_{j,k})_{k=1}^j)$ with $d_j = n_j d_{j+1}$ for $j \geq 2$. \square

[II] The 2nd Conclusion(A solution of The 2nd problem)

In preparation for finding a solution of The 2nd problem, follow the same properties and notations as in The 1st Conclusion in order to avoid the complexity of the terminology and notations of the statements between Theorem 1.4 and Theorem 1.16. For any $H_q \in \text{Family}[1]$ of The 1st Conclusion, it suffices to show by explicit algorithm in (1.4.1) of Theorem 1.4 that we can rewrite explicit algorithm for finding the standard Puiseux expansion, denoted by the curve $C(H_q : t)$, which satisfies $f \stackrel{\text{multiseq}}{\sim} H_q$ at $0 \in \mathbb{C}^2$ explicitly and rigorously, as in Sublemma 1.17.

Sublemma 1.17(Theorem 1.4:Algorithm for finding a one-to-one function from Family(1) into Family(2))

Assumptions Following the same properties and notations as in the assumption and The 1st Conclusion of Theorem 1.16, recall the following:

(i) We may assume by Lemma 1.8 that $f \in \mathbb{C}\{y\}[z]$ is an arbitrary W -poly of degree $n \geq 2$ in z , satisfying the following form:

$$(Eq.1) \quad f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i,$$

(ii) In the q -th step in The 1st Conclusion of this theorem(A solution of The 1st problem), we can compute $H_q \in \text{Family}[1]$ for any $f \in$ the type $[q]$ in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_q$ at $0 \in \mathbb{C}^2$, directly.

Conclusions

[I] By Algorithm 1.4.1 for Theorem 1.4, we can compute the standard Puiseux expansion

for the curve $C(H_q : t)$ such that $\text{Multiseq}(V(H_q)) \equiv \text{Multiseq}(C(H_q : t))$ as sequence:

(Algorithm 1.4.1 for Theorem 1.4)

$$(Eq.2) \quad C(H_q : t) := \begin{cases} y = t^n \\ z = t^{\gamma_1} + t^{\gamma_2} + \dots + t^{\gamma_q}, \end{cases}$$

such that $d_2 = \gcd(n, \gamma_1)$ with $n = n_1 d_2$ and $\gamma_1 = \alpha_{1,0,1} d_2$, and

for $1 \leq j \leq r$, $d_{j+1} = \gcd(d_j, \gamma_j - \gamma_{j-1})$ with $d_j = n_j d_{j+1}$ and $\gamma_j - \gamma_{j-1} = \widehat{\theta}_j d_j$

where $d_r = 1$ and $\widehat{\theta}_j = \theta_j(\beta_{j,0,k})_{k=1}^j - n_j n_{j-1} \theta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1}$ is a positive integer for $2 \leq j \leq r$ and $\theta_1(t) = t$. Note that $d_j = n_j d_{j+1}$ for $1 \leq j \leq r-1$.

Remark for the q -th step: To find $H_q \in \text{Family}(1)$ such that $f \stackrel{\text{multiseq}}{\sim} H_q$ for

any $f \in$ the type $[q]$ in the sense of Definition 2.5. Let $\gcd(d_q, \theta_q(\beta_{q,0,k}^{(\nu_{q-1}+1)})_{k=1}^q) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if (Eq.(3q-1).2) holds. In this case, if f is irreducible in ${}_2\mathcal{O}_0$, then $f \in$ the type $[q]$ in the sense of Definition 2.5 and also $f \stackrel{\text{multiseq}}{\sim} H_{q-1}^{d_q} + \Pi_{k=1}^q H_{k-2}^{\sigma_{q,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^{n_1} + y^{\beta_{1,0,1}}$, $H_2 = H_1^{n_2} + y^{\sigma_{2,1}} z^{\sigma_{2,2}}$, $H_j = H_{j-1}^{n_j} + \Pi_{k=1}^j H_{k-2}^{\sigma_{j,k}}$ for $3 \leq j \leq q-2$, $H_{q-1} = H_{q-2}^{d_{q-1}} + \Pi_{k=1}^{q-1} H_{k-2}^{\sigma_{q-1,k}}$, noting that for each fixed $j = 1, 2, \dots, q-2$, $\sigma_{j,k} = \beta_{j,0,k}^{(1)}$ for $1 \leq k \leq j$ and $\sigma_{q-1,k} = \beta_{q-1,0,k}^{(\nu_{q-1}+1)}$ for $1 \leq k \leq q-1$. \square

Assume that $f \in$ the type $[q]$ with $q \geq 3$ in the sense of Definition 2.5.

- (i) Define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by $\theta_2(t_k)_{k=1}^2 = t_2 \theta_1(\sigma_{1,1}) + n_1 t_1$. $\sigma_{1,1} = \alpha_{1,0,1}$.
- (ii) By induction on positive integers $j=2, 3, \dots, q-1$, define $\theta_j : \mathbb{N}_0^{(j)} \rightarrow \mathbb{N}_0$ by $\theta_j(t_k)_{k=1}^j = t_j \theta_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1} + n_{j-1} \theta_{j-1}(t_k)_{k=1}^{j-1}$, and then θ_j is one-to-one. By (Eq.3j-3), $\sigma_{j-1,k} = \beta_{j-1,d_{j-1}-n_{j-1},k}$ for $k = 1, 2$.
- (q) Define $\theta_q : \mathbb{N}_0^{(q)} \rightarrow \mathbb{N}_0$ by $\theta_q(t_k)_{k=1}^q = t_q \theta_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} + n_{q-1} \theta_{q-1}(t_k)_{k=1}^{q-1}$, and then θ_{q-1} is one-to-one. By (Eq.3q-3), $\sigma_{q-1,k} = \beta_{q-1,0,k}^{(\nu+1)}$ for $k = 1, 2$. \square

§1.10. Examples for The 2nd Algorithm(Theorem 1.15(Corollary 1.15.1)) and The 3rd Algorithm(Theorem 1.16)

As Examples for The 2nd Algorithm and The 3rd Algorithm, let $f(y, z)$ be a W-poly of two complex variables, which is given by the following:

$$(Eq.1) \quad f(y, z) = z^{16} + 4y^3 z^{14} + \{4y^5 + 6y^6\} z^{12} + \{12y^8 + 4y^9\} z^{10} + \{6y^{10} + 12y^{11} + y^{12}\} z^8 \\ + \{12y^{13} + 4y^{14} + y^{17}\} z^6 + \{6y^{10} + 4y^{15} + y^{20}\} z^4 + \{4y^{18} + y^{22}\} z^2 \\ + y^{24} z + \{y^{20} + y^{29}\}.$$

By the same method and notations as in Theorem 1.15(Corollary 1.15.1) and Theorem 1.16, we are going to find The 2nd Algorithm and The 3rd Algorithm for $f(y, z)$ of (Eq.1).

Example 1.10.1 for The 2nd Algorithm: To find irreducibility of f of (Eq.1) in ${}_2\mathcal{O}_0$

We can find The 2nd algorithm in both Theorem 1.15 and Corollary 1.15.1 to compute irreducibility of f of (Eq.1) in ${}_2\mathcal{O}_0$. To find a solution, we use the 1st method defined by The 2nd Algorithm in Theorem 1.15, and the 2nd method defined by The 2nd Algorithm in Corollary 1.15.1, respectively.

[The 1st Method] We use The 2nd algorithm in Theorem 1.15, to find a solution.

The 1st step: To find the irreducibility algorithm for $f \in$ the type $[\ell]$ with $\ell \geq 1$ in the sense of Definition 2.5.

If f is irreducible in ${}_2\mathcal{O}_0$, then f must satisfy the following necessary condition:

$$(Eq.2) \quad \frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n} \quad \text{and} \quad 1 \leq \gcd(n, \alpha_0) < n \quad \text{for} \quad 0 \leq i \leq n-2.$$

Note that $1 < \gcd(n, \alpha_0) = 4 < n$ where $n = 16$ and $\alpha_0 = 20$.

Remark. It is clear that $\frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n} = \frac{5}{4}$ for $0 \leq i \leq 14$, if exists. Since $1 < \gcd(n, \alpha_0) < n$, $f \notin$ the type [1] in the sense of Definition 2.5, and so either $f \in$ the type $[\ell]$ for $\ell \geq 2$ in the sense of Definition 2.5 or f is not irreducible in $\mathbb{C}\{y, z\}$. So, it suffices to follow the second step.

The 2nd step: To find the irreducibility algorithm for $f \in$ the type $[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5.

Let $d_2 = \gcd(n, \alpha_0)$, and then write $n = n_1 d_2$ and $\alpha_0 = \beta_{1,0,1} d_2$. Then, $n_1 = 4$ and $\beta_{1,0,1} = 5$ with $d_2 = 4$. If f is irreducible in ${}_2\mathcal{O}_0$, it suffices to show that f with g_1 in (Eq.1) can be represented as follows:

$$(Eq.3) \quad (a) \quad g_1 = z^{n_1} + \xi_1 y^{\beta_{1,0,1}} = z^4 + y^5 \quad \text{with} \quad \xi_1 = 1, \\ (b) \quad f = (z^{n_1} + \xi_1 y^{\beta_{1,0,1}})^{d_2} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} d_2,$$

Remark. It is trivial that (g_1, f) satisfies the inequalities in (Eq.3).

Whether the W-poly $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ or not, apply the WDT(Theorem 1.7) to $f(y, z)$ with a divisor $g_1(y, z)$. By The Division Alorithm for the W-polys(Theorem 1.8), we can show that (g_1, f) can be written in the form

$$(Eq.4)(Eq.4.1) \quad \begin{cases} g_1 &= z^4 + y^5 \quad \text{with} \quad h_{1,1} = g_1, \\ f &= g_1^{d_2} + \sum_{i=1}^{d_2-1} T_{2,i}^{(1)} g_1^i, \quad \text{with} \quad T_{2,d_2-1}^{(1)} \neq 0, \end{cases}$$

with the following property:

$$(Eq.4.1)(Eq.4.1.1) \quad T_{2,3}^{(1)} = \{4y^3 z^2 + 6y^6\}, \quad T_{2,2}^{(1)} = \{4y^9 z^2 - 6y^{11} + y^{12}\}, \\ T_{2,1}^{(1)} = \{-4y^{14} z^2 + y^{17} z^2 - 2y^{17} + y^{20}\}, \quad T_{2,0}^{(1)} = \{y^{24} z + y^{22} - y^{25} + y^{29}\}.$$

If f is irreducible in ${}_2\mathcal{O}_0$, it can be proved that (g_1, f) satisfies the necessary condition:

$$(Eq.4.1)(Eq.4.1.2) \quad \frac{\theta_2(\beta_{2,i,k}^{(1)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2}{d_2} > n_1 \beta_{1,0,1} \quad \text{for} \quad 0 \leq i \leq d_2 - 1$$

$$(Eq.4.1.3) \quad \gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 4 \geq 1.$$

Remark. It is easy to compute that (g_1, f) satisfies the inequalities in (Eq.4.1.2) and (Eq.4.1.3) because $\frac{\theta_2(\beta_{2,i,k}^{(1)})_{k=1}^2}{d_2 - i} = 22$ for $0 \leq i \leq 3$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = \gcd(4, 88) = 4 > 1$.

Since $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) > 1$ and the coefficient $T_{2,d_2-1}^{(1)} = 4y^3 z^2 + 6y^6 \neq 0$, to find a necessary and sufficient condition for f to be irreducible in ${}_2\mathcal{O}_0$, first of all, we can use the same notations and properties as in Case(B) of The 2nd step in the conclusions of Theorem 1.15. Then, it suffices to consider the following:

Subcase(B1) of Case(B) of the 2nd step: Since $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(1)})_{k=1}^2) = 4 > 1$ and $T_{2,d_2-1}^{(1)} = 4y^3 z^2 + 6y^6 \neq 0$, to find whether the above W-poly $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ or not, by the WDT(Theorem 1.7) only (g_1, f) can be rewritten by $(h_{1,2}, f)$

$$(Eq.5)(Eq.5.1) \quad \begin{cases} h_{1,2} &= h_{1,1} + \frac{1}{d_2} T_{2,d_2-1}^{(1)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(2)} z^i \\ f &= h_{1,2}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(2)} h_{1,2}^i \quad \text{with} \quad T_{2,d_2-1}^{(2)} \neq 0, \end{cases}$$

with the following property:

$$(Eq.5.1)(Eq.5.1.1) \quad h_{1,2} = g_1 + \frac{1}{4}T_{2,d_2-1}^{(1)} = z^4 + y^3z^2 + \{y^5 + \frac{3}{2}y^6\} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(2)}z^i,$$

$$\text{and } T_{2,3}^{(2)} = -6y^6, T_{2,2}^{(2)} = \frac{27}{2}y^{12}, T_{2,1}^{(2)} = -\frac{27}{2}y^{18} + y^{17}z^2, T_{2,0}^{(2)} = y^{24}z + \frac{81}{16}y^{24} - y^{25} + y^{29}.$$

If f is irreducible in ${}_2\mathcal{O}_0$, it can be proved that $(h_{1,2}, f)$ satisfies the necessary condition. In particular, if $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(2)})_{k=1}^2) = 1$ in (Eq.5.1.3), f must be irreducible in ${}_2\mathcal{O}_0$:

$$(Eq.5.1.2) \quad \frac{\theta_2(\beta_{2,i,k}^{(2)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(2)})_{k=1}^2}{d_2} > n_1\beta_{1,0,1} \quad \text{for } 0 \leq i \leq d_2 - 1$$

$$(Eq.5.1.3) \quad \gcd(d_2, \theta_2(\beta_{2,0,k}^{(2)})_{k=1}^2) > 1.$$

Remark. It is clear that $h_{1,2} \stackrel{\text{multiseq}}{\sim} h_{1,1} = g_1$ and note that $(h_{1,2}, f)$ satisfies the inequalities in (Eq.5.1.2) and (Eq.5.1.3) because $\theta_2(\beta_{2,3,k}^{(2)})_{k=1}^2 = 24$, $\theta_2(\beta_{2,2,k}^{(2)})_{k=1}^2 = 48$, $\theta_2(\beta_{2,1,k}^{(2)})_{k=1}^2 = 72$, and $\theta_2(\beta_{2,0,k}^{(2)})_{k=1}^2 = 96$. So, we need the next step to find that f is irreducible in ${}_2\mathcal{O}_0$.

Subcase(B2) of Case(B) of the 2nd step: Since $T_{2,d_2-1}^{(2)} = -6y^6 \neq 0$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(2)})_{k=1}^2) > 1$, to find whether the above W-poly $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ or not, if necessary, for the computation of this step, the proof of the inequalities in (Eq.5.1.2) and (Eq.5.1.3) may not be needed for (Eq.5.1.1) of **Subcase(B1) of Case(B) of the 2nd step**.

To find whether the above W-poly $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ or not, by the WDT(Theorem 1.7) only $(h_{1,2}, f)$ can be rewritten by $(h_{1,3}, f)$

$$(Eq.5)(Eq.5.2) \quad \begin{cases} h_{1,3} &= h_{1,2} + \frac{1}{d_2}T_{2,d_2-1}^{(2)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(3)}z^i \\ f &= h_{1,3}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(3)}h_{1,2}^i \text{ with } T_{2,d_2-1}^{(2)} = 0, \end{cases}$$

with the following property:

$$(Eq.5.2.1) \quad f = h_{1,3}^4 + y^{17}z^2h_{1,3}^2 + y^{24}z + y^{29} = h_{1,3}^{d_2} + \sum_{i=0}^{d_2-2} T_{2,i}^{(3)}h_{1,3}^i,$$

$$h_{1,3} = h_{1,2} + \frac{1}{4}T_{2,d_2-1}^{(2)} = h_{1,2} + \frac{1}{4}\{-6y^6\} = z^4 + y^5 + y^3z^2 = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(3)}z^i,$$

$$\text{where } T_{2,3}^{(3)} = 0, T_{2,2}^{(3)} = 0, T_{2,1}^{(3)} = y^{17}z^2, T_{2,0}^{(3)} = y^{24}z + y^{29}.$$

Now, if f is irreducible in ${}_2\mathcal{O}_0$, it can be proved that $(h_{1,2}, f)$ satisfies the necessary condition:

$$(Eq.5.2.2) \quad \frac{\theta_2(\beta_{2,i,k}^{(3)})_{k=1}^2}{d_2 - i} \geq \frac{\theta_2(\beta_{2,0,k}^{(3)})_{k=1}^2}{d_2} > n_1\beta_{1,0,1} \quad \text{for } 0 \leq i \leq d_2 - 2,$$

$$(Eq.5.2.3) \quad \gcd(d_2, \theta_2(\beta_{2,0,k}^{(3)})_{k=1}^2) = 1.$$

Remark. (i) It is clear that $h_{1,3} \stackrel{\text{multiseq}}{\sim} h_{1,1} = g_1$ and note that $(h_{1,3}, f)$ satisfies the inequalities in (Eq.5.2) because $\theta_2(\beta_{2,1,k}^{(3)})_{k=1}^2 = 98$, and $\theta_2(\beta_{2,0,k}^{(3)})_{k=1}^2 = 101$. So, f is irreducible in ${}_2\mathcal{O}_0$.

(ii) It can be computed that $T_{2,d_2-1}^{(2)} = -6y^6 \neq 0$ and $\gcd(d_2, \theta_2(\beta_{2,0,k}^{(3)})_{k=1}^2) = 1$. So $(f_1, f) = (h_{1,\nu+1}, f) = (h_{1,3}, f)$ is a generalized representation of f in the sense of Theorem 1.15. \square

[The 2nd Method] We use The 2nd algorithm in Corollary 1.15.1, to find a solution.

To compute irreducibility of f of (Eq.1) in ${}_2\mathcal{O}_0$ by The 2nd Algorithm in Corollary 1.15.1, it suffices to consider the following:

(a) Since $T_{2,3}^{(1)} = \{4y^3z^2 + 6y^6\} \neq 0$, in this case to compute the inequality in (Eq.4.1.2) and (Eq.4.1.2) may not be needed, if necessary, in order to find a solution,

(b) Since $T_{2,d_2-1}^{(2)} = -6y^6 \neq 0$ with $d_2 = 4$, in this case to compute the inequality in (Eq.5.1.2) and (Eq.5.1.3) may not be needed if necessary, in order to find a solution.

(c) Since $T_{2,d_2-1}^{(3)} = 0$ with $d_2 = 4$, in this case to compute the inequality in (Eq.5.2.2) and (Eq.5.2.3) must be needed if necessary, in order to find a solution.

Thus, without provig the inequality in (Eq.4.1.2) and (Eq.4.1.2) of (a) and the inequality in (Eq.5.1.2) and (Eq.5.1.3) of (b) in Theorem 15.5, following the same methods and notations as in (c) we can find the same solution as in Theorem 15.5. \square

Example 1.10.2 for The 3rd Algorithm: To find the standard Puiseux expansion

$C(f)$ for an irreducible W-poly f of (Eq.1) such that $f \stackrel{\text{multiseq}}{\sim} C(f)$ at $0 \in \mathbb{C}^2$

By Remark for the 2nd step in the conclusion of Theorem 1.15, we can find $H_2 \in \text{Family}[1]$ for any $f \in$ the type [2] in the sense of Definition 2.5 such that $f \stackrel{\text{multiseq}}{\sim} H_2 = H_1^{d_2} + \Pi_{k=1}^2 H_{k-2}^{\sigma_{2,k}}$ where $H_{-1} = y$, $H_0 = z$, $H_1 = z^4 + y^5 \in \text{Family}[1]$ and $\sigma_{2,k} = \beta_{2,0,k}^{(\nu_1+1)}$ for $1 \leq k \leq 2$ and for some integer $\nu_1 + 1 > 0$, because of the equations, (Eq.3), (Eq.4) and (Eq.5) in the conclusion of Theorem 1.15, noting that $d_2 = 4$, $\sigma_{2,1} = 24$ and $\sigma_{2,2} = 1$. Let $C(H_q : t)$ be the the standard Puiseux expansion defined by

$$(Eq.6) \quad C(H_q : t) := \begin{cases} y = t^n \\ z = t^{\gamma_1} + t^{\gamma_2} + \dots + t^{\gamma_q}, \end{cases}$$

such that $d_2 = \gcd(n, \gamma_1)$ with $n = n_1 d_2 = 12$ and $\gamma_1 = \alpha_{1,0,1} d_2 = 20$, and

$d_3 = \gcd(d_2, \gamma_2 - \gamma_1) = 1$ with $d_2 = n_2 d_3$ and $\gamma_2 - \gamma_1 = \hat{\theta}_2 d_3$

where $d_3 = 1$ and $\hat{\theta}_j = \theta_j(\beta_{j,0,k})_{k=1}^j - n_j n_{j-1} \theta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1}$ is a positive integer for $2 \leq j \leq r$ and $\theta_1(t) = t$. Note that $d_j = n_j d_{j+1}$ for $1 \leq j \leq r-1$.

By Theorem 1.4, it remains to compute $\gamma_2 - \gamma_1$, which follows from

$$\hat{\theta}_2 d_3 = \theta_2(\beta_{2,0,k})_{k=1}^2 - n_2 n_1 \theta_1(\beta_{1,0,1}) = \theta_2(\sigma_{2,1}, \sigma_{2,2}) - n_2 n_1 \alpha_{1,0,1} = \theta_2(24, 1) - 5 \cdot 4 \cdot 4 = 21.$$

Thus, $C(H_q : t)$ can be computed by $y = t^{16}$ and $z = t^{20} + t^{41}$. \square

Part[B](Chapter III, . . . , Chapter VII)

Explicit algorithm for computing the correspondence between the irreducible W-polys of two complex variables and the Puiseux expansions with proofs

Chapter III: Foundations

§2. New definitions for quasisingularity and a generalization of one coordinate patch covering of the local coordinates used in the standard resolution process of irreducible plane curve singularities

Definition 2.1. Let $\mathbb{C}\{y, z\}$ be the ring of convergent power series or analytic functions at $(y, z) = (0, 0)$. Let $V(F) = \{(y, z) : F = F(y, z) = 0\}$ be an analytic variety at $(y, z) = (0, 0)$ where F is in $\mathbb{C}\{y, z\}$. Assume that $V(F)$ has an isolated singular point at the origin in \mathbb{C}^2 as reduced variety. Note that F may have multiple factors as analytic function at the origin. Let $\pi_1 : M \rightarrow \mathbb{C}^2$ be the blow-up of \mathbb{C}^2 at $(y, z) = (0, 0)$, which is the singular point $(0, 0)$ of $V(F)$. Let $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ be coordinate patches for M with $\pi_1(v_1, u_1) = (y, z) = (v_1, v_1 u_1)$ and $\pi_1(v'_1, u'_1) = (y, z) = (v'_1 u'_1, v'_1)$ where $u'_1 = \frac{1}{u_1}$ and $v'_1 = v_1 u_1$. For brevity of notations, the above $F(y, z)$ is square-free in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. Let e be the multiplicity of $V(F)$ at $(0, 0)$, with $e \geq 2$.

Then $\pi_1^{-1}(V(F))$, the total transform of $V(F)$ under π_1 , is given locally by $(F \circ \pi_1)_{total} = F(v_1, v_1 u_1) = v_1^e F_1(v_1, u_1)$ along $v_1 = 0$ where $F_1(v_1, u_1)$ is in $\mathbb{C}\{v_1, u_1\}$ and $(F \circ \pi_1)_{total} = F(v'_1 u'_1, v'_1) = v'^e_1 F_2(v'_1, u'_1)$ along $v'_1 = 0$ where $F_2(v'_1, u'_1)$ is in $\mathbb{C}\{v'_1, u'_1\}$. We call $V^{(1)}(F)$ the proper transform of $V(F)$ under π_1 where $V^{(1)}(F) = \{(v_1, u_1) : F_1(v_1, u_1) = 0\} \cup \{(v'_1, u'_1) : F_2(v'_1, u'_1) = 0\}$. We say that $E_1 = \{(v_1, u_1) : v_1 = 0\} \cup \{(v'_1, u'_1) : v'_1 = 0\}$ is an exceptional curve of the first kind. In this case, each of $U_1 = \{(v_1, u_1)\}$ and $U_2 = \{(v'_1, u'_1)\}$ is called one coordinate patch of the given local coordinates for blow-up π_1 , respectively. Note that if $F(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$, then just one coordinate patch is needed for the study of $V^{(1)}(F)$.

After m iterations of blow-ups, let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$. Let $V^{(m)}(F)$ be the proper transform of $V(F)$ under τ_m . Let $E^{(m)} = \tau_m^{-1}(0, 0)$. Then $E^{(m)}$ is, by definition, an exceptional set of the first kind. Let $E^{(m)} = \cup_{i=1}^m E_i$ be the decomposition into irreducible components. Each E_i is called an exceptional curve of the first kind. Let $(F \circ \tau_m)_{divisor}$ be the divisor of $F \circ \tau_m$ defined by $(F \circ \tau_m)_{divisor} = V^{(m)}(F) + \sum_{i=1}^m e_i E_i$ where each e_i is the multiplicity of $F \circ \tau_m$ along E_i for $1 \leq i \leq m$.

Then, we have the following well-known theorem.

Theorem 2.2. Let $V(F) = \{(y, z) : F(y, z) = 0\}$ be an analytic variety at $(0, 0)$ where $F(y, z)$ is in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. There exists an analytic manifold M by using the composition of a finite number m of successive blow-ups, $\tau_m : M \rightarrow \mathbb{C}^2$, such that if R is the set of regular points on $V(F)$ then $\tau_m : \tau_m^{-1}(R) \rightarrow V(F)$ is a resolution of a singular point $(0, 0)$ of $V(F)$, where $\tau_m^{-1}(R)$ is the closure of $\tau_m^{-1}(R)$ in M .

Remark 2.2.1. (i) If V is an analytic set, a resolution of the singularities f consists of a complex manifold and a proper analytic map $\pi : M \rightarrow V$ such that π is biholomorphic on the inverse image of R , the regular points of V , and such that $\pi^{-1}(R)$ is dense in M .

(ii) Blow-ups are canonical. Namely, let $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a biholomorphic map, and let $\pi' : M' \rightarrow \mathbb{C}^2$ be a blow-up at $\phi(0, 0)$. Then, there is a unique induced biholomorphic map $\phi : M \rightarrow M'$ such that $\phi \circ \pi_1 = \pi' \circ \phi'$.

Corollary 2.3. Under the same assumption of Theorem 2.2, after additional blow-ups any two components of $V^{(m)}(F)$ and $\cup_{i=1}^m E_i$ meets with normal crossings whenever they meet and no three distinct components of $V^{(m)}(F)$ and $\cup E_i$ meet, where $V^{(m)}(F)$ and $\cup E_i$ are defined just before Theorem 2.2.

Remark 2.3.1. If $F(y, z)$ of Corollary 2.3 is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin, then each exceptional curve E_i of the first kind meets at most three distinct intersection points with other exceptional curves and the proper transform $V^{(m)}(F)$. It will be proved by Theorem 3.6 and Theorem 3.7.

Definition 2.4(The standard resolution, A homeomorphic resolution, Having the same divisor under two standard resolutions).

(I) For a singularity of a plane curve, the smallest resolution with normal crossings in the sense of Corollary 2.3 is called the standard resolution of a given singularity.

(II) As in Definition 2.1, let $V(F)$ and $V(G)$ be analytic varieties at $0 \in \mathbb{C}^2$ where $F = F(y, z)$ and $G = G(y, z)$ are in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. $V(F)$ and $V(G)$ are said to either have a homeomorphic resolution, or be equisingular, if $(F \circ \tau_m)_{divisor}$ and $(G \circ \tau_m)_{divisor}$ are equivalent in the sense of Definition 2.1 where $\tau_m = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ is the composition of the same number m of successive blow-ups at the origin, which is the standard resolution of the singularity $(0, 0)$ of both $V(F)$ and $V(G)$. Then, we denote this relation by either $V(F) \stackrel{\text{resol}}{\sim} V(G)$ at $0 \in \mathbb{C}^2$ or $(F \circ \tau_m)_{divisor} = (G \circ \tau_m)_{divisor}$ under the same standard resolution τ_m of both $V(F)$ and $V(G)$. In other words, if we write $(F \circ \tau_m)_{divisor} = V^{(m)}(F) + \sum_{i=1}^m e_i E_i$ and $(G \circ \tau_m)_{divisor} = V^{(m)}(G) + \sum_{i=1}^m \bar{e}_i E_i$ in the sense of Definition 2.1, then it is said that $V(F) \stackrel{\text{resol}}{\sim} V(G)$ at $0 \in \mathbb{C}^2$ under the same standard resolution τ_m of both $V(F)$ and $V(G)$ if and only if $e_i = \bar{e}_i$.

(III) In particular, let $f \in \text{Family}(0)$ and $g \in \text{Family}(0)$ in the sense of Definition 1.2, that is, f and g be analytically irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. Let $\tau_\xi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ be the composition of a finite number ξ of successive blow-ups π_i at $0 \in \mathbb{C}^2$, which is needed only to get the standard resolution of the singularity of $V(f)$. Using the same method as before, let $\mu_\eta = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \dots \circ \bar{\pi}_\eta : \bar{M}^{(\eta)} \rightarrow \mathbb{C}^2$ be the composition of a finite number η of successive blow-ups at $0 \in \mathbb{C}^2$, which is needed only to get the standard resolution of the singularity of $V(g)$. Following the same properties and notations as in Definition 2.1, we write $(f \circ \tau_m)_{divisor} = V^{(m)}(f) + \sum_{i=1}^m e_i E_i$ and $(g \circ \mu_\eta)_{divisor} = V^{(\eta)}(g) + \sum_{i=1}^\eta \bar{e}_i \bar{E}_i$ in the sense of Definition 2.1.

It is said that either $V(f)$ and $V(g)$ have the same divisor under the standard resolutions, or $(f \circ \tau_\xi)_{divisor}$ and $(g \circ \mu_\eta)_{divisor}$ are equivalent, denoted by either $V(f) \stackrel{\text{divisor}}{\sim} V(g)$ or $f \stackrel{\text{divisor}}{\sim} g$ under the standard resolutions or $(f \circ \tau_\xi)_{divisor} = (g \circ \mu_\eta)_{divisor}$ under the standard resolutions, if the following condition is satisfied:

$$(2.4.1) \quad \begin{aligned} &\text{either } \{(f \circ \tau_\xi)_{divisor}\}_{seq.} \equiv \{(g \circ \mu_\eta)_{divisor}\}_{seq.} \quad \text{as sequence,} \\ &\text{or } \{e_i : i = 1, 2, \dots, \xi\} \equiv \{\bar{e}_i \in N : i = 1, 2, \dots, \eta\} \quad \text{as an increasing sequence.} \end{aligned}$$

(IV) **Family(4)** is the fourth family, consisting of all the divisors of $(f \circ \tau)$ defined by the total transform of $V(f)$, denoted by $(f \circ \tau)_{divisor}$ where $\tau : M \rightarrow \mathbb{C}^2$ is the standard resolution of the singularity of $V(f)$ with $f \in \text{Family}(0)$, denoted by

$$(2.4.2) \quad \text{Family(4)} = \{(f \circ \tau)_{divisor} : f \in \text{Family}(0) \text{ where } \tau : M \rightarrow \mathbb{C}^2 \text{ is} \\ \text{the standard resolution of the singularity of } V(f)\}.$$

Remark 2.4.1. Let $f \in \text{Family}(0)$ and $g \in \text{Family}(0)$. By Definition 2.4, $V(f) \stackrel{\text{resol}}{\sim} V(g)$ at $0 \in \mathbb{C}^2$ implies $f \stackrel{\text{divisor}}{\sim} g$ under two standard resolutions. But, it remains to be proved that $f \stackrel{\text{divisor}}{\sim} g$ under two standard resolutions implies $V(f) \stackrel{\text{resol}}{\sim} V(g)$ at $0 \in \mathbb{C}^2$.

Definition 2.5. Suppose that $f \in \mathbb{C}\{y, z\}$ has an isolated singular point at $0 \in \mathbb{C}^2$. It is said that f is of the type $[j]$ or belongs to the type $[j]$ under the standard resolution if f satisfies the following properties (a) and (b) after m iterations of blow-ups which is needed only to get the standard resolution of the singularity of the curve defined by f : Sometimes, we write $V(f)$ is of the type $[j]$, instead of f is of the type $[j]$.

(a) There are exactly j exceptional curves of the first kind with $j \leq m$, each of which has three distinct intersections with other exceptional curves and the proper transform.

(b) Each of the remaining $(m-j)$ exceptional curves rather than the above j exceptional curves in (a) has at most two distinct intersection points with other exceptional curves and the proper transform.

Remark 2.5.1.

- (i) If f has a nonsingular point at the origin, we say that $f \in$ the type $[0]$.
- (ii) If $f \in$ the type $[j]$ for an integer $j \geq 1$, then f may not be irreducible by the following example:

Let $f = (z + y)(z + 2y)(z + 3y) + y^6$. Then $f \in$ the type $[1]$, but f is not irreducible in $\mathbb{C}\{y, z\}$.

- (iii) Using Definition 2.5, then the following will be proved by Theorem 12.0:

Let $V(f)$ be an analytic variety at $(y, z) = (0, 0)$ where $f = f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. As it has been in Definition 2.1, and Definition 2.4, let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the standard resolution of the singular point $(0, 0)$ of $V(f)$. Then, $f \in$ the type $[j]$ under the standard resolution τ_m in the sense of Definition 2.5 where j is a positive integer with $j < m$.

Definition 2.6(Quasisingularity, homeomorphic resolutions, Having the same divisor under two standard resolutions and equivalence of multiplicity sequences for irreducible plane curve singularities).

(I) Definition for quasisingularity: Let $V(F)$ be an analytic variety at $(y, z) = (0, 0)$ where $F = F(y, z)$ is in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. Assume that $V(F)$ satisfies the same properties and notations as in Definition 2.1. After m iterations of blow-ups, let $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ be an arbitrary composition of a finite number m of successive blow-ups at the origin in \mathbb{C}^2 , which is the singular point of $V(F)$. As in Definition 2.1, let $\tau_m^{-1}(0, 0) = \bigcup_{i=1}^m E_i$ where each E_i is called an exceptional curve of the first kind, and $V^{(m)}(F)$ be the proper transform under τ_m .

For a given τ_m , let $P \in \tau_m^{-1}(0, 0)$ be chosen arbitrary. It is said that P is a quasisingular point of $V^{(m)}(F)$ under τ_m if additional blow-ups at $P \in V^{(m)}(F)$ are still necessary after m iterations of blow-ups at $(y, z) = (0, 0)$, in order to get the standard resolution of the singular point of $V(F)$. For notation, $qs(V^{(m)}(F))$ is called the set of all quasisingular points of $V^{(m)}(F)$ under τ_m . Assuming that $qs(V^{(m)}(F)) \neq \emptyset$, note that $P \in qs(V^{(m)}(F))$ may not be a singular point of $V^{(m)}(F)$.

(II) Homeomorphic resolutions of some plane curve singularities:

Let $\phi(y, z) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ be irreducible in $\mathbb{C}\{y, z\}$, where a_0, a_n are units in $\mathbb{C}\{y, z\}$, and also each a_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers, and $1 \leq n < \alpha_n$.

Let $\psi(y, z) = b_0 z^\ell + b_1 y^{\beta_1} z^{\ell-1} + \cdots + b_\ell y^{\beta_\ell}$ be irreducible in $\mathbb{C}\{y, z\}$, where b_0, b_ℓ are units in $\mathbb{C}\{y, z\}$, and also each b_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the β_i are positive integers, and $1 \leq \ell < \beta_\ell$.

For example, let e be the multiplicity of ϕ at 0. Then, it was well-known by Hensel's Lemma or Lemma 3.1 that $e = n$ and $n < \alpha_i + n - i$ for all $i = 1, 2, \dots, n$.

Let $V(\Phi)$ and $V(\Psi)$ be analytic varieties at $(y, z) = (0, 0)$ defined by

$$(2.6.1) \quad \Phi = \Phi(y, z) = y^\zeta z^\eta \phi(y, z) \quad \text{and} \quad \Psi = \Psi(y, z) = y^{\zeta'} z^{\eta'} \psi(y, z),$$

where each of ζ, η, ζ' and η' is either a positive integer or 0.

In order for both $V(\Phi)$ and $V(\Psi)$ to have an isolated singular point at the origin as reduced varieties, in addition we may assume that the following property holds: Note by assumption that $\phi = \phi(y, z)$ and $\psi = \psi(y, z)$ are irreducible in $\mathbb{C}\{y, z\}$.

- (2.6.2) (i) If $n = 1$ from Φ , then either ζ or η is positive.
- (ii) If $\ell = 1$ from Ψ , then either ζ' or η' is positive.

As in Definition 2.1, let $V(\Phi)$ and $V(\Psi)$ be analytic varieties at $(y, z) = (0, 0)$ in \mathbb{C}^2 where $\Phi = \Phi(y, z)$ and $\Psi = \Psi(y, z)$ are in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. $V(\Phi)$ and $V(\Psi)$ are said to have a homeomorphic resolution if $(\Phi \circ \tau_m)_{\text{divisor}}$ and $(\Psi \circ \tau_m)_{\text{divisor}}$ are equivalent in the sense of Definition 2.1 where $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ is the composition of the same number m of successive blow-ups at the origin, which is the standard resolution of the singularity $(0, 0)$ of both $V(\Phi)$ and $V(\Psi)$. Then, we denote this relation by either $V(\Phi) \stackrel{\text{resol}}{\sim} V(\Psi)$ under the same standard resolution or $(\Phi \circ \tau_m)_{\text{divisor}} \stackrel{\text{resol}}{\sim} (\Psi \circ \tau_m)_{\text{divisor}}$ under the same standard resolution τ_m of both $V(\Phi)$ and $V(\Psi)$. In other words, if we write $(\Phi \circ \tau_m)_{\text{divisor}} = V^{(m)}(\Phi) + \sum_{i=1}^m e_i E_i$ and $(\Psi \circ \tau_m)_{\text{divisor}} = V^{(m)}(\Psi) + \sum_{i=1}^m \bar{e}_i E_i$

in the sense of Definition 2.1, then it is said that $V(\Phi) \stackrel{\text{resol}}{\sim} V(\Psi)$ at the origin in \mathbb{C}^2 under the same standard resolution τ_m of both $V(\Phi)$ and $V(\Psi)$ if and only if $e_i = \bar{e}_i$ under the same standard resolution τ_m .

As an application of Definition 2.4, we have the following definition for $V(\Phi)$ and $V(\Psi)$ to have a homeomorphic resolution at the origin in \mathbb{C}^2 , denoted by $V(\Phi) \stackrel{\text{resol}}{\sim} V(\Psi)$ at the origin in \mathbb{C}^2 :

$$(2.6.3) \quad \begin{aligned} & V(y^\zeta z^\eta \phi) \stackrel{\text{resol}}{\sim} V(y^{\zeta'} z^{\eta'} \psi) \text{ under the same standard resolution} \\ \iff & \text{the following conditions are satisfied:} \\ & \text{(i) } \zeta = \zeta' \text{ and } \eta = \eta'. \\ & \text{(ii) } e_j = \bar{e}_j \text{ for each } j = 1, 2, \dots, m. \end{aligned}$$

(III) Having the same divisor under two standard resolutions:

By the same way as in (II), assume that $\phi = \phi(y, z)$ and $\psi = \psi(y, z)$ are irreducible in $\mathbb{C}\{y, z\}$, satisfying the same properties and notations as in (II). Also, let $V(\Phi)$ and $V(\Psi)$ be analytic varieties at $(y, z) = (0, 0)$ defined by equalities in (2.6.1) and (2.6.2).

Let $\tau_\lambda = \pi_1 \circ \pi_2 \circ \dots \circ \pi_\lambda : M^{(\lambda)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ of successive blow-ups π_i at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singular point of an analytic variety $V(\Phi)$ where Φ is defined as above. Let $\tau_\lambda^{-1}(0, 0) = \bigcup_{i=1}^\lambda E_i$ where each E_i is called an exceptional curve of the first kind.

By Definition 2.1, let $(\Phi \circ \tau_\lambda)_{\text{divisor}}$ be the divisor of $\Phi \circ \tau_\lambda$ defined by

$$(2.6.4) \quad (\Phi \circ \tau_\lambda)_{\text{divisor}} = V^{(\lambda)}(\Phi) + \sum_{i=1}^\lambda e_i E_i,$$

where each e_i is the multiplicity of $\Phi \circ \tau_\lambda$ along E_i for $1 \leq i \leq \lambda$ and $V^{(\lambda)}(\Phi)$ is the proper transform of $V(\Phi)$ under τ_λ .

In more detail, let $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ be the i -th blow-up of τ_λ at a quasisingular point of $V^{(i-1)}(\Phi)$ for $1 \leq i < \lambda$ in the sense of (I) of Definition 2.6 where $M^{(0)} = \mathbb{C}^2$ and $V^{(0)}(\Phi) = V(\Phi)$. Then, it is clear by Remark 2.6.0, later that $V^{(i)}(\Phi)$ has one and only one quasisingular point along E_i for each $i = 1, 2, \dots, \lambda - 1$. So, it is well-defined that E_i is called the i -th exceptional curve from $\tau_\lambda^{-1}(0, 0) = \bigcup_{i=1}^\lambda E_i$. Therefore, it is clear from $(\Phi \circ \tau_\lambda)_{\text{divisor}}$ of (2.6.4) that $e_{i+1} > e_i$ for $1 \leq i \leq \lambda - 1$.

Using the same method as before, let $\mu_\sigma = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \dots \circ \bar{\pi}_\sigma : \bar{M}^{(\sigma)} \rightarrow \mathbb{C}^2$ be the composition of a finite number σ of successive blow-ups at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singular point of an analytic variety $V(\Psi)$ where Ψ is defined as above. Let $\mu_\sigma^{-1}(0, 0) = \bigcup_{j=1}^\sigma \bar{E}_j$ for $V(\Psi)$ where each \bar{E}_j is called an exceptional curve of the first kind.

By Definition 2.1, let $(\Psi \circ \mu_\sigma)_{\text{divisor}}$ be the divisor of $\Psi \circ \mu_\sigma$ defined by

$$(2.6.5) \quad (\Psi \circ \mu_\sigma)_{\text{divisor}} = V^{(\sigma)}(\Psi) + \sum_{j=1}^\sigma \bar{e}_j \bar{E}_j,$$

where each \bar{e}_j is the multiplicity of $\Psi \circ \mu_\sigma$ along \bar{E}_j for $1 \leq j \leq \sigma$ and $V^{(\sigma)}(\Psi)$ is the proper transform of $V(\Psi)$ under μ_σ .

As an application of Definition 2.4, we have the following definition for $V(\Phi)$ and $V(\Psi)$ to have the same divisor under two standard resolutions, denoted by $V(\Phi) \stackrel{\text{divisor}}{\sim} V(\Psi)$ under two standard resolutions:

$$(2.6.6) \quad \begin{aligned} & V(y^\zeta z^\eta \phi) \stackrel{\text{divisor}}{\sim} V(y^{\zeta'} z^{\eta'} \psi) \text{ under two standard resolutions} \\ \iff & \text{the following conditions are satisfied:} \\ & \text{(i) } \zeta = \zeta', \eta = \eta' \text{ and } \lambda = \sigma. \\ & \text{(ii) We may assume that } E_j \text{ and } \bar{E}_j \text{ are the same} \\ & \quad \text{and so } e_j = \bar{e}_j \text{ for each } j = 1, 2, \dots, \lambda. \end{aligned}$$

(IV) Equivalence of multiplicity sequences of irreducible plane curve singularities:

As in (II), let $\Phi(y, z) = \phi(y, z) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin, where a_0, a_n are units in $\mathbb{C}\{y, z\}$, and also each a_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers, and $2 \leq n < \alpha_n$. Let $\tau_\lambda = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_\lambda : M^{(\lambda)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ of successive blow-ups π_i at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singular point of an analytic variety $V(\Phi)$ where Φ is defined as above. Let $\tau_\lambda^{-1}(0, 0) = \cup_{i=1}^\lambda E_i$ where each E_i is called an exceptional curve of the first kind.

Now, we may use the same notations and properties as in (II). Since $\Phi(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin, then $V^{(i)}(\Phi)$ with has one and only one quasisingular point along E_i for each $i = 1, 2, \dots, \lambda - 1$.

In more detail, let $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ be the i -th blow-up of τ_λ at a quasisingular point of $V^{(i-1)}(\Phi)$ for $1 \leq i < \lambda$ in the sense of (I) of Definition 2.6 where $M^{(0)} = \mathbb{C}^2$ and $V^{(0)}(\Phi) = V(\Phi)$. Then, it may be assumed that $V^{(i)}(\Phi)$ has the multiplicity ν_{i+1} at one and only one quasisingular point along E_i for each $i=1, 2, \dots, \lambda - 1$. By definition, $\{\nu_i : i = 1, 2, \dots, \lambda\}$ is called a multiplicity sequence of $V(\Phi)$, denoted by $\text{Multiseq}(V(\Phi))$, where $\nu_1 = n$ is the multiplicity of $V(\Phi)$ at $(y, z) = (0, 0)$.

Let $\psi(y, z) = b_0 z^\ell + b_1 y^{\beta_1} z^{\ell-1} + \cdots + b_\ell y^{\beta_\ell}$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin, where b_0, b_ℓ are units in $\mathbb{C}\{y, z\}$, and also each b_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the β_i are positive integers, and $2 \leq \ell < \beta_\ell$. Using the same method as before, let $\mu_\sigma = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \cdots \circ \bar{\pi}_\sigma : \bar{M}^{(\sigma)} \rightarrow \mathbb{C}^2$ be the composition of a finite number σ of successive blow-ups at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singular point of an analytic variety $V(\Psi)$ where Ψ is defined as above. Let $\mu_\sigma^{-1}(0, 0) = \cup_{j=1}^\sigma \bar{E}_j$ for $V(\Psi)$ where each \bar{E}_j is called an exceptional curve of the first kind.

In more detail, let $\bar{\pi}_i : \bar{M}^{(i)} \rightarrow \bar{M}^{(i-1)}$ be the i -th blow-up of μ_σ at a quasisingular point of $V^{(i-1)}(\Psi)$ for $1 \leq i < \sigma$ in the sense of (I) of Definition 2.6 where $\bar{M}^{(0)} = \mathbb{C}^2$ and $V^{(0)}(\Psi) = V(\Psi)$. Then, it may be assumed that $V^{(i)}(\Psi)$ has the multiplicity η_{i+1} at one and only one quasisingular point along \bar{E}_i for each $i = 1, 2, \dots, \sigma - 1$. By definition, $\{\eta_i : i = 1, 2, \dots, \sigma\}$ is called a multiplicity sequence of $V(\Psi)$, denoted by $\text{Multiseq}(V(\Psi))$, where $\eta_1 = \ell$ is the multiplicity of $V(\Psi)$ at $(y, z) = (0, 0)$.

If either $\text{Multiseq}(V(\Phi)) = \text{Multiseq}(V(\Psi))$ as sequence, or $\nu_i = \eta_i$ for each $i = 1, 2, \dots, \lambda = \sigma$, then it is said that either $\Phi \stackrel{\text{multiseq}}{\sim} \Psi$, or $V(\Phi) \stackrel{\text{multiseq}}{\sim} V(\Psi)$ at $0 \in \mathbb{C}^2$. Otherwise, we write $\text{Multiseq}(V(\Phi)) \neq \text{Multiseq}(V(\Psi))$ as sequence.

Remark 2.6.0. (3) As in Definition 2.6, let $V(\Phi)$ and $V(\Psi)$ have a homeomorphic resolution at the origin in \mathbb{C}^2 .

(3a) Then, $V^{(\lambda)}(\Phi)$ and $V^{(\lambda)}(\Psi)$ have the same number of components, that is, $1 + \zeta + \eta = 1 + \zeta' + \eta'$. Also, $\Phi(y, z) = 0$ and $\Psi(y, z) = 0$ have the same multiplicity $e_1 = \zeta + \eta + n = \zeta' + \eta' + \ell = \bar{e}_1$ at $(y, z) = (0, 0)$ by construction. So, $n = \ell$.

(3b) Then, $V^{(\lambda)}(\Phi) \cap E_1$ and $V^{(\lambda)}(\Psi) \cap \bar{E}_1$ have the same number of elements as set, that is, $\zeta = \zeta'$ because $1 \leq n < \alpha_n$ and $1 \leq \ell < \beta_\ell$. So, $\eta = \eta'$ by (3a).

(3c) Whenever there are two integers n and α_n with $1 \leq n < \alpha_n$, there is a positive integer s such that $sn < \alpha_n \leq (s+1)n$. Also, whenever there are two integers ℓ and β_ℓ with $1 \leq \ell < \beta_\ell$, there is a positive integer \bar{s} such that $\bar{s}\ell < \beta_\ell \leq (\bar{s}+1)\ell$. Note that $e_{s+1} = \zeta + (s+1)\eta + (s+1)\alpha_n$ and $e_{\bar{s}+1} = \zeta' + (\bar{s}+1)\eta' + (\bar{s}+1)\beta_\ell$ with $s = \bar{s}$ are equal, and so $\alpha_n = \beta_\ell$. Therefore, $V^{(s)}(\Phi) = V^{(s)}(\phi)$ and $V^{(s)}(\Psi) = V^{(s)}(\psi)$. Then, $V^{(s)}(\phi) = V^{(s)}(\psi)$ implies that $V^{(s)}(\Phi) = V^{(s)}(\Psi)$, and so $V^{(s)}(\Phi) \stackrel{\text{resol}}{\sim} V^{(s)}(\Psi)$ at the origin in \mathbb{C}^2 implies that $V^{(s)}(\phi) \stackrel{\text{resol}}{\sim} V^{(s)}(\psi)$ at the origin in \mathbb{C}^2 .

Thus, Definition 2.6 is well-defined by (3a), (3b) and (3c).

Remark 2.6.1. A quasisingular point of $V^{(m)}(F)$ may not be a singular point of $V^{(m)}(F)$ by the following example:

Let $V(F) = \{(y, z) : F(y, z) = z^2 + y^3 = 0\}$. Then $(y, z) = (0, 0)$ is a singular point of $V(F)$. Let $\pi : M \rightarrow \mathbb{C}^2$ be a blow-up at $(0, 0)$ such that $\pi(v, u) = (y, z) = (v, vu)$ and $\pi(v', u') = (y, z) = (v'u', v')$ where $u' = \frac{1}{u}$ and $v' = vu$.

(a1) Along $v = 0$, $(F \circ \pi)_{total} = v^2 F_1(v, u) = v^2(u^2 + v)$ where $F_1(v, u)$ is in $\mathbb{C}\{v, u\}$.

(a2) Along $v' = 0$, $(F \circ \pi)_{total} = v'^2 F_2(v', u') = v'^2(1 + u'^3 v')$ where $F_2(v', u')$ is a unit in $\mathbb{C}\{v', u'\}$.

Then, $V^{(1)}(F)$ has no singularity everywhere, but $(v, u) = (0, 0)$ is only one quasisingular point of $V^{(1)}(F)$ because additional blow-ups at $(v, u) = (0, 0)$ are still necessary to get the standard resolutions of the singular point of $V(F)$, noting that $V^{(1)}(F)$ and $\{v = 0\}$ meet tangentially at $(v, u) = (0, 0)$.

Lemma 2.7. Assume that $V(F)$ satisfies the same properties as in Definition 2.6. If P is a quasisingular point of $V^{(m)}(F)$ under τ_m , then P satisfies at least one of the following three properties (i), (ii) and (iii):

- (i) $V^{(m)}(F)$ itself has a singular point at P as a reduced variety.
- (ii) There exists an exceptional curve of the first kind of $\tau_m^{-1}(0, 0)$ and at least one irreducible component of $V^{(m)}(F)$ which meet tangentially at P .
- (iii) There are two exceptional curves of the first kind of $\tau_m^{-1}(0, 0)$ and at least one irreducible component of $V^{(m)}(F)$ which meet at P .

Proof of Lemma 2.7. It is trivial.

Definition 2.8. Let $V(F)$ be an analytic variety at the origin. Assume that $V(F)$ satisfies the same properties and notations as in Definition 2.6. It is said that $F \in$ the type $[j]$ or belongs to the type $[j]$ under τ_m if F satisfies the following four properties: Sometimes, we say that $V(F) \in$ the type $[j]$ or belongs to the type $[j]$ under τ_m .

- (1) For any point of the set $\cup_{i=1}^{m-1} E_i$, no blow-ups are needed to get the standard resolution of the singular point of $V(F)$. That is, there are no quasisingular points on $\tau_m^{-1}(0, 0) = \cup_{i=1}^{m-1} E_i$.
- (2) At some point of $E_m - \cup_{i=1}^{m-1} E_i$, additional blow-ups may be needed for the standard resolution of the singular point of $V(F)$.
- (3) On the set $\cup_{i=1}^m E_i$, there are j exceptional curves of the first kind with $j \leq m$, each of which has three distinct intersection points with other exceptional curves and the proper transform $V^{(m)}(F)$.
- (4) Each of the $(m - j)$ remaining exceptional curves of the first kind from $\cup_{i=1}^m E_i$, rather than the above j exceptional curves as just mentioned in (3), has at most two distinct intersection points with other exceptional curves of the first kind and the proper transform $V^{(m)}(F)$.

Remark 2.8.1. Assume by Definition 2.8 that $F \in$ the type $[j]$ or belongs to the type $[j]$ under τ_m . Whenever $\cup_{i=1}^{m-1} E_i$ and $V^{(m)}(F)$ meet, then they meet with normal crossings, but no three distinct components of $\cup_{i=1}^{m-1} E_i$ and $V^{(m)}(F)$ meet.

Definition 2.9. Let $V(F)$ be an analytic variety at $(y, z) = (0, 0)$ where $F = F(y, z)$ is in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. Assume that $V(F)$ satisfies the same properties and notations as in Definition 2.1 and Definition 2.6.

(1) Then, each of $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ is called one coordinate patch of the given local coordinates for a blow-up π_1 , respectively. Also, $E_1 = E_1^+ \cup E_1^-$ is called an exceptional curve of the first kind where $E_1^+ = \{(v_1, u_1) : v_1 = 0\}$ and $E_1^- = \{(v'_1, u'_1) : v'_1 = 0\}$.

(2) For notation, let $qs(V^{(1)}(F))$ be the set of all quasisingular points of $V^{(1)}(F)$ under $\tau_1 = \pi_1$ in the sense of Definition 2.6. Observe that $qs(V^{(1)}(F))$ may be either empty or nonempty. Let $P \in qs(V^{(1)}(F))$ be chosen arbitrary. For the given two local coordinates for $M^{(1)}$ as above, it is said that the coordinate of P is written by one and only one of three different types :

(2a) $\{(v_1, u_1) = (0, 0)\} = E_1^+ - E_1^-$.

(2b) $\{(v'_1, u'_1) = (0, 0)\} = E_1^- - E_1^+$.

(2c) either $(v_1, u_1) = (0, \xi) \in E_1^+ \cap E_1^-$ or $(v'_1, u'_1) = (0, \frac{1}{\xi}) \in E_1^+ \cap E_1^-$ for a nonzero number ξ .

- (3) Now, let P be a quasisingular point of $V^{(1)}(F)$ in the sense of Definition 2.6.
- (3a) It is said that P satisfies the first type of coordinate under $\tau_1 = \pi_1$ if P satisfies either (2a) or (2b) in (2). Equivalently, P satisfies the first type of coordinate under π_1 if and only if the coordinate of P coincides with the origin in one and only one of the given two coordinate patches for M .
- (3b) It is said that P satisfies the second type of coordinate under π_1 if P satisfies (2c).

Lemma 2.10. *Assume that $V(F)$ satisfies the same properties and notations as in Definition 2.9. Suppose that $qs(V^{(1)}(F))$ is nonempty.*

- (1) *Then, $qs(V^{(1)}(F))$ satisfies one and only one of the following four properties (1a), (1b), (1c) and (1d):*
 - (1a) $qs(V^{(1)}(F)) \subset E_1^+$ and $qs(V^{(1)}(F)) \not\subset E_1^-$.
 - (1b) $qs(V^{(1)}(F)) \subset E_1^-$ and $qs(V^{(1)}(F)) \not\subset E_1^+$.
 - (1c) $qs(V^{(1)}(F)) \subset E_1^+ \cap E_1^-$.
 - (1d) $qs(V^{(1)}(F)) \subset E_1$, but $qs(V^{(1)}(F)) \not\subset E_1^+$ and $qs(V^{(1)}(F)) \not\subset E_1^-$.
- (2) *By (1), there are four subcases:*
 - (2a) *Suppose that $qs(V^{(1)}(F)) \subset E_1^+$ and $qs(V^{(1)}(F)) \not\subset E_1^-$. Then, there is one and only one point in $qs(V^{(1)}(F))$ satisfying the first type of coordinate.*
 - (2b) *Suppose that $qs(V^{(1)}(F)) \subset E_1^-$ and $qs(V^{(1)}(F)) \not\subset E_1^+$. Then, there is one and only one point in $qs(V^{(1)}(F))$ satisfying the first type of coordinate.*
 - (2c) *Suppose that $qs(V^{(1)}(F)) \subset E_1^+ \cap E_1^-$. Then, any $P \in qs(V^{(1)}(F))$ satisfies the second type of coordinate.*
 - (2d) *Suppose that $qs(V^{(1)}(F)) \subset E_1$, but $qs(V^{(1)}(F)) \not\subset E_1^+$ and $qs(V^{(1)}(F)) \not\subset E_1^-$. Then, there are two distinct points in $qs(V^{(1)}(F))$, each of which satisfies the first type of coordinate, respectively.*

The proof of lemma is clear.

Remark 2.10.1. Let $V(G)$ be an analytic variety at $(y, z) = (0, 0)$ where $G = G(y, z)$ is in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. As we have seen in Definition 2.1 and Definition 2.4, let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the composition of a finite number m of successive blow-ups at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singular point $(0, 0)$ of $V(G)$.

(1) First, let $G = G(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. Note that $qs(V^{(t)}(G))$ is a one-point set for each $t = 1, 2, \dots, (m-1)$ and $qs(V^{(m)}(G))$ is empty, because τ_m is the standard resolution of the singular point of $V(G)$ by assumption. In order to either study $V^{(t)}(G)$ or find $qs(V^{(t)}(G))$ under τ_t , it is possible to choose just one coordinate patch of the local coordinates for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$, where $1 \leq t \leq m$ and $M^{(0)} = \mathbb{C}^2$, by Definition 2.9 and Lemma 2.10, as it has been well-known in the standard resolution of irreducible plane curve singularities.

In order to study $\tau_s : M^{(s)} \rightarrow \mathbb{C}^2$ with $1 \leq s \leq m$, it is clear that τ_s can be viewed as a composition of a finite number s of successive analytic mappings from one coordinate patch of $M^{(t)}$ to another coordinate patch of $M^{(t-1)}$ in the sense of Definition 2.9, where $1 \leq t \leq s$ and $M^{(0)} = \mathbb{C}^2$.

(2) Next, without assuming that $G(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$, for $1 \leq t \leq m$ let $\tau_t = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ be defined by the composition of a finite number t of successive blow-ups at the origin in \mathbb{C}^2 , which is in process of the standard resolution of the singular point $(0, 0)$ of $V(G)$.

Assuming that $qs(V^{(t)}(G))$ is a one-point set for each $t = 1, 2, \dots, s-1$ if exists where s is a positive integer with $2 \leq s \leq m$, then by the similar way as in (1) it is easy to prove that $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ can be viewed as a composition of a finite number t of successive analytic mappings from one coordinate patch of $M^{(j)}$ to another coordinate patch of $M^{(j-1)}$ in the sense of Definition 2.9, where $1 \leq j \leq t$ and $M^{(0)} = \mathbb{C}^2$.

Now, whether or not $qs(V^{(s)}(G))$ is a one-point set, by using the definition of the quasisingularity and also following the definitions and lemmas in this section and the next section, we are going to generalize the possibility of the terminology about the choices of just one coordinate patch of the local coordinates for each blow-up π_t , in order to study the local defining equation for the total transform of $V(G)$ under τ_s .

Definition 2.11. Let $V(F)$ and $V(G)$ be analytic varieties at $(y, z) = (0, 0)$ where $F = F(y, z)$ and $G = G(y, z)$ are in $\mathbb{C}\{y, z\}$. Let $V(F)$ and $V(G)$ have an isolated singular point at the origin as reduced varieties. Assuming that $V(F)$ and $V(G)$ satisfy the same kind of properties and notations as in Definition 2.9 and Lemma 2.10, let $\pi_1 : M^{(1)} \rightarrow \mathbb{C}^2$ be the first blow-up of \mathbb{C}^2 at $(y, z) = (0, 0)$ in process of the standard resolution of the isolated singular point $(0, 0)$ of both $V(F)$ and $V(G)$. Let $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ be two coordinate patches with $u'_1 = 1/u_1$ and $v'_1 = v_1 u_1$ for the blow-up π_1 as before. Let $qs(V^{(1)}(F))$ and $qs(V^{(1)}(G))$ be the set of quasisingular points of $V^{(1)}(F)$ and $V^{(1)}(G)$, respectively. Note by Definition 2.9 that each of $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ is called one coordinate patch of the given local coordinates for a blow-up π_1 , respectively. As we have seen in Definition 2.9, each of $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ is called one coordinate patch of the given local coordinates for a blow-up π_1 , respectively. Also, $E_1 = E_1^+ \cup E_1^-$ is called an exceptional curve of the first kind where $E_1^+ = \{(v_1, u_1) : v_1 = 0\}$ and $E_1^- = \{(v'_1, u'_1) : v'_1 = 0\}$.

Then, we have the following new terminologies [I] and [II]:

[I] If $qs(V^{(1)}(F)) \neq \emptyset$ satisfies either $qs(V^{(1)}(F)) \subset E_1^+$ or $qs(V^{(1)}(F)) \subset E_1^-$, then we say that just one coordinate patch covering is needed for the blow-up π_1 , in order to either study $V^{(1)}(F)$ or find $qs(V^{(1)}(F))$, including the case that $qs(V^{(1)}(F)) = \emptyset$, whether or not $qs(V^{(1)}(F))$ is nonempty.

In more detail, we say that if $qs(V^{(1)}(F))$ satisfies one and only one of three properties (1a), (1b) and (1c) except for (1d) in Lemma 2.10 then it has one coordinate patch covering. Equivalently, depending on (1a), (1b) and (1c) in Lemma 2.10, then [I] can be divided into the following three subcases [Ia], [Ib] and [Ic]:

- [Ia] If $qs(V^{(1)}(F)) \neq \emptyset$ satisfies the property (1a) in Lemma 2.10, then it is said by definition that just one coordinate patch covering U_1 of E_1^+ is needed for the study of $V^{(1)}(F)$ under π_1 , because the other coordinate patch covering U_2 of E_1^- is not needed to find $qs(V^{(1)}(F))$, as a consequence.
- [Ib] If $qs(V^{(1)}(F)) \neq \emptyset$ satisfies (1b) in Lemma 2.10, then it is said by definition that just one coordinate patch covering U_2 of E_1^- is needed for the study of $V^{(1)}(F)$ under π_1 , because the other coordinate patch covering of U_1 of E_1^+ is not needed to find $qs(V^{(1)}(F))$, as a consequence.
- [Ic] If $qs(V^{(1)}(F)) \neq \emptyset$ satisfies (1c) in Lemma 2.10, then it is said by definition that either one coordinate patch covering U_1 of E_1^+ or another coordinate patch covering U_2 of E_1^- may be chosen arbitrary for the study of $V^{(1)}(F)$ under π_1 , because any one of two coordinate patches is needed enough to find $qs(V^{(1)}(F))$, as a consequence.

For example, if $qs(V^{(1)}(F))$ is a one-point set, then it is clear that we can use just one coordinate patch covering of the given local coordinates for the blow-up π_1 , in order to either study $V^{(1)}(F)$ or find $qs(V^{(1)}(F))$.

[Id] If $qs(V^{(1)}(F))$ satisfies (1d) in Lemma 2.10, then it is said that two coordinate patches U_1 and U_2 are all needed without using any nonsingular change of coordinates, in order to either study $V^{(1)}(F)$ or find $qs(V^{(1)}(F))$.

In other words, we say that if $qs(V^{(1)}(F))$ satisfies (1d) in Lemma 2.10 then it must have two coordinate patch coverings, but it does not have one coordinate patch covering under the given blow-up π_1 , in order to study $V^{(1)}(F)$.

[II] It is said that we may use a common one coordinates patch of the given local coordinates for the blow-up π_1 , in order to study both $V^{(1)}(F)$ and $V^{(1)}(G)$ simultaneously, if one of the following properties holds :

- (a) $qs(V^{(1)}(F)) \subset E_1^+$ and $qs(V^{(1)}(G)) \subset E_1^+$.
- (b) $qs(V^{(1)}(F)) \subset E_1^-$ and $qs(V^{(1)}(G)) \subset E_1^-$.

Equivalently, it is said that $qs(V^{(1)}(F))$ and $qs(V^{(1)}(G))$ may have the same one coordinate patch covering. Observe that either $qs(V^{(1)}(F))$ or $qs(V^{(1)}(G))$ may be empty.

Lemma 2.12. Assumptions Let $V(F)$ be an analytic variety at $(y, z) = (0, 0)$ where $F = F(y, z)$ is in $\mathbb{C}\{y, z\}$. Assume that $V(F)$ has an isolated singular point at the origin as a reduced variety. Let $\tau_\ell : M^{(\ell)} \rightarrow \mathbb{C}^2$ be the composition of the first finite number ℓ of successive blow-ups at the origin which is in process of the standard resolution of the singular point of $V(F)$. For each $t = 1, 2, \dots, \ell$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\{\pi_i : M^{(i)} \rightarrow M^{(i-1)} \text{ is a blow-up of } M^{(i-1)} \text{ at some point of } M^{(i-1)} \text{ for } 1 \leq i \leq \ell \text{ with } M^{(0)} = \mathbb{C}^2\}$.

Assume that $V(F)$ satisfies the following properties:

- (i) $F = F(y, z)$ may not be irreducible in $\mathbb{C}\{y, z\}$.
- (ii) $qs(V^{(t)}(F))$ is a one-point set for each $t = 1, 2, \dots, \ell - 1$, which is denoted by P_t .
- (iii) $qs(V^{(\ell)}(F))$ may be empty. If not empty, then it has a one coordinate patch covering in the sense of Definition 2.11.

Conclusions Each τ_t with $1 \leq t \leq \ell$ can be also viewed as the composition of the same number of analytic mappings from a two-dimensional complex analytic polydisc centered at the origin in \mathbb{C}^2 to itself, in order to either study $V^{(t)}(F)$ or find $(F \circ \tau_t)_{proper}$.

The Proof of Lemma 2.12 is clear.

Definition 2.13. Let $V(F)$ and $V(G)$ be analytic varieties at $(y, z) = (0, 0)$ where $F = F(y, z)$ and $G = G(y, z)$ are in $\mathbb{C}\{y, z\}$. Let $V(F)$ and $V(G)$ have an isolated singular point at the origin as reduced varieties. Assuming that $V(F)$ and $V(G)$ satisfy the same kind of properties and notations as in Definition 2.9 and Definition 2.11, let $\pi_1 : M^{(1)} \rightarrow \mathbb{C}^2$ be the first blow-up of \mathbb{C}^2 at $(y, z) = (0, 0)$ in process of the standard resolution of the isolated singular point $(0, 0)$ of both $V(F)$ and $V(G)$. Let $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ be two coordinate patches with $u'_1 = 1/u_1$ and $v'_1 = v_1 u_1$ for the blow-up π_1 as before. Let $qs(V^{(1)}(F))$ and $qs(V^{(1)}(G))$ be the set of quasisingular points of $V^{(1)}(F)$ and $V^{(1)}(G)$, respectively. Note by Definition 2.9 that each of $U_1 = (v_1, u_1)$ and $U_2 = (v'_1, u'_1)$ is called one coordinate patch of the given local coordinates for a blow-up π_1 , respectively.

It is said that we may use a common one coordinates patch of the given local coordinates for the blow-up π_1 , in order to study both $V^{(1)}(F)$ and $V^{(1)}(G)$ simultaneously, if one of the following properties holds : Recall that $E_1^+ = \{(v_1, u_1) : v_1 = 0\}$ and $E_1^- = \{(v'_1, u'_1) : v'_1 = 0\}$.

- (a) $qs(V^{(1)}(F)) \subset E_1^+$ and $qs(V^{(1)}(G)) \subset E_1^+$.
- (b) $qs(V^{(1)}(F)) \subset E_1^-$ and $qs(V^{(1)}(G)) \subset E_1^-$.

Equivalently, it is said that $qs(V^{(1)}(F))$ and $qs(V^{(1)}(G))$ may have the same one coordinate patch covering. Observe that either $qs(V^{(1)}(F))$ or $qs(V^{(1)}(G))$ may be empty.

Remark 2.13.1.

(i) In particular, if $qs(V^{(1)}(F)) = qs(V^{(1)}(G))$ is a one-point set, then we may use a common one coordinate patch of the local coordinates under the blow-up π_1 , in order to study both $V^{(1)}(F)$ and $V^{(1)}(G)$ at the same time.

(ii) Let $V(F) = \{(y, z) : F(y, z) = 0\}$, $V(G) = \{(y, z) : G(y, z) = 0\}$ and $V(H) = \{(y, z) : H(y, z) = 0\}$ be analytic varieties defined respectively, as follows:

$$(2.13.1) \quad \begin{aligned} F(y, z) &= z^2 + y^3, \\ G(y, z) &= (z^2 + y^2)^2 + y^5, \\ H(y, z) &= z^3 + y^2. \end{aligned}$$

Let $\pi_1 : M^{(1)} \rightarrow \mathbb{C}^2$ be the first blow-up of \mathbb{C}^2 at $(y, z) = (0, 0)$ which is an isolated singular point of $V(F)$, $V(G)$ and $V(H)$. Let (v, u) and (v', u') be the given local coordinates for $M^{(1)}$ where $\pi_1(v, u) = (y, z) = (v, vu)$ and $\pi_1(v', u') = (y, z) = (v'u', v')$ with $u' = 1/u$ and $v' = vu$.

Note that $E_1 = E_1^+ \cup E_1^-$ where $E_1^+ = \{(v, u) : v = 0\}$ and $E_1^- = \{(v', u') : v' = 0\}$. For notation of two coordinate patches, we write $U_1 = (v, u)$ and $U_2 = (v', u')$.

Observe that $qs(V^{(1)}(F)) \neq qs(V^{(1)}(G))$ and $qs(V^{(1)}(H)) \neq qs(V^{(1)}(G))$, because $qs(V^{(1)}(F)) \subset E_1^+ - E_1^-$, $qs(V^{(1)}(G)) \subset E_1^+ \cap E_1^-$ and $qs(V^{(1)}(H)) \subset E_1^- - E_1^+$.

For the study of $V^{(1)}(G)$, it is enough to choose either one of two coordinate patches $U_1 = (v, u)$ and $U_2 = (v', u')$ under the blow-up π_1 . So, in order to study both $V^{(1)}(F)$ and $V^{(1)}(G)$ simultaneously, it is possible to use one and only one coordinate $U_1 = (v, u)$ as a common one coordinate patch of the local coordinates under the blow-up π_1 . Also, in order to study both $V^{(1)}(H)$ and $V^{(1)}(G)$ simultaneously, it is possible to use one and only one coordinate $U_2 = (v', u')$ as a common one coordinate patch of the local coordinates under the blow-up π_1 . But, in order to find $V^{(1)}(F)$, $V^{(1)}(G)$ and $V^{(1)}(H)$ at the same time, we cannot choose a common one coordinate patch of the given two local coordinates under the blow-up π_1 without using a nonsingular change of coordinates.

Define a local nonsingular mapping ϕ from $(y, z) = (0, 0)$ to $(y', z') = (0, 0)$ as follows:

$$(2.13.2) \quad \begin{aligned} \phi(y, z) &= (y', z') \quad \text{with} \quad \phi(0, 0) = (0, 0), \\ y' &= y \quad \text{and} \quad z' = z + y. \end{aligned}$$

From (2.13.1) and (2.13.2), let $F^* = F \circ \phi$, $G^* = G \circ \phi$ and $H^* = H \circ \phi$, respectively. Then, $V(F^*)$, $V(G^*)$ and $V(H^*)$ have still the same isolated singular point at $(y, z) = (0, 0)$, as follows:

$$(2.13.3) \quad \begin{aligned} F^*(y, z) &= (z + y)^2 + y^3, \\ G^*(y, z) &= \{(z + y)^2 + y^2\}^2 + y^5, \\ H^*(y, z) &= (z + y)^3 + y^2. \end{aligned}$$

Let $\pi_1 : M^{(1)} \rightarrow \mathbb{C}^2$ be the first blow-up of \mathbb{C}^2 at $(y, z) = (0, 0)$ with the same local coordinates (v, u) and (v', u') for $M^{(1)}$ just as above. Then, F^* , G^* and H^* have a common one coordinate patch of the given two local coordinates under the blow-up π_1 , in order to find $V^{(1)}(F^*)$, $V^{(1)}(G^*)$ and $V^{(1)}(H^*)$ at the same time.

Lemma 2.14. Assumptions *As in Definition 2.1, let $V(F)$ and $V(G)$ be analytic varieties at $(y, z) = (0, 0)$ where $F = F(y, z)$ and $G = G(y, z)$ are in $\mathbb{C}\{y, z\}$. Assume that $V(F)$ and $V(G)$ have an isolated singular point at the origin in a two-dimensional manifold \mathbb{C}^2 , as reduced varieties. Suppose that $V(F)$ satisfies the same assumptions and notations as in Lemma 2.12. Let $\omega_q : L^{(q)} \rightarrow \mathbb{C}^2$ be an arbitrary composition of a finite number q of successive blow-ups at the origin which is in process of the standard resolution of the singular point of $V(G)$. For each $r = 1, 2, \dots, q$, write $\omega_q = \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_q : L^{(q)} \rightarrow \mathbb{C}^2$ with $\pi_1 = \sigma_1$, where $\{\sigma_r : L^{(r)} \rightarrow L^{(r-1)} \text{ is a blow-up of } L^{(r-1)} \text{ at some point of } L^{(r-1)} \text{ for } 1 \leq r \leq q\}$ with $L^{(0)} = \mathbb{C}^2$ and $L^{(1)} = M^{(1)}$.*

As we have seen in the assumption of Lemma 2.12, assume that $V(G)$ satisfies the following properties:

- (i) $G = G(y, z)$ may not be irreducible in $\mathbb{C}\{y, z\}$.
- (ii) $qs(V^{(r)}(G))$ is a one-point set for each $r = 1, 2, \dots, q - 1$, which is denoted by Q_r .
- (iii) $qs(V^{(q)}(G))$ may be empty. If not empty, then it has one coordinate patch covering in the sense of Definition 2.11.

Define $s = \min\{\ell, q\}$, and then we assume the following property:

$$(2.14.1) \quad \begin{aligned} &\text{There exists a positive integer } p \text{ with } 1 \leq p \leq s \text{ such that} \\ &P_i = Q_i \quad \text{for each } i = 1, 2, \dots, p - 1. \end{aligned}$$

Conclusions *We have the following:*

- (a) *For each $i = 1, 2, \dots, s$, it is possible to choose the same local coordinates for both π_i and σ_i simultaneously, without any nonsingular change of coordinates.*
- (b) *We may use a common one coordinate patch of the given local coordinates for each blow-up $\pi_p = \sigma_p$, in order to study both $V^{(p)}(F)$ and $V^{(p)}(G)$ simultaneously for each $p = 1, 2, \dots, s$. But, a common one coordinate patch may not be unique.*
- (c) *In more detail, for each $p = 1, 2, \dots, s$, $\tau_p = \omega_p$ can be considered as a local analytic mapping from a two-dimensional complex analytic manifold to an analytic polydisc centered at the origin in \mathbb{C}^2 , in order either to study both $V^{(s)}(F)$ and $V^{(s)}(G)$ simultaneously, or to find both $(F \circ \tau_s)_{\text{proper}}$ and $(G \circ \omega_s)_{\text{proper}}$ at the same time.*

Proof of Lemma 2.14. If $qs(V^{(1)}(F)) = qs(V^{(1)}(G))$ is a one-point set, then we may use a common one coordinate patch of the local coordinates under the blow-up π_1 for the study of both $V^{(1)}(F)$ and $V^{(1)}(G)$ at the same time. Then, the proof can be just finished by induction on the positive integer s where $s = \min\{\ell, q\}$, by using Lemma 2.12 and Definition 2.13 only.

§3. The representation for the local defining equations of irreducible plane curve singularities which have either the same multiplicity sequence or the homeomorphic resolution under the standard resolution as the curve singularity $(\{z^n + y^k = 0\})$ does and its generalizations

§3.0. Introduction

In §3.1, by Theorem 3.2 and Corollary 3.3, we will find the method how to construct the local defining equation for irreducible plane curve singularities, which have the homeomorphic resolution as an irreducible plane curve singularity defined by $z^n + y^k = 0$ does in the sense of Definition 2.4 and its generalizations.

In §3.2, by Theorem 3.6 and Theorem 3.7, we will find the method how to compute the local defining equation for the total transform of irreducible plane curve singularities, which have the homeomorphic resolution as an irreducible plane curve singularity defined by $z^n + y^k = 0$ does, after a finite number of successive blow-ups, which is just needed to get the standard resolution of the above singularity and its generalizations.

§3.1. How to find the necessary condition for any local defining equation defining plane curve singularities to be irreducible in $\mathbb{C}\{y, z\}$ and its applications

Lemma 3.1(Hensel's Lemma). *Let $f(y, z) = a_0z^n + a_1y^{\alpha_1}z^{n-1} + \cdots + a_ny^{\alpha_n}$ be irreducible in $\mathbb{C}\{y, z\}$, where $n \geq 2$ and $\alpha_n \geq 2$, and each a_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers. Let m be the multiplicity of f at 0. Then $m = n$ or α_n . If $m = n = \alpha_i + n - i$ for some i , then $n = \alpha_i + n - i$ for all $i = 1, \dots, n$, and so f can be written in a power series as follows: $f = f_n(y, z) + \text{terms of degree} > n$, where $f_n = f_n(y, z)$ is a homogeneous polynomial of degree n with $f_n = (ay + bz)^n$ for some $a, b \in \mathbb{C}$.*

Theorem 3.2(The generalized Hensel's lemma).

Assumptions *Let $f(y, z)$ be in $\mathbb{C}\{y, z\}$ with isolated singularity at $(0, 0)$, defined by the following:*

$$(3.2.1) \quad f(y, z) = a_0z^n + a_1y^{\alpha_1}z^{n-1} + \cdots + a_ny^{\alpha_n} \quad \text{with } n \geq 2 \text{ and } k = \alpha_n \geq 2,$$

where each $a_i = a_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers for all $i = 1, 2, \dots, n$, and also $a_0 = a_0(y, z)$ and $a_n = a_n(y, z)$ are units in $\mathbb{C}\{y, z\}$.

If necessary, we may assume without need of the proof that $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n$.

Conclusions

(1) *The necessary condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ is as follows:*

$$(3.2.2) \quad \frac{\alpha_i}{i} \geq \frac{k}{n} \quad \text{for all } i = 1, 2, \dots, n-1.$$

(2) *Equivalently, the inequality in (3.2.2) holds if and only if f is represented as follows:*

$$(3.2.3) \quad f(y, z) = a_{0,0}z^n + a_{n,0}y^k + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } n\alpha + k\beta \geq nk,$$

where the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β with $n\alpha + k\beta \geq nk$, satisfying (i) and (ii):

- (i) a_0 and a_n are units in $\mathbb{C}\{y, z\}$ with $a_{0,0} = a_0(0, 0)$ and $a_{n,0} = a_n(0, 0)$.
- (ii) if $c_{0, \beta} \neq 0$ for some integers $\beta > 0$, then $\beta > n$, and if $c_{\alpha, 0} \neq 0$ for some integers $\alpha > 0$, then $\alpha > k$.

(3) *In particular, as far as the inequality in (3.2.2) is concerned, if $k = np$ for some positive integer p , $\frac{k}{n} = \frac{\alpha_i}{i} = p$ for $i = 1, 2, \dots, n-1$ where all the a_i are units in $\mathbb{C}\{y, z\}$.*

Proof of Theorem 3.2. Assume that $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $(0, 0)$. The proof will be by induction on the multiplicity of f . Then, it suffices to prove that either the inequality in (3.2.2) or the representation in (3.2.3) is true. For the induction proof, assuming that $2 \leq n \leq k$, it suffices to consider two facts, respectively:

Fact(1) Let $n = k \geq 2$. If $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$, by Lemma 3.1 and by (3.2.1), $\alpha_i + (n - i) = n$ for all $i = 1, 2, \dots, n - 1$, and then $\alpha_i = i$. So, there is nothing to prove for the inequality in (3.2.2).

Fact(2) To prove that the inequality in (3.2.2) is true, by using Fact(1) it suffices to consider two cases by induction on the multiplicity n of the local defining equation of $f(y, z)$ at the origin.

Case(I) $n = 2 < k$ and Case(II) $2 < n < k$.

Case(I) Let $n = 2 < k$. The proof is clear.

Case(II) Let $2 < n < k$. Suppose we have shown that the theorem is true whenever the multiplicity of the local defining equation of $f(y, z)$ at the origin is less than n . Since $2 < n < k$, then there is a positive integer s such that $0 < k - sn \leq n$. For the induction proof, first of all, let τ_m be the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singularity of $V(f)$. Then, it will be shown by Sublemma 3.2.1 that $s < m$, and also if $V^{(s)}(f)$ is the proper transform under τ_s where τ_s is the composition of a finite number s of successive blow-ups at $(y, z) = (0, 0)$, then the local defining equation for $V^{(s)}(f)$ is of the multiplicity $k - sn$ where $0 < k - sn \leq n$. Then, there are two subcases (i) and (ii).

(i) If $0 < k - sn < n$, apply the induction method to the multiplicity of the local defining equation for $V^{(s)}(f)$ in (3.2.7) of Sublemma 3.2.1. After then, these will give the desired proof of Case(II) by Sublemma 3.2.1.

(ii) If $k - sn = n$, then the proof will be done by the same method as we have used in the proof of Fact(1).

Sublemma 3.2.1. Assumptions Suppose that $f \in \mathbb{C}\{y, z\}$ satisfies the same properties and notations as in the assumption of Theorem 3.2. In addition, let $2 \leq n < k$. Then, there is a unique integer s such that $sn < k \leq (s + 1)n$.

Conclusions Let τ_m be the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(f)$. For each $t = 1, 2, \dots, m$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\{\pi_i : M^{(i)} \rightarrow M^{(i-1)}\}$ is a blow-up of $M^{(i-1)}$ at some point of $M^{(i-1)}$ for $1 \leq i \leq t$ with $M^{(0)} = \mathbb{C}^2$. Let $V^{(t)}(f)$ be the proper transform under τ_t for $1 \leq t \leq m$. Let $E^{(m)} = \tau_m^{-1}(0, 0)$, and let $E^{(m)} = \cup E_i$, $1 \leq i \leq m$, be the decomposition of $E^{(m)}$ into irreducible components where each E_i is called an exceptional curve of the first kind.

As a conclusion, $1 \leq s < m$, and in order to study the proper transform $V^{(t)}(f)$ for each $t = 1, 2, \dots, s$, we can use one and only one coordinate patch of the local coordinates for each blow-up $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ of τ_t with $1 \leq i \leq t$, satisfying the following properties:

- (a) For each $t = 1, 2, \dots, s$, $qs(V^{(t)}(f))$ is a one-point set in the sense of Definition 2.6.
- (b) For each $i = 1, 2, \dots, t$, let (v_i, u_i) and (v'_i, u'_i) be the local coordinates for $M^{(i)}$ where $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ is a blow-up of $M^{(i-1)}$ at some point of $M^{(i-1)}$ for $1 \leq i \leq t$ where $u'_i = 1/u_i$ and $v'_i = v_i u_i$ and $M^{(0)} = \mathbb{C}^2$ with $(v_0, u_0) = (y, z)$. We write $E_i = \{(v_i, u_i) : v_i = 0\} \cup \{(v'_i, u'_i) : v'_i = 0\}$ for each $i = 1, 2, \dots, t$.
- (b1) Let $\pi_t(v_t, u_t) = (v_{t-1}, u_{t-1}) = (v_t, v_t u_t)$ and $\pi_t(v'_t, u'_t) = (v_{t-1}, u_{t-1}) = (v'_t u'_t, v'_t)$ for $i = 1, 2, \dots, t$ where $(v_0, u_0) = (y, z)$. Then, (v'_i, u'_i) is not needed for the study of the proper transform $V^{(t)}(f)$ under π_t by (3.2.6), and so $qs(V^{(t)}(f)) \cap E_t = \{(v_t, u_t) = (0, 0)\}$, whose proof just follows from (3.2.6) and (3.2.7), below.
- (c) For $1 \leq t \leq s$, let $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ be defined by the local coordinates in (b) where $E_t = \{v_t = 0\} \cup \{v'_t = 0\}$.

In order to study the proper transform $V^{(t)}(f)$ for each $t = 1, 2, \dots, s$, we may assume that $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ satisfies the same assumptions and notations as in (b). For each fixed $t = 1, 2, \dots, s$, τ_t can be rewritten in the form

$$(3.2.4) \quad \begin{aligned} \tau_t &= \pi_1 \circ \pi_2 \circ \dots \circ \pi_t, \\ M^{(t)} &\xrightarrow{\pi_t} M^{(t-1)} \xrightarrow{\pi_{t-1}} M^{(t-2)} \rightarrow \dots \rightarrow M^{(1)} \xrightarrow{\pi_1} M^{(0)} = \mathbb{C}^2, \\ (v_t, u_t) &\xrightarrow{\pi_t} (v_{t-1}, u_{t-1}) \xrightarrow{\pi_{t-1}} (v_{t-2}, u_{t-2}) \rightarrow \dots \rightarrow (v_1, u_1) \xrightarrow{\pi_1} (v_0, u_0), \\ (v'_t, u'_t) &\xrightarrow{\pi_t} (v_{t-1}, u_{t-1}) \xrightarrow{\pi_{t-1}} (v_{t-2}, u_{t-2}) \rightarrow \dots \rightarrow (v_1, u_1) \xrightarrow{\pi_1} (v_0, u_0), \\ &\text{where } (v_0, u_0) = (y, z). \end{aligned}$$

By (3.2.4), along E_t $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ can be represented, as a composition of analytic mappings, as follows:

$$(3.2.5) \quad \begin{aligned} \tau_t(v_t, u_t) &= (y, z) = (v_t, v_t^t u_t), \\ \tau_t(v'_t, u'_t) &= (y, z) = (v'_t u'_t, v'^t_t u'^{t-1}_t), \end{aligned}$$

where $u'_t = 1/u_t$ and $v'_t = v_t u_t$.

For each fixed $t = 1, 2, \dots, s$, along $v'_t = 0$, $(f \circ \tau_t)_{total}$ can be written in the form

$$(3.2.6) \quad \begin{aligned} (f \circ \tau_t)_{total} &= f(v'_t u'_t, v'^t_t u'^{t-1}_t) = v'^{tn}_t u'^{(t-1)n}_t f_t(v'_t, u'_t), \\ f_t(v'_t, u'_t) &= a_{0,0} + a_{\ell+1,0} v'^{k-tn}_t u'^{k-(t-1)n}_t + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v'^{\alpha+t\beta-tn}_t u'^{\alpha+(t-1)\beta-(t-1)n}_t \end{aligned}$$

with $\alpha + t\beta - tn > 0$.

For each $t = 1, 2, \dots, s$, $\{(v'_i, u'_i) : i = 1, 2, \dots, t\}$ is not needed for the study of the proper transform $V^{(t)}(f)$ under τ_t , because $\alpha + t\beta - tn > 0$ by (3.2.6) and $k - tn > 0$ imply that $f_t(v'_t, u'_t) = \varepsilon'_t$ where ε'_t is defined to be a unit in $\mathbb{C}\{v'_t, u'_t\}$.

For each fixed $t = 1, 2, \dots, s$, along $v_t = 0$, $(f \circ \tau_t)_{total}$ can be written in the form

$$(3.2.7) \quad \begin{aligned} (f \circ \tau_t)_{total} &= f(v_t, v_t^t u_t) = v_t^{tn} f_t(v_t, u_t), \\ f_t(v_t, u_t) &= a_{0,0} u_t^n + a_{\ell+1,0} v_t^{k-tn} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v_t^{\alpha+t\beta-tn} u_t^\beta, \end{aligned}$$

with $\alpha + t\beta - tn > 0$.

So, $qs(V^{(t)}(f)) \cap E_t = \{(v_t, u_t) = (0, 0)\}$, and also the multiplicity of the local defining equation for the proper transform $V^{(s)}(f)$ under τ_s is $k - ns \leq n$ at $(v_s, u_s) = (0, 0)$.

(c-1) If $1 \leq k - sn < n$ in (3.2.7), then $(k - sn)\beta + n(\alpha + s\beta - sn) \geq (k - sn)n$ if and only if $n\alpha + k\beta \geq nk$.

(c-2) If $k - sn = n$ in (3.2.7), then by Lemma 3.1 we have the following:

$$(3.2.8) \quad \text{There are nonzero complex numbers } c_{\alpha_i \beta_i} \text{ for all } i = 1, 2, \dots, n-1$$

$$\text{such that } \alpha_i + s\beta_i - sn + \beta_i = n, \text{ i.e., } \frac{\alpha_i}{n - \beta_i} = \frac{k}{n} = s + 1. \quad \square$$

Now, it is clear that Sublemma 3.2.1 is true, and so we finished the proof of the theorem.

Corollary 3.3. Assumptions Suppose that $f \in \mathbb{C}\{y, z\}$ with isolated singularity at $(0, 0)$ satisfies the same properties and notations as in the assumption of Theorem 3.2.

In addition, we assume that the following equality holds:

$$(3.3.1) \quad \gcd(n, k) = 1.$$

Conclusions

(1) The necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ is as follows:

$$(3.3.2) \quad \frac{\alpha_i}{i} > \frac{k}{n} \quad \text{for all } i = 1, 2, \dots, n-1.$$

(2) Equivalently, the inequality in (3.3.2) holds if and only if f is represented as follows:

$$(3.2.3) \quad f(y, z) = a_{0,0} z^n + a_{n,0} y^k + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } n\alpha + k\beta > nk,$$

where the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β with $n\alpha + k\beta \geq nk$, and $a_{0,0}$ and $a_{n,0}$ are nonzero constant.

Proof of Corollary 3.3. It was already proved by Theorem 3.2 that the inequality in (3.3.2) is the necessary condition for f to be irreducible in $\mathbb{C}\{y, z\}$. Now, in order to prove that the inequality in (3.3.2) is a sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$, assuming for convenience of proof that $n < k$, the proof of this corollary can be easily finished by the induction on the multiplicity of f at the origin, using the same method as we have seen in (3.2.6) and (3.2.7) of Sublemma 3.2.1. \square

§3.2. The algorithm for finding the total transform of an irreducible plane curve singularity under the standard resolution which has the homeomorphic resolution as the curve $(z^n + y^k = 0)$ does and its generalizations

Lemma 3.4. Assumptions Let $f(y, z) = a_n z^n + a_0 y^k + \sum_{i=1}^{n-1} a_i y^{\alpha_i} z^i$ be irreducible in $\mathbb{C}\{y, z\}$ where for $0 \leq i \leq n$, each $a_i = a_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ if exists and the α_i are positive integers. Let $d = \gcd(n, k)$ with $1 \leq n \leq k$, and write $n = n_1 d$ and $k = k_1 d$ with $\gcd(n_1, k_1) = 1$.

Conclusions Then, f can be written in the form

$$(3.4.1) \quad f = A \prod_{i=1}^d (z^{n_1} + \xi_i y^{k_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d,$$

where $A = a_n(0, 0)$ is a nonzero complex number, and for $1 \leq i \leq d$, all ξ_i are nonzero complex numbers, and the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β with $n_1 \alpha + k_1 \beta > n_1 k_1 d$.

Proof of Lemma 3.4. Assume that $d = \gcd(n, k) > 1$ with $2 \leq n \leq k$, otherwise there is nothing to prove. For any nonzero monomial $y^\alpha z^\beta$ of $f(y, z)$ in the assumption, by Theorem 3.2 it suffices to consider two cases, denoted by Case(I) $n\alpha + k\beta = nk$ and Case(II) $n\alpha + k\beta > nk$. Then, the proof of the remainder is trivial.

Theorem 3.5(The representation theorem for the local defining equations of irreducible plane curve singularities).

Assumptions Let $f(y, z) = a_n z^n + a_0 y^k + \sum_{i=1}^{n-1} a_i y^{\alpha_i} z^i$ be irreducible in $\mathbb{C}\{y, z\}$ where each $a_i = a_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $0 \leq i \leq n$, if exists and the α_i are positive integers. Let n be the multiplicity of f at the origin with $1 \leq n \leq k$. Let $d = \gcd(n, k)$, and write $n = n_1 d$ and $k = k_1 d$ with $\gcd(n_1, k_1) = 1$.

Conclusions Then, f can be represented as follows:

$$(3.5.1) \quad f = A(z^{n_1} + \xi y^{k_1})^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d,$$

where the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β , satisfying the following properties :

- (i) A and ξ are the unique nonzero complex numbers such that $A = a_n(0, 0) \neq 0$, $dA\xi = a_{n-n_1}(0, 0) \neq 0$, and $\binom{d}{i} A\xi^i = a_{n-in_1}(0, 0)$ for $1 \leq i \leq d$.
- (ii) $\frac{\alpha_i}{n-i} \geq \frac{k}{n} = \frac{k_1}{n_1}$ for $0 \leq i \leq n-1$.

The proof of Theorem 3.5 will be done by Lemma 3.4 and Theorem 3.6.

Remark 3.5.1. (a) If $d = \gcd(n, k) = 1$, then it is clear by Corollary 3.3 that the representation form in (3.5.1) is a sufficient condition for f to be irreducible in $\mathbb{C}\{y, z\}$.

(b) If f is defined by $f = (z^2 + y^3)^2 + y^2 z^4$, then note that f is not irreducible in $\mathbb{C}\{y, z\}$, satisfying an equation in (3.5.1).

Theorem 3.6. Assumptions Let $V(g_i) = \{(y, z) : g_i(y, z) = 0\}$ for $1 \leq i \leq d$, $V(f) = \{(y, z) : f(y, z) = 0\}$ and $V(F) = \{(y, z) : F(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively in the form,

$$(3.6.1) \quad \begin{aligned} g_i &= z^{n_1} + \xi_i y^{k_1}, \\ f &= \prod_{i=1}^d g_i + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d, \\ F &= y^{\delta_1} z^{\delta_2} f, \end{aligned}$$

satisfying the properties (i), (ii), (iii), (iv), (v) and (vi):

- (i) $\gcd(n_1, k_1) = 1$ with $1 \leq n_1 < k_1$ and d is a positive integer.
- (ii) The ξ_i are nonzero complex numbers for $1 \leq i \leq d$, and μ is the counting number of distinct elements in the set $\{\xi_1, \xi_2, \dots, \xi_d\}$, as a set, but not as a sequence.
- (iii) The $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + k_1 \beta > n_1 k_1 d$, if exist.
- (iv) Each δ_i is either a positive integer or 0 for $i = 1, 2$.
- (v) If $n_1 = d = 1$, assume additionally that $\delta_2 > 0$.
- (vi) If $n_1 = 1$ and $\delta_2 = 0$, note by (v) that $d \geq 2$. If $d \geq 2$ and $n_1 \geq 1$, then assume that $V(f)$ has an isolated singular point at the origin as a reduced variety.

Note. By the property (v), if $n_1 = d = 1$ then $V(f)$ does not have a singularity at $(0, 0)$, but $V(F)$ has an isolated singular point at $(0, 0)$ as a reduced variety.

In preparation for the construction of the statement in the conclusion, let $V(G) = \{(y, z) : G(y, z) = 0\}$ be another analytic variety with isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(3.6.2) \quad G = z^\gamma g_1 \quad \text{and} \quad g_1 = z^{n_1} + \xi_1 y^{k_1},$$

satisfying the properties (vii) and (viii):

(vii) If $n_1 = 1$, then $\gamma = 1$.

(viii) If $n_1 \geq 2$, then $\gamma = 0$.

Let τ_m be the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(G)$. For each $t = 1, 2, \dots, m$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\{\pi_i : M^{(i)} \rightarrow M^{(i-1)} \text{ is a blow-up of } M^{(i-1)} \text{ at some point of } M^{(i-1)} \text{ for } 1 \leq i \leq t\}$ with $M^{(0)} = \mathbb{C}^2$. For brevity of notation, let $V^{(t)}(G)$ be the proper transform under τ_t for $1 \leq t \leq m$.

Let $E^{(m)} = \tau_m^{-1}(0, 0)$, and let $E^{(m)} = \cup E_i$, $1 \leq i \leq m$, be the decomposition of $E^{(m)}$ into irreducible components where each E_i is called an exceptional curve of the first kind.

Conclusions We have the following facts.

Fact(1). In order to study $V^{(t)}(G)$ under τ_t , we can find just one coordinate patch of the local coordinates for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$, where $1 \leq t \leq m$ and $M^{(0)} = \mathbb{C}^2$, which will be proved in process of the standard resolution of the singular point of $V(G)$ by the following lemma, Lemma 4.1.

Fact(2). By Fact(1), we can use the same τ_m for the composition of the first finite number m of successive blow-ups in preparation for finding the standard resolution of the singular point $(0, 0)$ of either $V(f)$ or $V(F)$ if exists, as a reduced variety.

Fact(3). In order to study each proper transform of either $V(f)$ or $V(F)$ under τ_t , without using a nonsingular change of coordinates, we can use the common one coordinate patch of the same local coordinates simultaneously, as it has been already used for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$ in Fact(1), where $1 \leq t \leq m$.

Fact(4). If f is irreducible in $\mathbb{C}\{y, z\}$, then $\mu = 1$, that is, all the ξ_i are the same. But, the converse does not hold.

After m iterations of blow-ups, let (v_m, u_m) and (v'_m, u'_m) be the local coordinates for $M^{(m)}$ where by Fact(1) $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ was defined to be the m -th blow-up at some point of $M^{(m-1)}$ with $u'_m = 1/u_m$ and $v'_m = v_m u_m$. Note that $E_m = \{v_m = 0\} \cup \{v'_m = 0\}$.

For brevity of the notation, we write

$$(3.6.3) \quad (v, u) = (v_m, u_m) \quad \text{and} \quad (v', u') = (v'_m, u'_m).$$

Also, let $(F \circ \tau_m)_{\text{divisor}}$ be a divisor of $F \circ \tau_m$ defined by

$$(3.6.4) \quad (F \circ \tau_m)_{\text{divisor}} = V^{(m)}(F) + \sum_{i=1}^m e_i E_i,$$

where each e_i is the multiplicity of $F \circ \tau_m$ along E_i for $1 \leq i \leq m$ and $V^{(m)}(F)$ is the proper transform of $V(F)$ under τ_m .

Using the proofs for Fact(1), Fact(2), Fact(3) and Fact(4) of the above, and also for the next statements [I], [II] and [III] in the remainder of this theorem, then we will find some elementary solutions for the following problems:

Problem 1. Along E_m , construct $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a local analytic mapping and the local defining equation $(F \circ \tau_m)_{\text{total}}$ for the total transform of $V(F)$ under τ_m .

Problem 2. Find the necessary condition for f to be irreducible in $\mathbb{C}\{y, z\}$.

Problem 3. Find the conditions under which f and F belong to the type [1] under τ_m in the sense of Definition 2.8.

Along $v = 0$, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings and $(F \circ \tau_m)_{total}$ can be written in the following form:

$$(3.6.5) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{k_1} u^b), \\ (F \circ \tau_m)_{total} &= v^{e_m} u^\varepsilon (f \circ \tau_m)_{proper} \quad \text{with} \\ (f \circ \tau_m)_{proper} &= \prod_{i=1}^d (u + \xi_i) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^{n_1 \alpha + k_1 \beta - n_1 k_1 d} u^{\varepsilon_{\alpha, \beta}}, \\ (G \circ \tau_m)_{total} &= v^{k_1 \gamma + n_1 k_1} u^{b\gamma + ak_1} (u + \xi_1), \end{aligned}$$

where (i) a and $b > 0$ are nonnegative integers such that $bn_1 - ak_1 = 1$,
(ii) $e_m = n_1 \delta_1 + k_1 \delta_2 + n_1 k_1 d$, $\varepsilon = a\delta_1 + b\delta_2 + ak_1 d$ and $\varepsilon_{\alpha, \beta} = a\alpha + b\beta - ak_1 d \geq 0$,
(iii) by assumption of $V(G)$, if $n_1 = 1$ then $\gamma = 1$, and if $n_1 \geq 2$ then $\gamma = 0$.

[II]

Along $v' = 0$, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings and $(F \circ \tau_m)_{total}$ can be written in the following form:

$$(3.6.6) \quad \begin{aligned} \tau_m(v', u') &= (y, z) = (v'^{n_1} u'^p, v'^{k_1} u'^q), \\ (F \circ \tau_m)_{total} &= v'^{e_m} u'^{\varepsilon'} (f \circ \tau_m)_{proper} \quad \text{with} \\ (f \circ \tau_m)_{proper} &= \prod_{i=1}^d (1 + \xi_i u') + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v'^{n_1 \alpha + k_1 \beta - n_1 k_1 d} u'^{\varepsilon'_{\alpha, \beta}}, \\ (G \circ \tau_m)_{total} &= v'^{k_1 \gamma + n_1 k_1} u'^{q\gamma + qn_1} (G \circ \tau_m)_{proper} \quad \text{with} \\ (G \circ \tau_m)_{proper} &= (1 + \xi_1 u'), \end{aligned}$$

where (i) p and q are positive integers such that $pk_1 - qn_1 = 1$,
(ii) $e_m = n_1 \delta_1 + k_1 \delta_2 + n_1 k_1 d$, $\varepsilon' = p\delta_1 + q\delta_2 + qn_1 d > 1$ and $\varepsilon'_{\alpha, \beta} = p\alpha + q\beta - qn_1 d > 0$,
(iii) $q\gamma + qn_1 > 1$ because $q > 0$, noting that if $n_1 = 1$ then $\gamma = 1$ by assumption of $V(G)$.

[III]

After m iterations of blow-ups, denoted by τ_m , we have the following consequences:

- (i) $V(G) \in$ the type[1] under τ_m in the sense of Definition 2.8.
- (ii) If f is irreducible in $\mathbb{C}\{y, z\}$, then all the ξ_i are the same or $\mu = 1$.
- (iii) If f is irreducible in $\mathbb{C}\{y, z\}$, then F belongs to the type [1] under τ_m in the sense of Definition 2.8 whether or not f has a singularity at the origin.
- (iv) Let f be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin. If $n = 1$, then $f \in$ the type[0] under τ_m in the sense of Definition 2.8, and if $n \geq 2$, then $f \in$ the type[1] under τ_m in the sense of Definition 2.8. \square

The proof of Theorem 3.6 will be done in §4.

Remark 3.6.1. Let $\tau_m = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m$ be the composition of a finite number m of successive blow-ups at the origin in \mathbb{C}^2 which is needed to get the standard resolution of the singular point of $V(G)$, as we have seen in the assumption of Theorem 3.6.

(a) Let $f(y, z) = (z^2 + y^3)^2 + y^2 z^4$ with $G(y, z) = z^2 + y^3$. Then, note that G and f satisfies the same assumption as in Theorem 3.6. Then, $V(G) \in$ the type[1] under τ_m , and $V(f) \in$ the type[1] under τ_m , too. But, it can be easily shown that f is not irreducible in $\mathbb{C}\{y, z\}$.

(b) Let $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t$ for $t = 1, 2, \dots, m$. Suppose that G , f and F satisfy the same assumption as in Theorem 3.6. For each $t = 1, 2, \dots, m$, consider $qs(V^{(t)}(G))$, $qs(V^{(t)}(f))$ and $qs(V^{(t)}(F))$ under τ_t in the sense of Definition 2.6.

(b1) For each $t = 1, 2, \dots, m-1$, $qs(V^{(t)}(G))$ is a one-point set under τ_t .

(b2) If $t = m$, then $qs(V^{(m)}(G))$ is empty under τ_m .

(b3) For each $t = 1, 2, \dots, m-1$, $V^{(t)}(f)$ and $V^{(t)}(F)$ has the same quasisingular point as $V^{(t)}(G)$ does in (b1), if exists.

(b4) As in the conclusion of this theorem, after m iterations of blow-ups, let (v_m, u_m) and (v'_m, u'_m) be the local coordinates for $M^{(m)}$ where $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ was defined to be the m -th blow-up at some point of $M^{(m-1)}$ with $u'_m = 1/u_m$ and $v'_m = v_m u_m$.

Then, $qs(V^{(m)}(F)) = qs(V^{(m)}(f))$ such that $qs(V^{(m)}(F)) = qs(V^{(m)}(F)) \cap \{v_m = 0\} = qs(V^{(m)}(F)) \cap \{v'_m = 0\}$ and $qs(V^{(m)}(f)) = qs(V^{(m)}(f)) \cap \{v_m = 0\} = qs(V^{(m)}(f)) \cap \{v'_m = 0\}$. Note that $E_m = \{v_m = 0\} \cup \{v'_m = 0\}$.

Theorem 3.7(The representation theorem for some total transforms of the local defining equations of irreducible plane curve singularities in process of the standard resolution of their singularity).

Assumptions Let $V(g_i) = \{(y, z) : g_i(y, z) = 0\}$ for $1 \leq i \leq d$, $V(f) = \{(y, z) : f(y, z) = 0\}$ and $V(F) = \{(y, z) : F(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , satisfying the same properties and notations as in the assumption of Theorem 3.6.

In addition, for the necessity of the irreducibility of f in $\mathbb{C}\{y, z\}$, we may assume that the following holds:

$$(3.7.0) \quad \text{all the } \xi_i \text{ are the same numbers and so } g_i = g_1 \text{ for all } i.$$

Let $V(\phi) = \{(y, z) : \phi(y, z) = 0\}$ and $V(\Psi) = \{(y, z) : \Psi(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively in the form,

$$(3.7.1) \quad \begin{aligned} \phi &= g_1^{\ell d} + \sum_{p, q \geq 0} a_{p, q} y^p z^q & \text{with } n_1 p + k_1 q > n_1 k_1 d \ell, \\ \Phi &= y^{\ell \delta_1} z^{\ell \delta_2} \phi, \end{aligned}$$

satisfying the properties (ix) and (x):

- (ix) ℓ is a positive integer.
- (x) The $a_{p, q}$ are nonzero complex numbers for some nonnegative integers p and q such that $n_1 p + k_1 q > n_1 k_1 d \ell$, if exist.

By Theorem 3.6, we can use the same τ_m for the composition of the first finite number m of successive blow-ups in the process of the standard resolution of the singular point $(0, 0)$ of both $V(f)$ and $V(F)$ if exist, as reduced varieties. Let $E^{(m)} = \tau_m^{-1}(0, 0)$, and let $E^{(m)} = \cup E_i$, $1 \leq i \leq m$, be the decomposition of $E^{(m)}$ into irreducible components where each E_i is called an exceptional curve of the first kind under τ_m .

As in (3.6.4) of Theorem 3.6, let $(G \circ \tau_m)_{\text{divisor}}$, $(f \circ \tau_m)_{\text{divisor}}$, $(F \circ \tau_m)_{\text{divisor}}$ and $(\Phi \circ \tau_m)_{\text{divisor}}$ be the divisors of $G \circ \tau_m$, $f \circ \tau_m$, $F \circ \tau_m$ and $\Phi \circ \tau_m$ under τ_m , respectively, each of which can be written as follows:

$$(3.7.2) \quad \begin{aligned} (G \circ \tau_m)_{\text{divisor}} &= V^{(m)}(G) + \sum_{i=1}^m e_{0,i} E_i, & (f \circ \tau_m)_{\text{divisor}} &= V^{(m)}(f) + \sum_{i=1}^m e_{1,i} E_i, \\ (F \circ \tau_m)_{\text{divisor}} &= V^{(m)}(F) + \sum_{i=1}^m e_{2,i} E_i, & (\Phi \circ \tau_m)_{\text{divisor}} &= V^{(m)}(\Phi) + \sum_{i=1}^m e_{3,i} E_i, \end{aligned}$$

where $e_{0,i}$, $e_{1,i}$, $e_{2,i}$ and $e_{3,i}$ are the multiplicities of $(G \circ \tau_m)_{\text{divisor}}$, $(f \circ \tau_m)_{\text{divisor}}$, $(F \circ \tau_m)_{\text{divisor}}$ and $(\Phi \circ \tau_m)_{\text{divisor}}$ along E_i , respectively, and $V^{(m)}(G)$, $V^{(m)}(f)$, $V^{(m)}(F)$ and $V^{(m)}(\Phi)$ are the proper transform of $V(G)$, $V(f)$, $V(F)$ and $V(\Phi)$ under τ_m , respectively.

Conclusions We have three statements, [I], [II] and [III].

[I] In the sense of Lemma 2.14, using the same notations and properties as in Theorem 3.6, there is the composition of a finite number m of successive blow-ups, denoted by $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$, which can be viewed as a local analytic mapping from a polydisc $\Delta(r_{11}, r_{12})$ to a polydisc $\Delta(r_{21}, r_{22})$, with

$$(3.7.3) \quad \begin{aligned} \Delta(r_{11}, r_{12}) &= \{(v, u) \in \mathbb{M}^{(m)} : |v| < r_{11}, |u| < r_{12}\} \quad \text{and} \\ \Delta(r_{21}, r_{22}) &= \{(y, z) \in \mathbb{C}^2 : |y| < r_{21}, |z| < r_{22}\}, \end{aligned}$$

satisfying the following properties :

$$(3.7.4) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{k_1} u^b), \\ (F \circ \tau_m)(v, u) &= v^{e_m} u^\varepsilon (f \circ \tau_m)_{\text{proper}}, \\ (f \circ \tau_m)_{\text{proper}} &= (u + \xi)^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^{n_1 \alpha + k_1 \beta - n_1 k_1 d} u^{\varepsilon_{\alpha, \beta}}, \end{aligned}$$

where

- (i) a and $b > 0$ are nonnegative integers such that $bn_1 - ak_1 = 1$ and $E_m = \{(u, v) : v = 0\}$,
- (ii) $e_m = n_1 \delta_1 + k_1 \delta_2 + n_1 k_1 d$, $\varepsilon = a \delta_1 + b \delta_2 + ak_1 d$ and $\varepsilon_{\alpha, \beta} = a\alpha + b\beta - ak_1 d \geq 0$.

Also, if $n = 1$, then $f \in \text{the type}[0]$ and $F \in \text{the type}[1]$ under τ_m , but if $n \geq 2$, then $f \in \text{the type}[1]$ and $F \in \text{the type}[1]$ under τ_m in the sense of Definition 2.8.

Observe by the notation that $(f \circ \tau_m)_{\text{proper}}$ is the local defining equation for the proper transform $V^{(m)}(f)$ and that $(F \circ \tau_m)(v, u)$ is the local defining equation for the total transform of $V(F)$ under τ_m .

[II] Let $\tau_m^{-1}(0) = E^{(m)} = \cup E_i$, $1 \leq i \leq m$.

Fact(1) of [II]: In case of $V(G)$, for $1 \leq i \leq m$, each E_i belongs to exactly one of the following two types relative to $\tau_m^{-1}(V(G))$: For any $A \subset \tau_m^{-1}(V(G))$, \overline{A} is called the closure of A in $\tau_m^{-1}(V(G))$ where $\tau_m^{-1}(V(G)) = E^{(m)} \cup V^{(m)}(V(G))$.

Type(1) E_m of $\tau_m^{-1}(V(G))$ and $\overline{\tau_m^{-1}(V(G)) - E_m}$ of $\tau_m^{-1}(V(G))$ have three intersection points in $\tau_m^{-1}(V(G))$ in the sense of Definition 2.8.

Type(2) For any $j \neq m$, E_j of $\tau_m^{-1}(V(G))$ and $\overline{\tau_m^{-1}(V(G)) - E_j}$ of $\tau_m^{-1}(V(G))$ have at most two intersection points in $\tau_m^{-1}(V(G))$ in the sense of Definition 2.8.

By the same method as in Fact(1), each of $V(F)$ and $V(\Phi)$ have the same type of exceptional curves of the first kind relative to the corresponding one of $\tau_m^{-1}(V(F))$ and $\tau_m^{-1}(V(\Phi))$ respectively, as $V(G)$ does in Fact(1), as a reduced variety. Also, whenever $V(f)$ has an isolated singularity as a reduced variety, then $V(f)$ has the same type of exceptional curves of the first kind relative to $\tau_m^{-1}(V(f))$ as the above $V(G)$ does.

Fact(2) of [II]: In any case of $V(G)$, $V(f)$, $V(F)$ and $V(\Phi)$, for $1 \leq i \leq m$, each E_i belongs to exactly one of the following two types relative to $E^{(m)} = \cup E_i$, $1 \leq i \leq m$:

Type(1) For each $i = 1, 2, \dots, m$, there are two distinct exceptional curves of the first kind in $E^{(m)}$, denoted by E_1 and E_s with $1 < s \leq m$, which satisfy the same property in the following sense: Note that s is an integer such that $(s-1)n_1 < k_1 \leq sn_1$.

(1a) E_1 of $E^{(m)}$ and $\overline{E^{(m)} - E_1}$ of $E^{(m)}$ has one and only one intersection point in $E^{(m)}$.

(1b) For some integer $s \neq 1$, E_s of $E^{(m)}$ and $\overline{E^{(m)} - E_s}$ of $E^{(m)}$ has one and only one intersection point in $E^{(m)}$.

Type(2) If E_t of $E^{(m)}$ is neither E_1 nor E_s in Type(1), then E_t of $E^{(m)}$ and $\overline{E^{(m)} - E_t}$ of $E^{(m)}$ have two distinct intersection points in $E^{(m)}$.

[III] Then, we have the following:

$$(3.7.5) \quad \frac{e_{3,i}}{e_{2,i}} = \ell \quad \text{for all } i = 1, 2, \dots, m.$$

In particular, $e_{0,m} = k_1\gamma + n_1k_1$ and $e_{2,m} = n_1\delta_1 + k_1\delta_2 + n_1k_1d$.

Moreover, whenever an analytic variety $V(F)$ satisfies the conditions in (3.6.1), then a finite sequence $\{e_i : i = 1, 2, \dots, m\}$ is uniquely determined by $\{\delta_1, \delta_2, n_1, k_1, d\}$, without depending on the nonzero complex numbers $c_{\alpha\beta}$. \square

The proof of Theorem 3.7 will be done in §4.

Corollary 3.8. Assumptions As in Corollary 3.3 or Theorem 3.7, assume that $V(h_1) = \{(y, z) : h_1(y, z) = 0\}$ is represented as follows: Note that $2 \leq n_1 < k_1$.

$$(3.8.1) \quad h_1(y, z) = z^{n_1} + y^{k_1} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } n_1\alpha + k_1\beta > n_1k_1,$$

where the $c_{\alpha, \beta}$ are nonzero complex numbers for some integers $\alpha \geq 0$ and $\beta \geq 0$, if exist.

Conclusions

(i) Let $g_1(y, z) = z^{n_1} + y^{k_1}$ with $\gcd(n_1, k_1) = 1$. Then, $g_1(y, z) = 0$ and $h_1(y, z) = 0$ have the homeomorphic resolution at $(y, z) = (0, 0)$.

(ii) If $f(y, z)$ of Theorem 3.7 is irreducible in $\mathbb{C}\{y, z\}$, then we have the following:

$$(3.8.2) \quad \begin{aligned} \text{Multiseq}(V(f)) &= \text{Join}(\{[dn_1 : dk_1]\}, \text{Multiseq}(V^{(m)}(f))) \quad \text{by Definition 1.7,} \\ \text{Multiseq}(V(g)) &= \{[n_1 : k_1]\}, \end{aligned}$$

noting that if $\{[n_1 : k_1]\} = \{c_i : i = 1, 2, \dots, w\}$, write $\{[dn_1 : dk_1]\} = \{dc_i : i = 1, 2, \dots, w\}$ for notation.

§4. The proofs of Theorem 3.6 and Theorem 3.7 in §3

§4.0. Introduction

In preparation for the proof of Theorem 3.6, we prove three lemmas, Lemma 4.1, Lemma 4.2 and Lemma 4.3 in §4.1. In §4.2, using three lemmas, it suffices to prove Theorem 3.6 because Theorem 3.5 and Theorem 3.7 can be proved as an easy corollary of Theorem 3.6.

§4.1. Lemma 4.1, Lemma 4.2 and Lemma 4.3 with proofs

Lemma 4.1. Assumptions Let $V(G) = \{(y, z) : G(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by the form

$$(4.1.1) \quad G = z^\gamma g \quad \text{and} \quad g = z^{n_1} + y^{k_1} \quad \text{with} \quad \gcd(n_1, k_1) = 1,$$

satisfying the following properties:

- (i) $1 \leq n_1 < k_1$. (ii) If $n_1 = 1$, then $\gamma = 1$. (iii) If $n_1 \geq 2$, then $\gamma = 0$.

Conclusions We can find the standard resolution of the singular point $(0, 0)$ of $V(G)$, that is, the composition $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m$ of a finite number m of successive blow-ups at the origin in \mathbb{C}^2 such that for each blow-up π_j with $1 \leq j \leq m$, just one coordinate patch is needed in order to study the j -th proper transform $V^{(j)}(G)$ under $\tau_j = \pi_1 \circ \cdots \circ \pi_j$. \square

The proof of Lemma 4.1 is trivial.

Lemma 4.2. Assumptions Let $V(G) = \{(y, z) : G(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , satisfying the same properties and notations as the assumption of Lemma 4.1. In addition, for given integers n_1 and k_1 in (4.1.1), let s be a positive integer such that $sn_1 < k_1 \leq (s+1)n_1$. Since $\gcd(n_1, k_1) = 1$ with $1 \leq n_1 < k_1$, note that

$$(4.2.1) \quad \begin{aligned} &\text{if } n_1 = 1, \text{ then } k_1 = s + 1, \\ &\text{if } n_1 \geq 2, \text{ then } sn_1 < k_1 < (s+1)n_1. \end{aligned}$$

Conclusions Let τ_m be the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(G)$.

Then, $1 \leq s < m$, and in preparation for the proof of Theorem 3.6, we can construct the first t iterations of blow-ups, in order to study the t -th proper transform $V^{(t)}(G)$ of $V(G)$ under $\tau_t = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t$ for each $t = 1, 2, \dots, s$, as follows.

- (a) For each $t = 1, 2, \dots, s$, $qs(V^{(t)}(G))$ is a one-point set in the sense of Definition 2.6.

(b) In order to study the t -th proper transform $V^{(t)}(G)$ under $\tau_t = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t$, we can use just one coordinate patch of the local coordinates for each blow-up $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ at some point of $M^{(i-1)}$, where $M^{(0)} = \mathbb{C}^2$ with $(v_0, u_0) = (y, z)$, as follows: For each $i = 1, 2, \dots, t$, let (v_i, u_i) and (v'_i, u'_i) be the local coordinates for $M^{(i)}$ with $u'_i = 1/u_i$ and $v'_i = v_i u_i$. Then, $\pi_i(v_i, u_i) = (v_{i-1}, u_{i-1}) = (v_i, v_i u_i)$ and $\pi_i(v'_i, u'_i) = (v_{i-1}, u_{i-1}) = (v'_i u'_i, v'_i)$ where (v'_i, u'_i) is not needed for the study of $V^{(i)}(G)$ under π_i .

- (c) For $1 \leq t \leq s$, let $\tau_t = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ be defined by the local coordinates in (b) where $E_t = \{v_t = 0\} \cup \{v'_t = 0\}$.

Suppose that $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ satisfies the same assumptions and notations as in (b). For each fixed t , along E_t , τ_t can be represented, as a composition of analytic mappings, as follows: Note that $u'_t = 1/u_t$ and $v'_t = v_t u_t$.

$$(4.2.2) \quad \begin{aligned} \tau_t(v_t, u_t) &= (y, z) = (v_t, v_t^t u_t) \quad \text{and} \\ \tau_t(v'_t, u'_t) &= (y, z) = (v'_t u'_t, v'_t u'^{t-1}_t). \end{aligned}$$

So, using the first s iterations of blow-ups, denoted by τ_s , the sequence of coordinates $\{(v'_1, u'_1), (v'_2, u'_2), \dots, (v'_s, u'_s)\}$ is not needed for the study of $V^{(s)}(G)$ by (b).

In addition, $qs(V^{(t)}(G)) = \{(v_t, u_t) = (0, 0)\}$ for each $t = 1, 2, \dots, s$.

Moreover, a divisor of $G \circ \tau_s$ under τ_s , denoted by $(G \circ \tau_s)_{\text{divisor}}$, can be written as follows:

$$(4.2.3) \quad (G \circ \tau_s)_{\text{divisor}} = V^{(s)}(G) + \sum_{j=1}^s e_j E_j$$

$$\text{where } e_j = (\gamma + n_1)j \quad \text{for } j = 1, 2, \dots, s.$$

Note that $V^{(s)}(G)$ is the proper transform of $V(G)$ under τ_s . \square

Proof of Lemma 4.2. Following the same method as we have used in the proof of Sublemma 3.2.1 of Theorem 3.2 for each $t = 1, 2, \dots, s$, it is easy to find the proof of (a), (b) and (c) at the same time. Also, there is nothing to prove for (4.2.3). Thus, the proof can be finished. \square

Remark 4.2.1. Recall by Theorem 3.6 that $g_1(y, z) = z^{n_1} + \xi_1 y^{k_1}$ with $n_1 < k_1$. Then, $g(y, z) = g_1(y, z)$ with $\xi_1 = 1$ and $G(y, z) = z^\gamma g = z^\gamma g_1$ where γ is either 1 or zero. In particular, $G(y, z)$ can be defined by $F(y, z)$ of (3.6.1) with some coefficients. Also, note that $\{z = 0\}$ and $\{G(y, z) = 0\}$ meet tangentially at the origin.

Lemma 4.3. Assumptions Suppose that $V(g_i)$, $1 \leq i \leq d$, $V(f)$, $V(F)$ and $V(G)$ satisfy the same properties and notations as in the assumption of Theorem 3.6 and Lemma 4.2.

Let s be a positive integer such that $sn_1 < k_1 \leq (s+1)n_1$, and τ_m be the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(G)$, where $s < m$. Using the same properties and notations as in the conclusion of Lemma 4.2, then we proved by Lemma 4.2 that we can construct the first t iterations of blow-ups, denoted by τ_t , in order to study the t -th proper transform $V^{(t)}(G)$ under $\tau_t = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t$ for each $t = 1, \dots, s$, as follows:

Property(a). For the study of the t -th proper transform $V^{(t)}(G)$ with $1 \leq t \leq s$, we can use one and only one coordinate patch of the local coordinates for each blow-up $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ at some point of $M^{(i-1)}$, where $1 \leq i \leq t$ and $M^{(0)} = \mathbb{C}^2$.

Property(b). Then, $qs(V^{(t)}(G)) = \{(v_t, u_t) = (0, 0)\}$ for each $t = 1, 2, \dots, s$.

Conclusions We can get five facts:

Fact(1). Whenever $V(F)$ has the singular point at the origin as a reduced variety, then we can use the same τ_s for the composition of the first finite number s of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of $V(F)$, as we have seen in the assumption of this lemma.

Fact(2). In order to study the t -th proper transform $V^{(t)}(F)$ of $V(F)$ under τ_t , without using a nonsingular change of coordinates, we can use one and only common one coordinate patch of the same local coordinates simultaneously, as it was already used for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$ in the above Property(a), where $1 \leq t \leq s$.

Fact(3). Each t -th proper transform $V^{(t)}(F)$ under τ_t has the same quasisingular point $(v_t, u_t) = (0, 0)$ as $V^{(t)}(G)$ does in the above Property(b), as a reduced variety for $1 \leq t \leq s$.

Fact(4). Along E_t , we can construct $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ as an analytic mapping in the sense of (4.2.2) and the local defining equation $(F \circ \tau_t)_{\text{total}}$ for the total transform of $V(F)$ under τ_t , in order to study the t -th proper transform $V^{(t)}(F)$ where $1 \leq t \leq s$.

Fact(5). Then, $F \in \text{the type}[0]$ under τ_s in the sense of Definition 2.8.

In order to prove the above five facts, in the remainder of this lemma, it suffices to construct the following three statements [I], [II] and [III], respectively:

[I] For any $t = 1, 2, \dots, s$, along $v_t = 0$, $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings in the sense of (4.2.2) and $(F \circ \tau_t)_{\text{total}}$ can be written, respectively as follows:

$$\begin{aligned}
(4.3.1) \quad & \tau_t(v_t, u_t) = (y, z) = (v_t, v_t^t u_t), \\
& (F \circ \tau_t)_{\text{total}} = v_t^{e_t} (F \circ \tau_t)_{\text{proper}} \quad \text{with} \quad (F \circ \tau_t)_{\text{proper}} = u_t^{\delta_2} (f \circ \tau_t)_{\text{proper}}, \\
& (f \circ \tau_t)_{\text{proper}} = \prod_{i=1}^d (u_t^{n_1} + \xi_i v_t^{k_1 - tn_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v_t^{\lambda_t} u_t^\beta, \\
& (f \circ \tau_t)_{\text{total}} = v_t^{tn_1 d} (f \circ \tau_t)_{\text{proper}}, \\
& (G \circ \tau_t)_{\text{total}} = v_t^{t\gamma + tn_1} (G \circ \tau_t)_{\text{proper}} \quad \text{with} \quad (G \circ \tau_t)_{\text{proper}} = u_t^\gamma (u_t^{n_1} + \xi_1 v_t^{k_1 - tn_1}),
\end{aligned}$$

where

- (i) $e_t = \delta_1 + t\delta_2 + tn_1 d$ and $\lambda_t = \alpha + t\beta - tn_1 d > 0$ for $1 \leq t \leq s$, respectively,
- (ii) $n_1 \lambda_t + (k_1 - tn_1) \beta > n_1(k_1 - tn_1) d$, that is, $n_1 \alpha + k_1 \beta > n_1 k_1 d$ for all α and β .

Moreover, $V^{(t)}(G)$ has one and only one quasisingularity at $(v_t, u_t) = (0, 0)$ along $v_t = 0$. Also, $V^{(t)}(F)$ has one and only one quasisingularity at $(v_t, u_t) = (0, 0)$ along $v_t = 0$, as a reduced variety.

[II] For any $t = 1, 2, \dots, s$, along $v'_t = 0$, $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings in the sense of (4.2.2) and $(F \circ \tau_t)_{total}$ can be written, respectively as follows:

$$\begin{aligned}
(4.3.2) \quad \tau_t(v'_t, u'_t) &= (y, z) = (v'_t u'_t, v'^t_t u'^{t-1}_t), \\
(F \circ \tau_t)_{total} &= \begin{cases} v'^{e_1}_1 (F \circ \tau_1)_{proper}, & \text{if } t = 1 \text{ or } s = 1, \\ v'^{e_t}_t u'^{e_t-1}_t (F \circ \tau_t)_{proper}, & \text{if } t \geq 2, \end{cases} \\
(F \circ \tau_t)_{proper} &= \begin{cases} u'^{\delta_1}_1 (f \circ \tau_1)_{proper}, & \text{if } t = 1 \text{ or } s = 1, \\ (f \circ \tau_t)_{proper}, & \text{if } t \geq 2, \end{cases} \\
(f \circ \tau_t)_{proper} &= \prod_{i=1}^d (1 + \xi_i v'^{k_1 - tn_1}_t u'^{k_1 - (t-1)n_1}_t) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v'^{\lambda_t}_t u'^{\lambda_t-1}_t \quad \text{for } t \geq 1, \\
(f \circ \tau_t)_{total} &= v'^{tn_1 d}_t (f \circ \tau_t)_{proper} \quad \text{for } t \geq 1, \\
(G \circ \tau_t)_{total} &= v'^{t\gamma + tn_1}_t u'^{(t-1)\gamma + (t-1)n_1}_t (G \circ \tau_t)_{proper} \quad \text{for } t \geq 1, \\
(G \circ \tau_t)_{proper} &= 1 + \xi_1 v'^{k_1 - tn_1}_t u'^{k_1 - (t-1)n_1}_t \quad \text{for } t \geq 1,
\end{aligned}$$

where $e_t = \delta_1 + t\delta_2 + tn_1 d$ and $\lambda_0 = \alpha$ are well-defined, being compared with $\lambda_t = \alpha + t\beta - tn_1 d > 0$ for $1 \leq t \leq s$, respectively.

But, note that $(F \circ \tau_t)_{proper}$ is represented as follows :

- (i) If $t = 1$, then $(F \circ \tau_1)_{proper} = u'^{\delta_1}_1 (f \circ \tau_1)_{proper}$.
- (ii) If $t \geq 2$, then $(F \circ \tau_t)_{proper} = (f \circ \tau_t)_{proper}$.

Moreover, there is no quasisingular point of $V^{(t)}(G)$ along $v'_t = 0$. Also, there is no quasisingular point of $V^{(t)}(F)$ along $v'_t = 0$, as a reduced variety.

[III] After t iterations of blow-ups for any $t = 1, 2, \dots, s$, $V(G) \in$ the type $[0]$ and also $V(F) \in$ the type $[0]$ under τ_t whether or not f is irreducible in $\mathbb{C}\{y, z\}$, satisfying the following properties : Let $\cup_{i=1}^t E_i = \tau_t^{-1}(0, 0)$ where each E_i is called an exceptional curve of the first kind under τ_t .

(a) $qs(V^{(t)}(G)) = qs(V^{(t)}(F)) = \{(v_t, u_t) = (0, 0)\}$.

(b) For $1 \leq i \leq t$, each E_i has at most two distinct intersection points with any other exceptional curves and $V^{(t)}(F)$. Also, no three distinct components of $\cup_{i=1}^t E_i$ meet.

(c) In more detail, for $2 \leq i \leq t$, each E_i has two distinct intersection points with any other exceptional curves and $V^{(t)}(F)$. But, E_1 has at most two distinct intersection points with any other exceptional curves and $V^{(t)}(F)$. In particular, if $\delta_1 > 0$, then E_1 has two distinct intersection points with $\cup_{i=2}^t E_i$ and $V^{(t)}(F)$, and if $\delta_1 = 0$, then E_1 has just one intersection point with $\cup_{i=2}^t E_i$ and $V^{(t)}(F)$, whether or not $t = 1$. \square

The proofs of three statements [I], [II] and [III] in Lemma 4.3 can be easily done by induction on t with $1 \leq t \leq s$, and so the proof of the lemma is completely finished.

§4.2. The proofs of Theorem 3.6 and Theorem 3.7

Proof of Theorem 3.6. For proof of this theorem, we are going to follow continuously the same notations and indices as we have used in the proof of Lemma 4.3.

By a given integer s with $sn_1 < k_1 \leq (s+1)n_1$, recall by (4.3.1) of Lemma 4.3 that along $v_s = 0$, $\tau_s : M^{(s)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings, and $(F \circ \tau_s)_{total}$ and $(G \circ \tau_s)_{total}$ can be written, respectively in the form

$$\begin{aligned}
(3.6.7) \quad \tau_s(v_s, u_s) &= (y, z) = (v_s, v_s^s u_s), \\
(F \circ \tau_s)_{total} &= v_s^{e_s} (F \circ \tau_s)_{proper} \quad \text{with} \quad (F \circ \tau_s)_{proper} = u_s^{\delta_2} (f \circ \tau_s)_{proper}, \\
(f \circ \tau_s)_{proper} &= \prod_{i=1}^d (u_s^{n_1} + \xi_i v_s^{k_1 - sn_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v_s^{\lambda_s} u_s^{\beta}, \\
(f \circ \tau_s)_{total} &= v_s^{sn_1 d} (f \circ \tau_s)_{proper}, \\
(G \circ \tau_s)_{total} &= v_s^{s\gamma + sn_1} (G \circ \tau_s)_{proper} \quad \text{with} \quad (G \circ \tau_s)_{proper} = u_s^{\gamma} (u_s^{n_1} + \xi_1 v_s^{k_1 - sn_1}),
\end{aligned}$$

where (i) $e_s = \delta_1 + s\delta_2 + sn_1d$ and $\lambda_s = \alpha + s\beta - sn_1d > 0$ by (4.3.1),

(ii) $n_1\lambda_s + (k_1 - sn_1)\beta > n_1(k_1 - sn_1)d$, that is, $n_1\alpha + k_1\beta > n_1k_1d$.

Note that $qs(V^{(s)}(F)) = qs(V^{(s)}(G)) = \{(v_s, u_s) = (0, 0)\}$ along E_s , as a reduced variety.

For the proof of the theorem in more detail, in order to apply the induction method to the multiplicity n_1 of $g_1 = z^{n_1} + \xi_1 y^{k_1}$ at $(y, z) = (0, 0)$ where $1 \leq n_1 < k_1$ and $G(y, z) = z^\gamma g_1 = z^\gamma(z^{n_1} + \xi_1 y^{k_1})$, it is enough to consider two cases:

Case(A): Let $n_1 = 1$. Then $\gamma = 1$ by assumption.

Case(B): Let $n_1 \geq 2$. Then $\gamma = 0$ by assumption.

Therefore, first we will show that Case(A) satisfies [I], [II] and [III], respectively, and next we will show that Case(B) satisfies [I], [II] and [III], respectively, where [I], [II] and [III] are three distinct statements in the conclusion of this theorem.

Case(A): Let $n_1 = 1$. In order to prove that this case satisfies [I], [II] and [III], let $V(f)$, $V(F)$ and $V(G)$ satisfy the same properties and notations as in (3.6.1) and (3.6.2) of the assumption of Theorem 3.6, each of which was already defined by the following form:

$$(3.6.8) \quad f = \prod_{i=1}^d (z + \xi_i y^{k_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with } \alpha + k_1\beta > k_1d,$$

$$F = y^{\delta_1} z^{\delta_2} f \quad \text{and} \quad G = z(z + \xi_1 y^{k_1}).$$

Since $sn_1 < k_1 \leq (s+1)n_1$ with $n_1 = 1$, note that $k_1 = (s+1)n_1 = s+1$.

Along $v_s = 0$, $(F \circ \tau_s)_{total}$ can be rewritten as follows, using (3.6.7) and (3.6.8):

$$(3.6.9) \quad (F \circ \tau_s)_{total} = v_s^{e_s} (F \circ \tau_s)_{proper} \quad \text{with} \quad (F \circ \tau_s)_{proper} = u_s^{\delta_2} (f \circ \tau_s)_{proper},$$

$$(f \circ \tau_s)_{proper} = \prod_{i=1}^d (u_s + \xi_i v_s) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v_s^\alpha u_s^\beta,$$

$$(G \circ \tau_s)_{total} = v_s^{2s} (G \circ \tau_s)_{proper} \quad \text{with} \quad (G \circ \tau_s)_{proper} = u_s(u_s + \xi_1 v_s),$$

where $e_s = \delta_1 + s\delta_2 + sd$, and also $\lambda_s = \alpha + s\beta - sd > 0$ by (4.3.1), because $n_1 = 1$ implies that $\gamma = 1$ by assumption of the theorem.

It is interesting to observe from (3.6.9) that $n_1\lambda_s + (k_1 - sn_1)\beta = 1(\alpha + s\beta - sd) + 1 \cdot \beta > 1 \cdot d$ if and only if $\alpha + k_1\beta > k_1d$ for all $\alpha \geq 0$ and $\beta \geq 0$.

For the proof, let $\pi_{s+1} : M^{(s+1)} \rightarrow M^{(s)}$ be a blow-up at $(v_s, u_s) = (0, 0)$ defined by

$$(3.6.10-a) \quad \pi_{s+1}(v_{s+1}, u_{s+1}) = (v_s, u_s) = (v_{s+1}, v_{s+1}u_{s+1}),$$

$$(3.6.10-b) \quad \pi_{s+1}(v'_{s+1}, u'_{s+1}) = (v_s, u_s) = (v'_{s+1}u'_{s+1}, v'_{s+1}),$$

where (v_{s+1}, u_{s+1}) and (v'_{s+1}, u'_{s+1}) are the local coordinates for $M^{(s+1)}$ with $u'_{s+1} = 1/u_{s+1}$ and $v'_{s+1} = v_{s+1}u_{s+1}$, and write $E_{s+1} = \{v_{s+1} = 0\} \cup \{v'_{s+1} = 0\}$.

Let $\tau_{s+1} = \tau_s \circ \pi_{s+1}$, and so (3.6.7), (3.6.10-a) and (3.6.10-b) imply that

$$(3.6.11-a) \quad \tau_{s+1}(v_{s+1}, u_{s+1}) = (\tau_s \circ \pi_{s+1})(v_{s+1}, u_{s+1}) = \tau_s(v_{s+1}, v_{s+1}u_{s+1})$$

$$= (v_{s+1}, v_{s+1}^{s+1}u_{s+1}) = (y, z) \quad \text{and}$$

$$(3.6.11-b) \quad \tau_{s+1}(v'_{s+1}, u'_{s+1}) = (\tau_s \circ \pi_{s+1})(v'_{s+1}, u'_{s+1}) = \tau_s(v'_{s+1}u'_{s+1}, v'_{s+1})$$

$$= (v'_{s+1}u'_{s+1}, v'_{s+1}^{s+1}u'_{s+1}^s) = (y, z).$$

Since $k_1 = s+1$, then (3.6.11-a) and (3.6.11-b) are rewritten as follows: For brevity of notations, write $(v, u) = (v_{s+1}, u_{s+1})$ and $(v', u') = (v'_{s+1}, u'_{s+1})$.

$$(3.6.12-a) \quad \tau_{k_1}(v, u) = (y, z) = (v, v^{k_1}u),$$

$$(3.6.12-b) \quad \tau_{k_1}(v', u') = (y, z) = (v'u', v'^{k_1}u'^s).$$

Thus, it can be easily proved by (3.6.9) and (3.6.10-a) and (3.6.10-b) that τ_{k_1} is the desired standard resolution of the singular point of $V(G)$.

Now, we will prove [I], [II] and [III] for Case(A), respectively as follows.

The proof of [I] for Case(A): By (3.6.5) and (3.6.12-a), compare the following:

$$(3.6.13) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{k_1} u^b), \\ \tau_{k_1}(v, u) &= (y, z) = (v, v^{k_1} u), \end{aligned}$$

where m is, by definition, a finite number of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(G)$. Then, $m = k_1$, $n_1 = 1$, $a = 0$ and $b = 1$. So, $bn_1 - ak_1 = 1 - 0 = 1$.

Assuming that $n_1 = 1$, then apply (3.6.12-a) to (3.6.1) and (3.6.2), respectively. Or, we may apply (3.6.10-a) to (3.6.9), if convenient.

Then, along $v = 0$, $(F \circ \tau_{k_1})_{total} = (F \circ \tau_{k_1})(v, u)$ can be written in the form

$$(3.6.14) \quad \begin{aligned} (F \circ \tau_{k_1})_{total} &= v^{\delta_1} (v^{k_1} u)^{\delta_2} \left\{ \prod_{i=1}^d (v^{k_1} u + \xi_i v^{k_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^\alpha (v^{k_1} u)^\beta \right\} \\ &= v^{e_{k_1}} (F \circ \tau_{k_1})_{proper} \quad \text{with} \quad (F \circ \tau_{k_1})_{proper} = u^{\delta_2} (f \circ \tau_{k_1})_{proper}, \\ (f \circ \tau_{k_1})_{proper} &= \prod_{i=1}^d (u + \xi_i) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^{\alpha + k_1 \beta - k_1 d} u^\beta, \\ (G \circ \tau_{k_1})_{total} &= v^{2k_1} (G \circ \tau_{k_1})_{proper} \quad \text{with} \quad (G \circ \tau_{k_1})_{proper} = u(u + \xi_1), \end{aligned}$$

where (i) $a = 0$ and $b = 1$ with $n_1 = 1$ imply that $bn_1 - ak_1 = 1$,

(ii) $e_{k_1} = \delta_1 + k_1 \delta_2 + k_1 d = n_1 \delta_1 + k_1 \delta_2 + n_1 k_1 d$, $\varepsilon = \delta_2$ and $\varepsilon_{\alpha\beta} = \beta$, as compared with (3.6.5).

Note by assumption in (3.6.1) that $\alpha + k_1 \beta - k_1 d = n_1 \alpha + k_1 \beta - n_1 k_1 d > 0$ because $n_1 = 1$. So, the proof of [I] is finished for Case (A).

The proof of [II] for Case(A): By (3.6.6) and (3.6.12-b), compare the following:

$$(3.6.15) \quad \begin{aligned} \tau_m(v', u') &= (y, z) = (v'^{n_1} u'^p, v'^{k_1} u'^q), \\ \tau_{k_1}(v', u') &= (y, z) = (v' u', v'^{k_1} u'^s), \end{aligned}$$

where m is, by definition, a finite number of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(G)$. Then, $m = k_1$, $n_1 = 1$, $p = 1$ and $q = s = k_1 - 1$ because $k_1 = s + 1$. So, $pk_1 - qn_1 = k_1 - s = 1$.

Since $n_1 = 1$, then apply (3.6.12-b) to (3.6.1) and (3.6.2), respectively. Or, we may apply (3.6.10-b) to (3.6.9), if convenient.

Then, along $v' = 0$, $(F \circ \tau_{k_1})_{total} = (F \circ \tau_{k_1})(v', u')$ can be written in the form

$$(3.6.16) \quad \begin{aligned} (F \circ \tau_{k_1})_{total} &= (v' u')^{\delta_1} (v'^{k_1} u'^s)^{\delta_2} \left(\prod_{i=1}^d (v'^{k_1} u'^s + \xi_i (v' u')^{k_1}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} (v' u')^\alpha (v'^{k_1} u'^s)^\beta \right) \\ &= v'^{e_{k_1}} u'^{\varepsilon'_1} (F \circ \tau_{k_1})_{proper} \quad \text{with} \quad (F \circ \tau_{k_1})_{proper} = (f \circ \tau_{k_1})_{proper}, \\ (f \circ \tau_{k_1})_{proper} &= \prod_{i=1}^d (1 + \xi_i u') + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v'^{\alpha + k_1 \beta - k_1 d} u'^{\lambda_s}, \\ (G \circ \tau_{k_1})_{total} &= v'^{2k_1} u'^{2s} (G \circ \tau_{k_1})_{proper} \quad \text{with} \quad (G \circ \tau_{k_1})_{proper} = 1 + \xi_1 u', \end{aligned}$$

where (i) $p = 1$ and $q = s = k_1 - 1$ imply that $pk_1 - qn_1 = 1$,

(ii) $e_{k_1} = \delta_1 + k_1 \delta_2 + k_1 d$ and $\varepsilon'_1 = \delta_1 + s \delta_2 + s d = e_s = e_{(k_1-1)} \geq 2$,

(iii) $\lambda_s = \alpha + s \beta - s d > 0$ by (4.3.1) or (3.6.9).

From (ii) of (3.6.16), if $d \geq 2$, then $\varepsilon'_1 \geq 2$. Also, if $d = 1$, then $\delta_2 > 0$ by assumption of the theorem, and so $\varepsilon'_1 \geq 2$. Thus, note by (3.6.16) that $(f \circ \tau_{k_1})_{proper} = (F \circ \tau_{k_1})_{proper}$. As compared with (3.6.6), $\varepsilon' = \delta_1 + (k_1 - 1)\delta_2 + (k_1 - 1)d = \delta_1 + s\delta_2 + sd = \varepsilon'_1$ and $\varepsilon'_{\alpha\beta} = \alpha + s\beta - sd = \lambda_s$, since $k_1 = s + 1$. Thus, the proof of [II] is done for Case (A).

The proof of [III] for Case(A): For brevity of notation in the conclusion of Theorem 3.6, recall that $\widehat{\delta}_2$ is defined as follows:

If δ_2 is positive, $\widehat{\delta}_2 = 1$, and if δ_2 is zero, $\widehat{\delta}_2 = 0$.

Now, to prove [III] for Case(A), note that $n_1 = 1$ and $m = k_1 = s + 1$ as in the proof of [I] and [II] for Case(A) where τ_m is the composition of a finite number m of successive blow-ups which is needed only to get the standard resolution of the singularity of $V(G)$.

First, $G \in$ the type[0] under τ_s by Lemma 4.3, and after $s + 1$ iterations of blow-ups, $E_{s+1} = E_m$ has three distinct intersection points with other exceptional curves and the proper transform $V^{(m)}(G)$ by (3.6.14) and (3.6.16), and so $G \in$ the type[1] under τ_m .

Next, as far as $V(F)$ is concerned, after s iterations of blow-ups, we proved by [III] of Lemma 4.3 that any irreducible component of $\cup_{i=1}^s E_i$ has at most two distinct intersection points with any other exceptional curves and $V^{(s)}(F)$, and that no three distinct components of $\cup_{i=1}^s E_i$ meet. Rigorously speaking, note by Lemma 4.3 or (3.6.9) that for any $s \geq 2$, $E_s \cap E_{s-1} \neq \emptyset$, $E_s \cap V^{(s)}(F)$ is a one-point subset and $E_s \cap E_{s-1} \cap V^{(s)}(F) = \emptyset$, and that for any $s \geq 1$, E_s and $V^{(s)}(F)$ meet transversely at $E_s \cap V^{(s)}(F)$, noting that $qs(V^{(s)}(F)) = qs(V^{(s)}(G)) = E_s \cap V^{(s)}(F)$ is a one-point subset under τ_s and that $F \in$ the type[0], as a reduced variety, and also $G \in$ the type[0] under τ_s .

Now, after k_1 iterations of blow-ups, by (3.6.14) and (3.6.16) $E_{k_1} = E_{s+1}$ has $(\mu + 1 + \widehat{\delta}_2)$ distinct intersection points with other exceptional curves and the proper transform $V^{(k_1)}(F)$, because E_{k_1} and $V^{(k_1)}(f)$ have μ distinct intersection points by (ii) of (3.6.1) in the assumption of this theorem, as a reduced variety. So, we get that $F \in$ the type[1] if and only if $\mu + 1 + \widehat{\delta}_2 = 3$, as a reduced variety. In particular, if $\mu = 1$, then $f \in$ the type[0] under τ_m , assuming that f has a singularity at the origin as a reduced variety.

(i) If $d = 1$, then $\mu = 1$ and also $\delta_2 > 0$ by assumption because $n_1 = 1$. In this case, F belongs to the type[1] under τ_m , as a reduced variety because $\mu + 1 + \widehat{\delta}_2 = 3$. In particular, if $n_1 = d = 1$, then it is clear that f has no singularity at the origin, and so f is irreducible in $\mathbb{C}\{y, z\}$.

(ii) Let $d \geq 2$. If $\mu = 1$, then it is trivial that $F \in$ the type[1] under τ_m if and only if $\widehat{\delta}_2 = 1$, as a reduced variety. If f is irreducible in $\mathbb{C}\{y, z\}$, then $\mu = 1$, but the converse does not hold, because $f = (z^2 + y^3)^2 + y^6 z^2$ is reducible in $\mathbb{C}\{y, z\}$, for example.

Thus, the proof of [III] is done, and so we finished the proof for Case(A).

Case(B): Let $n_1 \geq 2$. Using the same methods and results as we have seen in the proof of Lemma 4.3 and Case(A), it can be shown computationally by induction on the integer n_1 that Case(B) implies the truth of three statements [I], [II] and [III] where n_1 is the multiplicity of $g_1(y, z) = z^{n_1} + \xi_1 y^{k_1}$ at the origin. Thus, the proof of Case(B) is done.

Therefore, the proof of theorem is completely finished. \square

Chapter IV: How to construct the Puiseux convergent power series of the recursive type in $\mathbb{C}\{y, z\}$ (irreducible W-polys of two complex variables of recursive types in $\mathbb{C}\{y, z\}$)

§5. To find the necessary and sufficient condition for the semi-quasi-Puiseux convergent power series in $\mathbb{C}\{y, z\}$ of the recursive type to be irreducible in $\mathbb{C}\{y, z\}$

§5.0. Introduction

In order to succeed in the computation of Explicit algorithm, in this section it is very interesting and important to prove that we can construct the new terminology, irreducible Weierstrass polynomials of two complex variables of the recursive type, called “the standard Puiseux polynomial in $\mathbb{C}[y, z]$ of the recursive type” throughout this paper, which will be shown to be equivalent to the standard Puiseux expansion with Explicit Algorithm of §11, as far as the multiplicity sequences of irreducible plane curve singularities are concerned.

In preparation, we need Definition 5.0.0 and Theorem 5.0.

Definition 5.0.0. Let N_0 be the set of nonnegative integers and N_0^k be its k -dimensional copy. Let r be an arbitrary positive integer.

[A] $g_r \in \mathbb{C}\{y, z\}$ is called a semi-quasi-Puiseux convergent power series of the recursive r-type if there are sequences $\{\bar{X}_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$ satisfying the following four conditions:

Four conditions are denoted by The 1st Cond⁽⁰⁾, ..., The 4-th Cond⁽⁰⁾ for notation.

The 1st Cond⁽⁰⁾ Let $\{X_j : j = 1, 2, \dots, r\}$ with $X_j \subset N_0$ be defined as follows:

- (1) (1a) $X_1 = \{n_1, \beta_{1,1}\}$ with $n_1 \geq 2$ and $\beta_{1,1} \geq 1$.
- (1b) $X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with $n_j \geq 2$ where $j = 2, \dots, r$.

If $j \geq 2$, then assume that at least one of $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}$ is nonzero.

The 2nd Cond⁽⁰⁾ For each $j = 1, 2, \dots, r$, let $g_j = g_j(y, z)$ be in $\mathbb{C}\{y, z\}$, each of which is defined by the following way:

- (2) (2a) $g_1 = z^{n_1} + \varepsilon_1 y^{\beta_{1,1}}$.
- (2b) $g_j = g_{j-1}^{n_j} + \varepsilon_j y^{\beta_{j,1}} z^{\beta_{j,2}} g_1^{\beta_{j,3}} \dots g_{j-2}^{\beta_{j,j}}$ with $g_{-1} = y$ and $g_0 = z$, where $j = 2, \dots, r$.

Note that each $\varepsilon_i = \varepsilon_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq i \leq r$.

The 3rd Cond⁽⁰⁾ Let $\{\Delta_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r\}$ be a sequence such that each Δ_k is an integer-valued function defined by the following:

- (3) (3a) $\Delta_1(t) = t$ for each $t \in N_0$.
- (3b) $\Delta_j(t_j)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, r$.

The 4-th Cond⁽⁰⁾ Then, the following inequalities hold: Note that $2 \leq j \leq r$.

- (4) (4a) $\Delta_1(\beta_{1,1}) = \beta_{1,1} > 0$ with $n_1 \geq 2$.
- (4b) $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ where $j = 2, \dots, r$.

[B] Let $g_r \in \mathbb{C}\{y, z\}$ be a semi-quasi-Puiseux convergent power series of the recursive r-type as in [A].

There are two additional conditions, denoted by The 5-th Cond⁽⁰⁾ and The 6-th Cond⁽⁰⁾.

The 5-th Cond⁽⁰⁾ The following inequalities hold:

- (5)(5a) $\gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq r$.

The 6-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq r$.

(6a) $2 \leq n_1 < \beta_{1,1}$.

(6b) $n_j \geq 2$, $\beta_{j,1} > 0$, and $0 \leq \beta_{j,k} < n_{k-1}$ for $2 \leq j \leq r$ and $2 \leq k \leq j$.

Now, we define the new terminology in [B1], [B2], [B3] and [B4] of [B].

[B1] $g_r \in \mathbb{C}\{y, z\}$ is called the quasi-Puiseux convergent power series of the recursive r-type in $\mathbb{C}\{y, z\}$ if g_r in [A] satisfies an additional condition, denoted by The 5-th Cond⁽⁰⁾.

Namely, it is said that $g_r \in \mathbb{C}\{y, z\}$ is the quasi-Puiseux convergent power series of the recursive r-type if g_r satisfies The 1st Cond⁽⁰⁾, ..., The 4-th Cond⁽⁰⁾, The 5-th Cond⁽⁰⁾.

[B2] $g_r \in \mathbb{C}\{y, z\}$ is called the Puiseux convergent power series of the recursive r-type in $\mathbb{C}\{y, z\}$ if g_r in [A] satisfies The 5-th Cond⁽⁰⁾ and an inequality in (6a) of The 6-th Cond⁽⁰⁾.

[B3] $g_r \in \mathbb{C}\{y, z\}$ is called the standard Puiseux convergent power series of the recursive r-type in $\mathbb{C}\{y, z\}$ if g_r in [A] satisfies The 5-th Cond⁽⁰⁾ and The 6-th Cond⁽⁰⁾.

[B4] Let $g_r \in \mathbb{C}\{y, z\}$ be the standard Puiseux convergent power series of the recursive r-type as in [B3]. Then, g_r is called the standard Puiseux polynomial of the recursive r-type if each unit $\varepsilon_i = \varepsilon_i(y, z)$ is equal to an integer one for $1 \leq i \leq r$ in The 2-th Cond⁽⁰⁾ of [A].

Theorem 5.0(To find the necessary and sufficient condition for the semi-quasi-Puiseux convergent power series in $\mathbb{C}\{y, z\}$ of the recursive type to be irreducible in $\mathbb{C}\{y, z\}$).

Assumptions Let g_r be a semi-quasi-Puiseux convergent power series of the recursive r-type, satisfying the same properties and notations as in [A] of Definition 5.0.0.

Conclusions For each $j = 1, 2, \dots, r$, let $(0, 0)$ be the singularity of an analytic variety $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ except that $g_1 = z^{n_1} + \varepsilon_1 y^{\beta_{1,1}}$ with $\beta_{1,1} = 1$. Then, we get two statements [A] and [B] as follows:

$$\begin{aligned}
 [A] \quad & g_r \text{ is irreducible in } \mathbb{C}\{y, z\} \\
 \iff & g_1, g_2, \dots, g_{r-1} \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \gcd(n_r, \Delta_r(\beta_{r,k})_{k=1}^r) = 1 \\
 \iff & \gcd(n_1, \beta_{1,1}) = 1, \gcd(n_2, \Delta_2(\beta_{2,1}, \beta_{2,2})) = 1, \dots, \gcd(n_r, \Delta_r(\beta_{r,k})_{k=1}^r) = 1. \\
 \iff & g_r \text{ is the quasi-Puiseux convergent power series of the recursive r-type}
 \end{aligned}$$

[B] Let g_r be irreducible in $\mathbb{C}\{y, z\}$.

[B1] Let $V(y^\gamma g_r) = \{(y, z) : y^\gamma g_r(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by

$$(5.0.0) \quad y^\gamma g_r(y, z) \quad \text{such that} \quad \begin{cases} \gamma = 1, & \text{if } \beta_{1,1} = 1, \\ \gamma = 0, & \text{if } \beta_{1,1} > 1. \end{cases}$$

Then, $y^\gamma g_r \in$ the type $[r]$ under the standard resolution, denoted by τ , in the sense of Definition 2.5. Also, if $\beta_{1,1} = 1$ then $g_r \in$ the type $[r-1]$ under the standard resolution.

[B2] In particular, $z^\delta y g_r \in$ the type $[r]$ under the same standard resolution τ , whether δ is either one or zero. \square

§5.1. In preparation for the proof of Theorem 5.0

For the proof of Theorem 5.0, in this section, we will prepare the statements of five sublemmas, consisting of Sublemma 5.1, Sublemma 5.2, ..., Sublemma 5.5 without proofs. After then, we will finish the proofs of these sublemmas and Theorem 5.0 in §6.

Moreover, as a corollary of the above theorem and sublemmas, we can easily get Corollary 5.6 and Corollary 5.7 with no need of proofs.

Sublemma 5.1. Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 5.0 hold.

For any integer $r \geq 2$, let $\Delta_2^\sharp(\beta_{2,1}, \beta_{2,2})$ and $\Delta_j^\sharp(\beta_{j,k})_{k=1}^j$ with $3 \leq j \leq r$ be the notations defined as follows : Note that $\Delta_2(t_1, t_2) = n_1 t_1 + \beta_{1,1} t_2$ for each $(t_1, t_2) \in N_0^2$.

$$(5.1.1) \quad \begin{aligned} \Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) &= \Delta_2(\beta_{2,1}, \beta_{2,2}), \\ \Delta_j^\sharp(\beta_{j,k})_{k=1}^j &= \Delta_2(\beta_{j,1}, \beta_{j,2}) + n_1 \beta_{1,1} \beta_{j,3} + n_1 \beta_{1,1} n_2 \beta_{j,4} \\ &\quad + n_1 \beta_{1,1} n_2 n_3 \beta_{j,5} + \cdots + n_1 \beta_{1,1} n_2 \cdots n_{j-2} \beta_{j,j}. \end{aligned}$$

Conclusions Then, we have the following:

$$(5.1.2) \quad \begin{aligned} \Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) &> n_1 \beta_{1,1} n_2 \quad \text{on } g_2, \\ \Delta_j^\sharp(\beta_{j,k})_{k=1}^j &> n_1 \beta_{1,1} n_2 n_3 \cdots n_j \quad \text{on } g_j. \quad \square \end{aligned}$$

Sublemma 5.2. Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 5.0 hold. Let r be any integer with either $r \geq 2$ or $r = 1$.

Conclusions Then, we get the following:

(a) For any $r \geq 2$, $g_r = g_r(y, z)$ can be written in the form

$$(5.2.1) \quad g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1 \text{ and} \\ \text{with } n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r,$$

where a unit $\varepsilon_1 = \varepsilon_1(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β .

(b) For each $r \geq 2$, we have the following:

(b1) The multiplicity of $g_r(0, z)$ at $z = 0$ is $n_1 \prod_{k=2}^r n_k$ when $g_r = g_r(y, z)$.

(b2) The multiplicity of $g_r(y, 0)$ at $y = 0$ is $\beta_{1,1} \prod_{k=2}^r n_k$ when $g_r = g_r(y, z)$.

(c) For each $r \geq 2$, we have the following:

(c1) If $n_1 < \beta_{1,1}$ then $\alpha + \beta > n_1 n_2 \cdots n_r$, and so the multiplicity of g_r at $(y, z) = (0, 0)$ is $n_1 \prod_{k=2}^r n_k$.

(c2) If $n_1 > \beta_{1,1}$ then $\alpha + \beta > \beta_{1,1} n_2 n_3 \cdots n_r$, and so the multiplicity of g_r at $(y, z) = (0, 0)$ is $\beta_{1,1} \prod_{k=2}^r n_k$.

(d) If g_r is irreducible in $\mathbb{C}\{y, z\}$ for any $r \geq 2$, then either $\gcd(n_1, \beta_{1,1}) = 1$ or g_1 in The 2-th Cond⁽⁰⁾ is irreducible in $\mathbb{C}\{y, z\}$.

(e) In particular, from (5.2.1) let $h_1 = h_1(y, z)$ be defined in the form

$$(5.2.1.1) \quad h_1 = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}}) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(1)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1 \text{ and} \\ \text{with } n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1},$$

where a unit $\varepsilon_1 = \varepsilon_1(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(1)}$ are nonzero complex numbers for some nonnegative integers α and β , if exist.

Then, h_1 satisfies the same kind of results as g_r of (a) does in (b), (c) and (d). \square

Sublemma 5.3. Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 5.0 hold. Let r be an arbitrary integer with $r \geq 2$.

In addition, we need the following assumptions:

$$(5.3.0) \quad \gcd(n_1, \beta_{1,1}) = 1 \text{ or } g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}.$$

Since $\gcd(n_1, \beta_{1,1}) = 1$ with $n_1 \geq 2$ and $\beta_{1,1} \geq 1$, then there are two nonnegative integers $a > 0$ and $b \geq 0$ such that $a\beta_{1,1} - bn_1 = 1$.

For given two integers $a > 0$ and $b \geq 0$, let $\Omega_2 : N_0^2 \rightarrow N_0$ be a function defined by

$$(5.3.1) \quad \Omega_2(t_1, t_2) = at_1 + bt_2.$$

Let $\Omega_2^\#(\beta_{2,1}, \beta_{2,2})$ and $\Omega_j^\#(\beta_{j,k})_{k=1}^j$ with $3 \leq j \leq r$ be the notations defined as follows:

$$(5.3.2) \quad \Omega_2^\#(\beta_{2,1}, \beta_{2,2}) = \Omega_2(\beta_{2,1}, \beta_{2,2}).$$

$$\Omega_j^\#(\beta_{j,k})_{k=1}^j = \Omega_2(\beta_{j,1}, \beta_{j,2}) + bn_1\beta_{j,3} + bn_1n_2\beta_{j,4} + \cdots + bn_1n_2 \cdots n_{j-2}\beta_{j,j}.$$

Conclusions Then, we get the following:

$$(5.3.3) \quad \Omega_2^\#(\beta_{2,1}, \beta_{2,2}) \geq bn_1n_2,$$

$$\Omega_j^\#(\beta_{j,k})_{k=1}^j \geq bn_1n_2n_3 \cdots n_j. \quad \square$$

Sublemma 5.4. Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 5.0 hold. As in Sublemma 5.3, additionally assume that we have the following properties:

$$(*1) \quad g_1 \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ or } \gcd(n_1, \beta_{1,1}) = 1.$$

$$(*2) \quad g_r \text{ may not be irreducible in } \mathbb{C}\{y, z\} \text{ for some } r \geq 2, \text{ but note by The 4-th Cond}^{(0)} \text{ in the assumption of this theorem that } \Delta_r(\beta_{r,k})_{k=1}^r > n_r n_{r-1} \Delta_{r-1}(\beta_{r-1,k})_{k=1}^{r-1}.$$

Conclusions For each $j = 1, 2, \dots, r$, let $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ be an analytic variety at the origin in \mathbb{C}^2 . For the construction of the statement of the conclusion, let $V(G) = \{(y, z) : G(y, z) = 0\}$ be another analytic variety with isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(5.4.0) \quad g_1 = z^{n_1} + \varepsilon_1 y^{\beta_{1,1}} \quad \text{with a unit } \varepsilon_1 \in \mathbb{C}\{y, z\}, \\ G = y^\gamma g_1,$$

satisfying the properties (i) and (ii):

$$(i) \quad \text{If } \beta_{1,1} = 1, \text{ then } \gamma = 1.$$

$$(ii) \quad \text{If } \beta_{1,1} \geq 2, \text{ then } \gamma = 0.$$

Let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$. If $V(g_1)$ has the singular point at the origin, then as compared with the above τ_m , exactly the same τ_m can be also used for the standard resolution of the singular point of $V(yg_1)$ as far as the blow-ups process is concerned.

(a)(a1) We can use just one coordinate patch of the local coordinates for each blow-up π_i of τ_m with $1 \leq i \leq m$ in the sense of Lemma 2.12.

(a2) Just as above, we can use the same τ_m for the composition of the first finite number m of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of $V(g_j)$ for all $j = 2, 3, \dots, r$.

(a3) Also, we can use just the common one coordinate patch of the given local coordinates for each blow-up π_i of the above τ_m in (a1), in order to study any of $V^{(i)}(g_j)$ for all $j = 2, 3, \dots, r$ and all $i = 1, 2, \dots, m$ in the sense of Lemma 2.14.

(b) For brevity of notations, let (v, u) be the common one of the local coordinates for the m -th blow-up $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ at $(0, 0)$ which is the quasisingular point of $V^{(m-1)}(y^\gamma g_1)$ in the sense of Definition 2.6. Being viewed as an analytic mapping, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ can be written in the form

$$(5.4.1) \quad \tau_m(v, u) = (y, z) = (v^{n_1} u^a, v^{\beta_{1,1}} u^b),$$

where

- (b₁) $a > 0$ and $b \geq 0$ are some nonnegative integers such that $a\beta_{1,1} - bn_1 = 1$,
- (b₂) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.

(c) Use the same notations for $\Delta_q^\#(\beta_{q,k})_{k=1}^q$ and $\Omega_q^\#(\beta_{q,k})_{k=1}^q$ as in Sublemma 5.1, and Sublemma 5.3 where $\Omega_2 : N_0^2 \rightarrow N_0$ is a function defined by $\Omega_2(t_1, t_2) = at_1 + bt_2$ for given two nonnegative integers a and b in (b₁) of (5.4.1), and we may start with assuming that $\varepsilon_1 = 1$ in $V(y^\gamma g_1) = \{y^\gamma(z^{n_1} + \varepsilon_1 y^{\beta_{1,1}}) = 0\}$, in order to study $V^{(i)}(g_j)$ for all $i = 1, 2, \dots, m$, and all $j = 1, 2, \dots, r$. Whether $\beta_{1,1} \geq 2$ or $\beta_{1,1} = 1$, we may write that $(g_1 \circ \tau_m)_{proper} = (1 + \varepsilon_1 u)$ with $\varepsilon_1 = 1$, without complexity of the notation if necessary, noting that if $\beta_{1,1} = 1$ then $V(g_1)$ has no singularity at the origin.

Now, along $v = 0$, $(g_j \circ \tau_m)_{total}$ with $(g_j \circ \tau_m)_{proper}$ can be written as follows:

$$(5.4.2) \quad \begin{aligned} ((y^\gamma g_1) \circ \tau_m)_{total} &= v^{(\gamma + \beta_{1,1})n_1} u^{bn_1} ((y^\gamma g_1) \circ \tau_m)_{proper} \quad \text{with} \\ ((y^\gamma g_1) \circ \tau_m)_{proper} &= u^{a\gamma} (1 + u), \\ (g_1 \circ \tau_m)_{total} &= v^{n_1 \beta_{1,1}} u^{bn_1} (g_1 \circ \tau_m)_{proper} \quad \text{with} \\ (g_1 \circ \tau_m)_{proper} &= (1 + u), \\ (g_j \circ \tau_m)_{total} &= v^{n_1 \beta_{1,1} n_2 \cdots n_j} u^{bn_1 n_2 \cdots n_j} (g_j \circ \tau_m)_{proper} \quad \text{with} \\ (g_j \circ \tau_m)_{proper} &= (g_{j-1} \circ \tau_m)_{proper}^{n_j} + \{\varepsilon'_j v^{\Delta_j^\#(\beta_{j,k})_{k=1}^j - n_1 \beta_{1,1} n_2 \cdots n_j} \times \\ &\quad u^{\Omega_j^\#(\beta_{j,k})_{k=1}^j - bn_1 n_2 \cdots n_j} (1 + u)^{\beta_{j,3}} (g_2 \circ \tau_m)_{proper}^{\beta_{j,4}} \cdots (g_{j-2} \circ \tau_m)_{proper}^{\beta_{j,j}}\}, \end{aligned}$$

where each $\varepsilon'_j = \varepsilon_j \circ \tau_m(v, u)$ is a unit in $\mathbb{C}\{v, 1 + u\}$ for $2 \leq j \leq r$.

Note by Sublemma 5.1 and Sublemma 5.3 that for $j = 2, 3, \dots, r$,

$$(5.4.3) \quad \begin{aligned} \Delta_j^\#(\beta_{j,k})_{k=1}^j &> n_1 \beta_{1,1} n_2 \cdots n_j \quad \text{and} \\ \Omega_j^\#(\beta_{j,k})_{k=1}^j &\geq bn_1 n_2 n_3 \cdots n_j. \end{aligned}$$

Moreover, $(y^{\delta_{r,1}} z^{\delta_{r,2}} g_1^{\delta_{r,3}} g_2^{\delta_{r,4}} \cdots g_{r-2}^{\delta_{r,r}}) \circ \tau_m(v, u)$ can be viewed as

$$(5.4.4) \quad u^{\Omega_r^\#(\delta_{r,k})_{k=1}^r} v^{\Delta_r^\#(\delta_{r,k})_{k=1}^r} (1 + u)^{\delta_{r,3}} (g_2 \circ \tau_m)_{proper}^{\delta_{r,4}} \cdots (g_{r-2} \circ \tau_m)_{proper}^{\delta_{r,r}},$$

where

- (c₁) the $\delta_{r,i}$ are nonnegative integers for $1 \leq i \leq r$,
- (c₂) $\Delta_r^\#(\delta_{r,k})_{k=1}^r = \Delta_2(\delta_{r,1}, \delta_{r,2}) + n_1 \beta_{1,1} \delta_{r,3} + n_1 \beta_{1,1} n_2 \delta_{r,4} + \cdots + n_1 \beta_{1,1} n_2 \cdots n_{r-2} \delta_{r,r}$ as in the definition of $\Delta_r^\#(\beta_{r,k})_{k=1}^r$ of Sublemma 5.1,
- (c₃) $\Omega_r^\#(\delta_{r,k})_{k=1}^r = \Omega_2(\delta_{r,1}, \delta_{r,2}) + bn_1 \delta_{r,3} + bn_1 n_2 \delta_{r,4} + \cdots + bn_1 n_2 \cdots n_{r-2} \delta_{r,r}$ as in the definition of $\Omega_r^\#(\beta_{r,k})_{k=1}^r$ of Sublemma 5.3.

(d) Note again that τ_m is the composition of a finite number m of successive blow-ups, which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ or $V(g_1)$. Let $\tau_m^{-1}(0, 0) = \cup_{i=1}^m E_i$ where E_i is an exceptional curve of the first kind under τ_m . For $j = 1, 2, \dots, r$, let

$$(5.4.5) \quad (g_j \circ \tau_m)_{divisor} = V^{(m)}(g_j) + \sum_{i=1}^m e_{j,i} E_i,$$

where $V^{(m)}(g_j)$ is the proper transform of $V(g_j)$ under τ_m .

Then we have the following:

- (d1) If $\beta_{1,1} \geq 2$, then $e_{j+1,i} = n_{j+1} e_{j,i}$ for any $j \geq 1$ and for $i = 1, 2, \dots, m$.
If $\beta_{1,1} = 1$, then $e_{j+1,i} = n_{j+1} e_{j,i}$ for any $j \geq 2$ and for $i = 1, 2, \dots, m$.
In particular, $e_{j,m} = n_1 \beta_{1,1} n_2 \cdots n_j$ for $j = 2, \dots, r$, and $e_{1,m} = n_1 \beta_{1,1}$, if exists.

- (d2) $V^{(m)}(g_j) \cap (\cup_{i=1}^m E_i) = V^{(m)}(g_j) \cap E_m = \{(v, 1 + u) = (0, 0)\}$ for any $j = 2, \dots, r$.

- (d3) If $\beta_{1,1} \geq 2$, then for any $j = 1, 2, \dots, r$, $g_j \in$ the type [1] under τ_m .
If $\beta_{1,1} = 1$, then for any $j = 1, 2, \dots, r$, $g_j \in$ the type [0] under τ_m .

In particular, if $\beta_{1,1} \geq 1$, note that for all $j = 1, 2, \dots, r$, $z^\delta y g_j \in$ the type [1] under τ_m whether $\delta = 1$ or $\delta = 0$, by Theorem 3.6. \square

Sublemma 5.5. Assumptions Suppose that the same properties and notations as in the assumptions of Theorem 5.0 hold. As in Sublemma 5.3, additionally assume that we have the following properties:

$$(5.5.0) \quad \gcd(n_1, \beta_{1,1}) = 1 \text{ or } g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}.$$

Let r be an arbitrary positive integer with $r \geq 2$. Throughout this sublemma, we will use the same notations and consequences as in Sublemma 5.4, in order to get the representation for the conclusion of this sublemma.

Conclusions As $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ satisfies four conditions in the assumptions of Theorem 5.0, denoted by **The 1st Cond⁽⁰⁾**, \dots , **The 4-th Cond⁽⁰⁾**, then $\{(g_k \circ \tau_m)_{\text{proper}} : k = 2, 3, \dots, r\}$ with $(g_k \circ \tau_m)_{\text{proper}}$ in $\mathbb{C}\{v, 1+u\}$ satisfies the same kind of four conditions, which will be denoted by **The 1st Cond⁽¹⁾**, \dots , **The 4-th Cond⁽¹⁾**. Note that $\{(g_k \circ \tau_m)_{\text{proper}} : k = 2, 3, \dots, r\}$ has been already well-defined by Sublemma 5.4.

In more detail, in order to construct four conditions recursively, which will be denoted by **The 1st Cond⁽¹⁾**, \dots , **The 4-th Cond⁽¹⁾**, we prefer to add one more condition to the above four conditions, denoted by **The 5-th Cond⁽¹⁾**, for convenience of representation. By using the same kind of properties and notations as in Theorem 5.0, the desired construction is as follows:

There are sequences:

$$\begin{aligned} &\{Y_k : k = 1, 2, \dots, r-1\} \quad \text{with } Y_k \subset N_0, \\ &\{h_k : k = 1, 2, \dots, r-1\} \quad \text{with } h_k = (g_{k+1} \circ \tau_m)_{\text{proper}} \text{ in } \mathbb{C}\{v, u+1\} \text{ and} \\ &\{\Xi_k : N_0^k \rightarrow N_0 \text{ is an integer-valued function for } k = 1, 2, \dots, r-1\}, \\ &\text{satisfying the following four conditions for each } k: \end{aligned}$$

Such conditions are denoted by **The 1st Cond⁽¹⁾**, \dots , **The 4-th Cond⁽¹⁾**.

The 1st Cond⁽¹⁾: Let $\{Y_k : k = 1, 2, \dots, r-1\}$ with $Y_k \subset N_0$ be defined by

$$\begin{aligned} (1a) \quad &Y_1 = \{s_1, \gamma_{1,1}\} \quad \text{with } s_1 \geq 2 \text{ and } \gamma_{1,1} \geq 1, \\ (1b) \quad &Y_j = \{s_j, \gamma_{j,1}, \gamma_{j,2}, \dots, \gamma_{j,j}\} \quad \text{with } s_j \geq 2, \quad \text{where } 2 \leq j \leq r-1. \end{aligned}$$

such that for each $j = 1, 2, \dots, r-1$,

$$\begin{aligned} (5.5.1) \quad &e_{1,m} = n_1 \Delta_1(\beta_{1,1}) = n_1 \beta_{1,1}, \\ &s_j = n_{j+1} \geq 2, \quad \gamma_{j,1} = \Delta_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} - n_1 \beta_{1,1} n_2 n_3 \cdots n_{j+1} > 0, \\ &\gamma_{j,2} = \beta_{j+1,3}, \quad \gamma_{j,3} = \beta_{j+1,4}, \quad \dots, \gamma_{j,j} = \beta_{j+1,j+1}, \end{aligned}$$

noting that $\gamma_{1,1}, \gamma_{2,1}, \dots, \gamma_{r-1,1}$ are positive by Sublemma 5.1.

The 2nd Cond⁽¹⁾: Let $(g_2 \circ \tau_m)_{\text{proper}}, (g_3 \circ \tau_m)_{\text{proper}}, \dots, (g_r \circ \tau_m)_{\text{proper}}$ be denoted by h_1, h_2, \dots, h_{r-1} , respectively in $\mathbb{C}\{v, u+1\}$ as follows:

$$\begin{aligned} (2a) \quad &(g_2 \circ \tau_m)_{\text{total}} = v^{n_2 e_{1,m}} (g_2 \circ \tau_m)_{\text{proper}} \quad \text{with} \\ &(g_2 \circ \tau_m)_{\text{proper}} = (u+1)^{s_1} + \eta_1 v^{\gamma_{1,1}} \quad \text{with } \eta_1 = 1, \\ (2b) \quad &(g_j \circ \tau_m)_{\text{total}} = v^{n_j n_{j-1} \cdots n_2 e_{1,m}} (g_j \circ \tau_m)_{\text{proper}} \quad \text{with} \\ &(g_j \circ \tau_m)_{\text{proper}} = (g_{j-1} \circ \tau_m)_{\text{proper}}^{s_{j-1}} + \eta_{j-1} v^{\gamma_{j-1,1}} (1+u)^{\gamma_{j-1,2}} \\ &\quad \times (g_2 \circ \tau_m)_{\text{proper}}^{\gamma_{j-1,3}} \cdots (g_{j-2} \circ \tau_m)_{\text{proper}}^{\gamma_{j-1,j-1}}, \end{aligned}$$

where $\eta_i = \eta_i(v, u+1)$ is a unit in $\mathbb{C}\{v, u+1\}$ for $1 \leq i \leq r-1$, noting that $\eta_i = \varepsilon'_{i+1} u^{\Omega_{i+1}^\#(\beta_{i+1,k})_{k=1}^{i+1} - b n_1 n_2 \cdots n_{i+1}}$. Here, we may assume by a nonsingular change of coordinates that η_1 can be equal to an integer one for the standard resolution of the quasisingular point $(v, u+1) = (0, 0)$ of $V((g_k \circ \tau_m)_{\text{proper}})$ for $2 \leq k \leq r$.

The 3rd Cond⁽¹⁾: Let $\{\Xi_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k=1,2, \dots, r-1\}$ be a sequence defined by the following:

$$(3a) \quad \Xi_1(t) = t \text{ for each } t \in N_0.$$

$$(3b) \quad \Xi_{j-1}(t_k)_{k=1}^{j-1} = t_{j-1} \Xi_{j-2}(\gamma_{j-2,k})_{k=1}^{j-2} + s_{j-2} \Xi_{j-2}(t_k)_{k=1}^{j-2} \text{ for each } (t_k)_{k=1}^{j-1} \in N_0^{j-1}.$$

The (4 α)-th Cond⁽¹⁾: For each $q = 1, 2, 3, \dots, r-1$, we have the following:

$$(5.5.4\alpha) \quad \Xi_1(\gamma_{1,1}) = \gamma_{1,1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 > 0,$$

$$\begin{aligned} & \Xi_q(\gamma_{q,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \\ & = \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,k})_{k=1}^q > 0 \quad \text{for } 2 \leq q \leq r-1. \end{aligned}$$

The 4-th Cond⁽¹⁾: For each $q = 1, 2, 3, \dots, r-1$, we have the following:

$$(5.5.4) \quad \Xi_1(\gamma_{1,1}) = \gamma_{1,1} > 0,$$

$$\Xi_q(\gamma_{q,k})_{k=1}^q > s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \quad \text{for each } q = 2, 3, \dots, r-1.$$

The (5 α)-th Cond⁽¹⁾: For each $j = 1, 2, 3, \dots, r-1$, we have the following:

$$(5.5.5\alpha) \quad \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}).$$

Remark 5.5.1. For brevity of the proof of **Sublemma 5**, suppose that **The 1st Cond⁽¹⁾**, **The 2nd Cond⁽¹⁾** and **The 3rd Cond⁽¹⁾** have proved in the conclusions of this sublemma. Then, for the remaining proof, it suffices to show that **The (4 α)-th Cond⁽¹⁾** is true, because of the following two facts:

Fact(1): If **The 4 α -th Cond⁽¹⁾** is true, then it is clear that **The 4-th Cond⁽¹⁾** is true.

Fact(2): If **The 4 α -th Cond⁽¹⁾** is true, then we can easily prove by **The 1-th Cond⁽¹⁾** that **The 5 α -th Cond⁽¹⁾** is true, by using the following elementary computation:

- (i) $\gcd(s_1, \gamma_{1,1}) = \gcd(n_2, \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2) = \gcd(n_2, \Delta_2(\beta_{2,1}, \beta_{2,2}))$.
- (ii) For each $j = 2, 3, \dots, r-1$, $s_j = n_{j+1}$, and

$$\begin{aligned} (5.5.5\alpha) \quad & \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j - s_j s_{j-1} \Xi_{j-1}(\gamma_{j-1,k})_{k=1}^{j-1}) \\ & = \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} - n_{j+1} n_j \Delta_j(\beta_{j,k})_{k=1}^j) \quad \text{by (5.5.4}\alpha) \\ & = \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}). \quad \square \end{aligned}$$

Corollary 5.6. Assumptions Let $g_r \in \mathbb{C}\{y, z\}$ be a semi-quasi-Puiseux convergent power series of the recursive r-type, as either in [A] of Definition 5.0.0 or in the assumption of Theorem 5.0.

Conclusions For any $r \geq 2$, $g_r = g_r(y, z)$ can be written in the form

$$(5.6.1) \quad g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1,$$

where a unit $\varepsilon_1 = \varepsilon_1(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r$.

Corollary 5.7. Assumptions As we have seen in Definition 5.0.0, let $f \in \mathbb{C}\{y, z\}$ be a semi-quasi-Puiseux convergent power series of the recursive r-type.

Conclusions Then, f is irreducible in $\mathbb{C}\{y, z\}$ if and only if f is the quasi-Puiseux convergent power series of the recursive r-type in $\mathbb{C}\{y, z\}$.

Corollary 5.8. Assumptions Let $g_r \in \mathbb{C}\{y, z\}$ be a semi-quasi-Puiseux convergent power series of the recursive r-type, as either in [A] of Definition 5.0.0 or in the assumption of Theorem 5.0. In addition, assume that g_r is the quasi-Puiseux convergent power series of the recursive r-type in $\mathbb{C}\{y, z\}$. Let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$.

Conclusions Then, $(g_{r+1} \circ \tau_m)_{\text{proper}}$ in $\mathbb{C}\{v, 1+u\}$ is a quasi-Puiseux convergent power series of the recursive r-type satisfying four conditions as we have seen in Sublemma 5.5.

§6. The proofs of Theorem 5.0 with five sublemmas and corollaries in §5

§6.0. Introduction

In preparation for the proof of Theorem 5.0, first we prove five sublemmas, Sublemma 5.1, Sublemma 5.2, ..., Sublemma 5.5, respectively in §6.1. After then, we will finish the proof of Theorem 5.0 with corollaries in §6.2.

§6.1. The proofs of five sublemmas

Proof of Sublemma 5.1. If $r = 2$, then it is trivial to prove that $\Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) > n_1\beta_{1,1}n_2$ on g_2 , because $\Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) = \Delta_2(\beta_{2,1}, \beta_{2,2})$ by (5.1.1) and $\Delta_2(\beta_{2,1}, \beta_{2,2}) > n_1\beta_{1,1}n_2$ by The 4-th Cond⁽⁰⁾ in the assumptions of Theorem 5.0.

Let $r \geq 3$. For any $\ell = 3, 4, \dots, r$, it is trivial to note by definition of $\Delta_\ell^\sharp(\beta_{\ell,k})_{k=1}^\ell$ in (5.1.1) that the following three equalities are the same, and so we can write $c = \Delta_\ell^\sharp(\beta_{\ell,k})_{k=1}^\ell - n_1\beta_{1,1}n_2n_3 \cdots n_\ell$ for convenience of notation:

$$(5.1.3) \quad c = \Delta_\ell^\sharp(\beta_{\ell,k})_{k=1}^\ell - n_1\beta_{1,1}n_2n_3 \cdots n_\ell$$

$$\Longleftrightarrow$$

$$(5.1.4) \quad c = \Delta_2(\beta_{\ell,1}, \beta_{\ell,2}) + n_1\beta_{1,1}\beta_{\ell,3} + n_1\beta_{1,1}n_2\beta_{\ell,4} + n_1\beta_{1,1}n_2n_3\beta_{\ell,5} \\ + \cdots + n_1\beta_{1,1}n_2 \cdots n_{\ell-2}\beta_{\ell,\ell} - n_1\beta_{1,1}n_2n_3 \cdots n_\ell$$

$$\Longleftrightarrow$$

$$(5.1.5) \quad c = \Delta_2(\beta_{\ell,1}, \beta_{\ell,2}) - (n_\ell n_{\ell-1} \cdots n_2 - \beta_{\ell,\ell} n_{\ell-2} n_{\ell-3} \cdots n_2 \\ - \beta_{\ell,\ell-1} n_{\ell-3} n_{\ell-4} \cdots n_2 - \cdots - \beta_{\ell,4} n_2 - \beta_{\ell,3}) n_1 \beta_{1,1}.$$

So, for any integer $\ell \geq 3$, it suffices to show that the above integer c of (5.1.5) can be equal to an integer $\xi_{\ell-2} > 0$ such that $\xi_{\ell-2}$ is the $(\ell-2)$ -th element of a positive sequence $\{\xi_j : j = 1, 2, \dots, \ell-2\}$, where each ξ_j satisfies the following properties:

$$(5.1.6) \quad \begin{aligned} \xi_0 &= \Delta_\ell(\beta_{\ell,k})_{k=1}^\ell - \{n_\ell\} n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1} > 0, \\ \xi_1 &= \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} - \{n_\ell n_{\ell-1} - \beta_{\ell,\ell}\} n_{\ell-2} \Delta_{\ell-2}(\beta_{\ell-2,k})_{k=1}^{\ell-2} > 0, \\ \xi_2 &= \Delta_{\ell-2}(\beta_{\ell,k})_{k=1}^{\ell-2} - \{n_\ell n_{\ell-1} n_{\ell-2} - \beta_{\ell,\ell} n_{\ell-2} - \beta_{\ell,\ell-1}\} n_{\ell-3} \Delta_{\ell-3}(\beta_{\ell-3,k})_{k=1}^{\ell-3} > 0, \\ \xi_j &= \Delta_{\ell-j}(\beta_{\ell,k})_{k=1}^{\ell-j} - \{n_\ell n_{\ell-1} \cdots n_{\ell-j} - \beta_{\ell,\ell} n_{\ell-2} n_{\ell-3} \cdots n_{\ell-j} \\ &\quad - \beta_{\ell,\ell-1} n_{\ell-3} n_{\ell-4} \cdots n_{\ell-j} - \cdots - \beta_{\ell,\ell-j+2} n_{\ell-j} \\ &\quad - \beta_{\ell,\ell-j+1}\} \times n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1} > 0 \quad \text{for } 3 \leq j \leq \ell-2. \end{aligned}$$

Let $\ell \geq 3$ be chosen arbitrary. Now, we will show by the induction method on the nonnegative integer $j \leq \ell-2$ that ξ_j is positive for all j .

It is trivial by (d) in the assumption of Theorem 5.0 that $\xi_0 > 0$.

To prove that ξ_1 is positive, first of all, it is easy to observe the following by the definition of $\Delta_\ell(\beta_{\ell,k})_{k=1}^\ell$:

$$(5.1.7) \quad \begin{aligned} 0 < \xi_0 &= \Delta_\ell(\beta_{\ell,k})_{k=1}^\ell - n_\ell n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1} \\ &= \beta_{\ell,\ell} \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1} + n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} - n_\ell n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1} \\ &= n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} - (n_\ell n_{\ell-1} - \beta_{\ell,\ell}) \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1}. \end{aligned}$$

Since $\Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1} > n_{\ell-1} n_{\ell-2} \Delta_{\ell-2}(\beta_{\ell-2,k})_{k=1}^{\ell-2}$ by The 4-th Cond⁽⁰⁾ in the assumption of Theorem 5.0, then the third inequality of (5.1.7) and $\beta_{\ell,\ell} < n_{\ell-1}$ imply that

$$(5.1.8) \quad \begin{aligned} n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} &> (n_\ell n_{\ell-1} - \beta_{\ell,\ell}) \Delta_{\ell-1}(\beta_{\ell-1,k})_{k=1}^{\ell-1}, \quad \text{and so} \\ n_{\ell-1} \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} &> (n_\ell n_{\ell-1} - \beta_{\ell,\ell}) n_{\ell-1} n_{\ell-2} \Delta_{\ell-2}(\beta_{\ell-2,k})_{k=1}^{\ell-2}, \end{aligned}$$

whether or not $n_\ell n_{\ell-1} - \beta_{\ell,\ell} > 0$.

Dividing both sides on (5.1.8) by $n_{\ell-1}$, then

$$(5.1.9) \quad \Delta_{\ell-1}(\beta_{\ell,k})_{k=1}^{\ell-1} > (n_{\ell}n_{\ell-1} - \beta_{\ell,\ell})n_{\ell-2}\Delta_{\ell-2}(\beta_{\ell-2,k})_{k=1}^{\ell-2},$$

which is equivalent to the fact that $\xi_1 > 0$.

By the induction assumption on the positive integer $j \leq \ell - 2$, suppose we have shown that ξ_j is positive with $1 \leq j \leq \ell - 3$. To prove that ξ_{j+1} is positive, for convenience of notations, let ξ_j of (5.1.6) be written again in the form

$$(5.1.10) \quad \begin{aligned} \xi_j &= \Delta_{\ell-j}(\beta_{\ell,k})_{k=1}^{\ell-j} - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1} > 0 \quad \text{with} \\ \omega_j &= n_{\ell}n_{\ell-1} \cdots n_{\ell-j} - \beta_{\ell,\ell}n_{\ell-2}n_{\ell-3} \cdots n_{\ell-j} \\ &\quad - \beta_{\ell,\ell-1}n_{\ell-3}n_{\ell-4} \cdots n_{\ell-j} - \cdots - \beta_{\ell,\ell-j+2}n_{\ell-j} - \beta_{\ell,\ell-j+1}. \end{aligned}$$

Now, by (5.1.10) and the definition of $\Delta_{\ell-j}(\beta_{\ell,k})_{k=1}^{\ell-j}$ only, it is easy to prove that

$$(5.1.11) \quad \begin{aligned} 0 < \xi_j &= \Delta_{\ell-j}(\beta_{\ell,k})_{k=1}^{\ell-j} - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1} \\ &= \beta_{\ell,\ell-j} \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1} + n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} \\ &\quad - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1} \\ &= n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} - (\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j}) \Delta_{\ell-j-1}(\beta_{\ell-j-1,k})_{k=1}^{\ell-j-1}. \end{aligned}$$

Since $\Delta_{r-j-1}(\beta_{r-j-1,k})_{k=1}^{r-j-1} > n_{r-j-1}n_{r-j-2}\Delta_{r-j-2}(\beta_{r-j-2,k})_{k=1}^{r-j-2}$ by The 4-th Cond⁽⁰⁾ in the assumption of Theorem 5.0, then we get the following from the last equality in (5.1.11):

$$(5.1.12) \quad \begin{aligned} n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} &> (\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j}) n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} \text{ and so} \\ n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} &> (\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j}) n_{\ell-j-1} n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,k})_{k=1}^{\ell-j-2}, \end{aligned}$$

whether or not $\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j} > 0$.

Dividing both sides of (5.1.12) by $n_{\ell-j-1}$, then we get

$$(5.1.13) \quad \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} > (\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j}) n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,k})_{k=1}^{\ell-j-2}.$$

Before we prove that $\xi_{j+1} > 0$, then it is trivial to observe by (5.1.10) that ξ_{j+1} of (5.1.6) can be rewritten as follows:

$$(5.1.14) \quad \xi_{j+1} = \Delta_{\ell-j-1}(\beta_{\ell,k})_{k=1}^{\ell-j-1} - (\omega_j n_{\ell-j-1} - \beta_{\ell,\ell-j}) n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,k})_{k=1}^{\ell-j-2}.$$

Now, we can show by (5.1.13) that ξ_{j+1} is positive. Thus, the proof is done. \square

Proof of Sublemma 5.2. We will prove (a), (b), (c), (d) and (e), respectively.

(a) In preparation for the proof of an equality in (5.2.1), it suffices to show that for any integer $r \geq 2$, $g_r = g_r(y, z)$ in the assumption of this theorem can be generally represented in the following form:

$$(5.2.2) \quad \begin{aligned} g_r &= \Sigma_{r,0} + \Sigma_{r,1}, \\ \Sigma_{r,0} &= (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r} = (g_1)^{n_2 n_3 \cdots n_r} \quad \text{with } \varepsilon_1 = 1 \text{ or a unit in } \mathbb{C}\{y, z\}, \\ \Sigma_{r,1} &= \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(1)} y^{\gamma} z^{\delta} \quad \text{with } n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r, \quad \text{and} \end{aligned}$$

$$(5.2.3) \quad \begin{aligned} g_r^{\ell} &= \Sigma_{r,0}^{(\ell)} + \Sigma_{r,1}^{(\ell)} \quad \text{for any integer } \ell \geq 2, \\ \Sigma_{r,0}^{(\ell)} &= (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r \ell} = (g_1)^{n_2 n_3 \cdots n_r \ell} = (\Sigma_{r,0})^{\ell}, \\ \Sigma_{r,1}^{(\ell)} &= \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(\ell)} y^{\gamma} z^{\delta} \quad \text{with } n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r \ell, \end{aligned}$$

where $\varepsilon_1 = \varepsilon_1(y, z)$ is assumed to be one in $\mathbb{C}\{y, z\}$ if necessary, and the $c_{\gamma, \delta}^{(1)}$ are nonzero complex numbers for some nonnegative integers γ and δ , if exists, and the $c_{\gamma, \delta}^{(\ell)}$ are nonzero complex numbers for some nonnegative integers γ and δ if exists.

For the induction proof of the equalities in (5.2.2) and (5.2.3), it suffices to consider the following two cases:

Case(I) $r = 2$ and Case(II) $r > 2$.

Case(I) Assuming that $r = 2$, recall by The 2nd Cond⁽⁰⁾ and The 4-th Cond⁽⁰⁾ in the assumptions of this theorem and by (5.1.2) of Sublemma 5.1 that g_2 can be written in the form

$$(5.2.4) \quad g_2 = g_1^{n_2} + \varepsilon_2 y^{\beta_{2,1}} z^{\beta_{2,2}} \quad \text{and} \\ \Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) = \Delta_2(\beta_{2,1}, \beta_{2,2}) = n_1 \beta_{2,1} + \beta_{1,1} \beta_{2,2} > n_1 \beta_{1,1} n_2 \quad \text{on } g_2,$$

where ε_2 is a unit in $\mathbb{C}\{y, z\}$.

In order to prove the equalities in (5.2.2) and (5.2.3) for $r = 2$, it remains to show that the equality in (5.2.3) holds, because the proof of the equality in (5.2.2) was already proved by (5.2.4).

So, in preparation for the proof of the equality in (5.2.3) with $r = 2$, then it is clear by (5.2.2) or (5.2.4) that $g_2(y, z)$ can be rewritten in the form

$$(5.2.5) \quad g_2 = \Sigma_{2,0} + \Sigma_{2,1}, \\ \Sigma_{2,0} = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2} \quad \text{with } \varepsilon_1 = 1 \text{ or a unit in } \mathbb{C}\{y, z\}, \\ \Sigma_{2,1} = \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(1)} y^\gamma z^\delta \quad \text{with } n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2,$$

where the $c_{\gamma, \delta}^{(1)}$ are nonzero complex numbers for some nonnegative integers γ and δ .

For the proof of an inequality in (5.2.3), note by (5.2.5) that g_2^ℓ can be written in the form

$$(5.2.6) \quad g_2^\ell = (\Sigma_{2,0} + \Sigma_{2,1})^\ell \quad \text{for any integer } \ell \geq 2 \\ = (\Sigma_{2,0})^\ell + \sum_{k=1}^{\ell-1} \binom{\ell}{k} (\Sigma_{2,0})^k (\Sigma_{2,1})^{\ell-k} + (\Sigma_{2,1})^\ell \\ = \Sigma_{2,0}^{(\ell)} + \Sigma_{2,1}^{(\ell)},$$

where $\Sigma_{2,0}^{(\ell)} = (\Sigma_{2,0})^\ell = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 \ell} \quad \text{with } \varepsilon_1 = 1 \text{ or a unit in } \mathbb{C}\{y, z\},$

$$\Sigma_{2,1}^{(\ell)} = \sum_{k=1}^{\ell-1} \binom{\ell}{k} (\Sigma_{2,0})^k (\Sigma_{2,1})^{\ell-k} + (\Sigma_{2,1})^\ell \quad \text{for notation.}$$

To prove that an inequality in (5.2.3) is true for $r = 2$, apply the equalities of (5.2.5) and Sublemma 5.1 to equalities of (5.2.6). After then, it suffices to show that for any nonzero monomial $y^\alpha z^\beta \in \Sigma_{2,1}^{(\ell)}$,

$$(5.2.7) \quad n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 \ell.$$

To prove (5.2.7), it is enough to show that the following two claims hold by using the defining equation for $\Sigma_{2,1}^{(\ell)}$ in (5.2.6):

$$(5.2.8) \quad \text{Claim(i)} \quad \text{For any monomial } y^\alpha z^\beta \in (\Sigma_{2,1})^\ell, \\ n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 \ell. \\ \text{Claim(ii)} \quad \text{For any monomial } y^\gamma z^\delta \in (\Sigma_{2,0})^k (\Sigma_{2,1})^{\ell-k}, \\ n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 k + n_1 \beta_{1,1} n_2 (\ell - k) = n_1 \beta_{1,1} n_2 \ell.$$

Note by (5.2.5) that the proof of two claims in (5.2.8) is trivial because $\ell - k > 0$, and so the proof of (5.2.3) for $r = 2$ is done. So, the proof of Case(I) is done.

Case(II) Let $r > 2$. Now, suppose we have proved by induction assumption on the positive integer $j < r$ that the representation of g_j^ℓ in (5.2.2) and (5.2.3) is true for $2 \leq j < r$ and for any integer $\ell \geq 1$. Then, recall by The 2nd Cond⁽⁰⁾ and The 4-th Cond⁽⁰⁾ in the assumptions of this theorem and by (5.1.2) of Sublemma 5.1 that g_{j+1} can be rewritten as follows:

$$(5.2.9) \quad g_{j+1} = g_j^{n_{j+1}} + \varepsilon_{j+1} y^{\beta_{j+1,1}} z^{\beta_{j+1,2}} g_1^{\beta_{j+1,3}} \cdots g_{j-1}^{\beta_{j+1,j+1}} \quad \text{with} \\ \Delta_{j+1}^\sharp (\beta_{j+1,k})_{k=1}^{j+1} > n_1 \beta_{1,1} n_2 n_3 \cdots n_j n_{j+1} \quad \text{on } g_{j+1},$$

where ε_{j+1} is defined to be a unit in $\mathbb{C}\{y, z\}$ by (5.2.9).

Now, applying the induction assumption on the integer j , we may assume that for each $k = 1, 2, \dots, j$, and any integer $\ell > 0$, we have

$$(5.2.10) \quad g_k^\ell = \Sigma_{k,0}^{(\ell)} + \Sigma_{k,1}^{(\ell)} \quad \text{for any integer } \ell \geq 2, \\ \Sigma_{k,0}^{(\ell)} = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_k \ell} = (g_1)^{n_2 n_3 \cdots n_k \ell} = (\Sigma_{k,0}^{(1)})^\ell, \\ \Sigma_{k,1}^{(\ell)} = \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(\ell)} y^\gamma z^\delta \quad \text{with} \quad n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 n_3 \cdots n_k \ell,$$

where the $c_{\gamma, \delta}^{(\ell)}$ are nonzero complex numbers for some nonnegative integers γ and δ .

To prove that an inequality in (5.2.2) is true for $k = j + 1$, apply the equalities of (5.2.10) and Sublemma 5.1 to the equalities of g_{j+1} in (5.2.9). Since it is clear by (5.2.10) that $g_j^{n_{j+1}}$ can be written in the form

$$(5.2.11) \quad g_j^{n_{j+1}} = \Sigma_{j,0}^{(n_{j+1})} + \Sigma_{j,1}^{(n_{j+1})} \quad \text{for any integer } n_{j+1} \geq 2, \\ \Sigma_{j,0}^{(n_{j+1})} = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_j n_{j+1}} = (g_1)^{n_2 n_3 \cdots n_j n_{j+1}}, \\ \Sigma_{j,1}^{(n_{j+1})} = \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(n_{j+1})} y^\gamma z^\delta \quad \text{with} \quad n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 n_3 \cdots n_j n_{j+1},$$

where the $c_{\gamma, \delta}^{(n_{j+1})}$ are nonzero complex numbers for some nonnegative integers γ and δ .

For the proof of an inequality in (5.2.2) for $k = j + 1$, applying (5.2.11) to (5.2.9), it remains to show the following: for any nonzero monomial $y^\alpha z^\beta \in g_{j+1} - g_j^{n_{j+1}}$,

$$(5.2.12) \quad n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_j n_{j+1}.$$

First of all, it is clear by (5.2.10) that for any nonzero monomial $y^{\alpha_k} z^{\beta_k} \in g_k^\ell$,

$$(5.2.13) \quad n_1 \alpha_k + \beta_{1,1} \beta_k \geq n_1 \beta_{1,1} n_2 n_3 \cdots n_k \ell \quad \text{for } k = 1, 2, \dots, j.$$

So, in order to prove that either (5.2.2) or (5.2.12) is true on g_{j+1} , it is clear by (5.2.9), (5.2.10) and (5.2.13) that any nonzero monomial $y^\gamma z^\delta \in \frac{1}{\varepsilon_{j+1}} \{g_{j+1} - g_j^{n_{j+1}}\}$ can be represented as follows:

$$(5.2.14) \quad y^\gamma z^\delta = y^{\beta_{j+1,1}} z^{\beta_{j+1,2}} \prod_{k=1}^{j-1} y^{\alpha_k} z^{\beta_k} \quad \text{such that} \\ n_1 \alpha_k + \beta_{1,1} \beta_k \geq n_1 \beta_{1,1} n_2 n_3 \cdots n_k \beta_{j+1,k+2} \quad \text{for some } y^{\alpha_k} z^{\beta_k} \in g_k^{\beta_{j+1,k+2}},$$

where ε_{j+1} is a unit in $\mathbb{C}\{y, z\}$.

In other words, whenever $y^\gamma z^\delta \in \frac{1}{\varepsilon_{j+1}}\{g_{j+1} - g_j^{n_{j+1}}\}$ is chosen arbitrary, then it may be assumed by (5.2.14) that

$$(5.2.15) \quad \begin{aligned} \gamma &= \beta_{j+1,1} + \alpha_1 + \cdots + \alpha_{j-1} \quad \text{and} \quad \delta = \beta_{j+1,2} + \beta_1 + \cdots + \beta_{j-1}, \\ \text{where} \quad &\text{for each } k = 1, 2, \dots, j-1, y^{\alpha_k} z^{\beta_k} \in g_k^{\beta_{j+1,k+2}} \\ &\text{with } n_1 \alpha_k + \beta_{1,1} \beta_k \geq n_1 \beta_{1,1} n_2 n_3 \cdots n_k \beta_{j+1,k+2}. \end{aligned}$$

Therefore, by (5.2.14) and (5.2.15) and by Sublemma 5.1 again, for any nonzero monomial $y^\gamma z^\delta \in \frac{1}{\varepsilon_{j+1}}\{g_{j+1} - g_j^{n_{j+1}}\}$, we can prove the following:

$$(5.2.16) \quad \begin{aligned} n_1 \gamma + \beta_{1,1} \delta &= n_1 \beta_{j+1,1} + \beta_{1,1} \beta_{j+1,2} + \sum_{k=1}^{j-1} (n_1 \alpha_k + \beta_{1,1} \beta_k) \\ &\geq n_1 \beta_{j+1,1} + \beta_{1,1} \beta_{j+1,2} + \sum_{k=1}^{j-1} (n_1 \beta_{1,1} n_2 n_3 \cdots n_k \beta_{j+1,k+2}) \\ &= \Delta_{j+1}^\sharp (\beta_{j+1,k})_{k=1}^{j+1} > n_1 \beta_{1,1} n_2 n_3 \cdots n_j n_{j+1}. \end{aligned}$$

Thus, if $r = j + 1$, then the proof of (5.2.2) is done by (5.2.11) and (5.2.16). Also, if $r = j + 1$, then the proof of (5.2.3) is trivial by the same method as we have used in the proof for (5.2.3) with $r = 2$. Then, the proof of Case(II) is done.

So, we finished the proof of (a) by Case(I) and Case(II).

(b) To prove (b1), it suffices to consider $g_r(0, z)$ from $g_r(y, z)$ of (5.2.1). Then

$$(5.2.17) \quad \begin{aligned} g_r(0, z) &= z^{n_1 n_2 \cdots n_r} + \sum c_{0,\beta}^{(r)} z^\beta \quad \text{with} \\ \beta_{1,1} \beta &> n_1 \beta_{1,1} n_2 n_3 \cdots n_r. \end{aligned}$$

Thus, $\beta > n_1 n_2 \cdots n_r$, and so it is done. Also, the proof of (b2) can be done similarly.

(c) To prove (c1), suppose that $n_1 < \beta_{1,1}$. If then, by (a), $\beta_{1,1}(\alpha + \beta) \geq n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r$, which implies that $\alpha + \beta > n_1 \prod_{k=2}^r n_k$. Thus, the proof of (c1) is done. Similarly, (c2) can be proved.

(d) By (a) and Theorem 3.6, it is clear.

(e) There is nothing to prove. Thus, the proof of this sublemma is finished. \square

Proof of Sublemma 5.3. To prove (5.3.3) for any $r \geq 2$, it is enough to show by (5.3.2) that the following equation in (5.3.4) is nonnegative:

$$(5.3.4) \quad \begin{aligned} &\Omega_r^\sharp(\beta_{r,k})_{k=1}^r - b n_1 n_2 n_3 \cdots n_r \\ &= a \beta_{r,1} + b \beta_{r,2} + b n_1 \beta_{r,3} + b n_1 n_2 \beta_{r,4} + \cdots + b n_1 n_2 \cdots n_{r-2} \beta_{r,r} - b n_1 n_2 n_3 \cdots n_r \\ &= a \beta_{r,1} + b D \geq 0 \quad \text{with} \\ &D = \beta_{r,2} + n_1 \beta_{r,3} + n_1 n_2 \beta_{r,4} + \cdots + n_1 n_2 \cdots n_{r-2} \beta_{r,r} - n_1 n_2 n_3 \cdots n_r, \end{aligned}$$

by definition of $\Omega_r^\sharp(\beta_{r,k})_{k=1}^r$ in (5.3.2) where $a > 0$ and $b \geq 0$.

Then, it remains to prove by (5.3.4) that $a \beta_{r,1} + b D \geq 0$, independently of D . For the proof, it suffices to consider two cases, Case(i) $D \geq 0$ and Case(ii) $D < 0$.

Case(i) Let $D \geq 0$. It is clear that $a \beta_{r,1} + b D \geq 0$ because $a > 0$, and also $b \geq 0$ with $\beta_{r,1}$ nonnegative. Thus, the proof of Case(i) is done.

Case(ii) Let $D < 0$. In preparation for proof of the inequality, first of all, note that the inequality $\Delta_r^\#(\beta_{r,k})_{k=1}^r - n_1\beta_{1,1}n_2n_3 \cdots n_r > 0$ of Sublemma 5.1 can be equivalently rewritten as follows:

$$\begin{aligned}
(5.3.5) \quad & \Delta_r^\#(\beta_{r,k})_{k=1}^r - n_1\beta_{1,1}n_2n_3 \cdots n_r \\
&= n_1\beta_{r,1} + \beta_{1,1}\beta_{r,2} + n_1\beta_{1,1}\beta_{r,3} + n_1\beta_{1,1}n_2\beta_{r,4} \\
&\quad + \cdots + n_1\beta_{1,1}n_2 \cdots n_{r-2}\beta_{r,r} - n_1\beta_{1,1}n_2n_3 \cdots n_r \quad \text{by (5.1.1)} \\
&= n_1\beta_{r,1} + \beta_{1,1}D \quad \text{is positive,} \\
&\text{where } D = \beta_{r,2} + n_1\beta_{r,3} + n_1n_2\beta_{r,4} + \cdots + n_1n_2 \cdots n_{r-2}\beta_{r,r} - n_1n_2n_3 \cdots n_r \text{ by (5.3.4).}
\end{aligned}$$

Since $-D > 0$ and $n_1 \geq 2 > 0$, then the inequality $n_1\beta_{r,1} + \beta_{1,1}D > 0$ in (5.3.5) can be equivalently represented as follows:

$$(5.3.6) \quad \frac{\beta_{r,1}}{-D} > \frac{\beta_{1,1}}{n_1}.$$

Also, $a\beta_{1,1} - bn_1 = 1$ with $a > 0$ implies that $\frac{\beta_{1,1}}{n_1} > \frac{b}{a}$. Therefore, we proved by (5.3.6)

that $\frac{\beta_{r,1}}{-D} > \frac{b}{a}$, that is, $a\beta_{r,1} + bD > 0$. Thus, the proof of Case(ii) is done.

Therefore, we showed by Case(i) and Case(ii) that the equation in (5.3.4) is nonnegative, and so the proof of this sublemma is finished. \square

Proof of Sublemma 5.4. Following the same assumptions and notations as in Sublemma 5.1, Sublemma 5.2 and Sublemma 5.3, then (a) in the conclusion of Sublemma 5.2 is true, and so for each $j = 2, 3, \dots, r$, $g_j = g_j(y, z)$ of (5.2.1) can be easily rewritten as follows:

$$\begin{aligned}
(5.4.6) \quad & g_j = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1 \text{ and} \\
& \text{with } n_1\alpha + \beta_{1,1}\beta > n_1\beta_{1,1}n_2n_3 \cdots n_j,
\end{aligned}$$

where $\varepsilon_1 = \varepsilon_1(y, z)$ is assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + \beta_{1,1}\beta > n_1\beta_{1,1}n_2n_3 \cdots n_j$.

First, we will show how to apply Theorem 3.6 to the proof of (a), (b) and (d) in this sublemma, and after then, the remaining part (c) of this sublemma will be proved computationally.

In preparation for the proof of (a), (b) and (d) in this sublemma, it is clear that the equation of g_j of (5.4.6) satisfies the same kind of properties as f does in the assumption of Theorem 3.6, which can be represented as follows:

g_j of (5.4.6) satisfies the same kind of assumption as in Theorem 3.6 Let $V(g_1) = \{(y, z) : g_1(y, z) = 0\}$, $V(f) = \{(y, z) : f(y, z) = 0\}$ and $V(G) = \{(y, z) : G(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively as follows: For convenience of notation, substitute g_j of (5.4.6) by f , for an application of Theorem 3.6.

$$\begin{aligned}
(5.4.7) \quad & g_1 = z^{n_1} + \varepsilon_1 y^{\beta_{1,1}} \quad \text{with } \varepsilon_1 = 1, \\
& f = g_1^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} y^\alpha z^\beta \quad \text{with } n_1\alpha + \beta_{1,1}\beta > n_1k_1d_j, \\
& F = y^{\delta_1} z^{\delta_2} f, \\
& G = y^\gamma g_1
\end{aligned}$$

satisfying the properties (i), (ii), (iii), (iv) and (v):

- (i) $\gcd(n_1, \beta_{1,1}) = 1$ with $n_1 \geq 2$ and $\beta_{1,1} \geq 1$.
- (ii) $d_j = n_2n_3 \cdots n_j$ is a positive integer with $d_j \geq 2$.
- (iii) ε_1 is assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + k_1\beta > n_1k_1d_j$, if exist.
- (iv) Assume that $V(f)$ has an isolated singular point at the origin as a reduced variety.
- (v) If $\beta_{1,1} = 1$, then $\gamma = 1$, and if $\beta_{1,1} \geq 2$, then $\gamma = 0$.
- (vi) In addition, assume that each δ_i is either a positive integer or 0 for $i = 1, 2$, as far as $V(F)$ has an isolated singular point at the origin as a reduced variety, even if $d_j \geq 1$.

So, we have the same kind of conclusion as we have seen in Theorem 3.6, up to change of notations:

The same kind of conclusion as in Theorem 3.6 Let $\tau_m = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(G) = V(y^\gamma g_1)$.

Therefore, by the conclusion of Theorem 3.6, there is nothing to prove for (a), (b), (d2) and (d3) in this sublemma. Also, the proof of (d1) is trivial, applying Corollary 3.8 to the defining equation of g_j and the defining equation of g_{j+1} in (5.4.6).

So, in order to prove (c) throughout this sublemma we can use the same kind of notations and properties as in (a), (b), (d2) and (d3) as follows:

For (a), (b) and (d2), along $v = 0$ $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings and $(f \circ \tau_m)_{total}$ can be rewritten in the following form: Note that $2 \leq j \leq r$.

$$(5.4.8) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{\beta_{1,1}} u^b), \\ (f \circ \tau_m)_{total} &= (f \circ \tau_m)(v, u) = v^{e_{j,m}} u^{\rho_{j,m}} (f \circ \tau_m)_{proper} \quad \text{with } g_j = f, \\ (f \circ \tau_m)_{proper} &= (1 + \varepsilon_1 u)^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} v^{n_1 \alpha + \beta_{1,1} \beta - n_1 \beta_{1,1} d_j} u^{a\alpha + b\beta - b n_1 d_j}, \end{aligned}$$

where

- (i) a and b are some nonnegative integers such that $a\beta_{1,1} - b n_1 = 1$ and ε_1 is assumed to be one in $\mathbb{C}\{y, z\}$,
- (ii) $e_{j,m} = n_1 \beta_{1,1} d_j$ and $\rho_{j,m} = b n_1 d_j$ and $\rho_{\alpha, \beta} = a\alpha + b\beta - b n_1 d_j \geq 0$,
- (iii) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.
- (iv) $V^{(m)}(g_j) \cap (\cup_{i=1}^m E_i) = V^{(m)}(g_j) \cap E_m = \{(v, 1 + \varepsilon_1 u) = (0, 0)\}$ for any $j = 2, \dots, r$.

For (d3), after m iterations of blow-ups, denoted by τ_m , we have the following consequences:

- $$(5.4.9) \quad \begin{aligned} (i) \quad & \text{If } \beta_{1,1} = 1, \text{ then } f \in \text{the type}[0] \text{ under } \tau_m, \text{ and} \\ & \text{if } \beta_{1,1} \geq 2, \text{ then } f \in \text{the type}[1] \text{ under } \tau_m. \\ (ii) \quad & \text{Whether } \beta_{1,1} = 1 \text{ or } \beta_{1,1} \geq 2, \text{ then } F \in \text{the type}[1] \text{ under } \tau_m. \end{aligned}$$

Remark 5.4.1

(i) In the assumption of Theorem 3.6, the construction for $G(y, z) = z^\gamma g_1$ with $g_1 = z^{n_1} + y^{k_1}$ was defined as follows: Note that $\gcd(n_1, k_1) = 1$.

Let $1 \leq n_1 < k_1$, and if $n_1 = 1$, then $\gamma = 1$, and if $n_1 \geq 2$, then $\gamma = 0$.

(ii) In the conclusion of Theorem 3.6, whether $n_1 = 1$ or $2 \leq n_1 < k_1$, or $2 \leq k_1 < n_1$, there are some nonnegative integers a and b such that $b n_1 - a k_1 = 1$ because of [I] and [II] in Theorem 3.6.

In preparation for the proof of (c), apply the above conclusion with (5.4.8), to $g_{j+1} = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \dots n_{j+1}} + \sum c_{\alpha, \beta}^{(j+1)} y^\alpha z^\beta$ in (5.4.6). Then for any $j = 1, 2, \dots, r-1$, we have the following:

$$(5.4.10) \quad \begin{aligned} (g_{j+1} \circ \tau_m)_{total} &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_1 \circ \tau_m)_{proper}^{d_{j+1}} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j+1)} (v^{n_1} u^a)^\alpha (v^{\beta_{1,1}} u^b)^\beta \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} \{ (g_1 \circ \tau_m)_{proper}^{d_{j+1}} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j+1)} v^{n_1 \alpha + \beta_{1,1} \beta - n_1 \beta_{1,1} d_{j+1}} u^{a\alpha + b\beta - b n_1 d_{j+1}} \} \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_{j+1} \circ \tau_m)_{proper}, \end{aligned}$$

where $d_{j+1} = n_2 n_3 \dots n_{j+1}$, $e_{j+1,m} = n_1 \beta_{1,1} d_{j+1}$ and $\rho_{j+1,m} = b n_1 d_{j+1}$, noting by (5.4.7) and (ii) of (5.4.8) that $n_1 \alpha + \beta_{1,1} \beta - n_1 \beta_{1,1} d_{j+1} > 0$ and $a\alpha + b\beta - b n_1 d_{j+1} \geq 0$.

On the other hand, recall that for $j = 2, 3, \dots, r-1$,

$$(5.4.11) \quad \begin{aligned} g_{j+1} &= g_j^{n_{j+1}} + \varepsilon_{j+1} y^{\beta_{j+1,1}} z^{\beta_{j+1,2}} g_1^{\beta_{j+1,3}} \dots g_{j-1}^{\beta_{j+1,j+1}}, \\ g_j &= g_1^{n_2 \dots n_j} + \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(j)} y^\gamma z^\delta, \\ g_1 &= z^{n_1} + \varepsilon_1 y^{\beta_{1,1}} \quad \text{with } \varepsilon_1 = 1, \end{aligned}$$

$$\begin{aligned} \text{where} \quad (i) \quad \Delta_2(\gamma, \delta) &= n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,1} n_2 \dots n_j && \text{by (5.4.6),} \\ (ii) \quad \Delta_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} &> n_1 \beta_{1,1} n_2 n_3 \dots n_{j+1} && \text{by (5.1.2),} \\ (iii) \quad \Omega_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} &\geq b n_1 n_2 n_3 \dots n_{j+1} && \text{by (5.3.3),} \\ (iv) \quad \varepsilon_{j+1} &\text{is a unit in } \mathbb{C}\{y, z\}. \end{aligned}$$

Now, apply (5.4.8) and (5.4.10) to $g_{j+1} = g_j^{n_{j+1}} + \varepsilon_{j+1} y^{\beta_{j+1,1}} z^{\beta_{j+1,2}} g_1^{\beta_{j+1,3}} \dots g_{j-1}^{\beta_{j+1,j+1}}$ in (5.4.11). Then, we have the following:

$$\begin{aligned} (5.4.12) \quad (g_{j+1} \circ \tau_m)_{total} &= ((g_j \circ \tau_m)(v, u))^{n_{j+1}} + (\varepsilon_{j+1} \circ \tau_m)(v, u)((y \circ \tau_m)(v, u))^{\beta_{j+1,1}} \\ &\quad \times ((z \circ \tau_m)(v, u))^{\beta_{j+1,2}} ((g_1 \circ \tau_m)(v, u))^{\beta_{j+1,3}} \dots ((g_{j-1} \circ \tau_m)(v, u))^{\beta_{j+1,j+1}} \\ &= \{v^{e_{j,m}} u^{\rho_{j,m}} (g_j \circ \tau_m)_{proper}\}^{n_{j+1}} + \varepsilon'_{j+1} (v^{n_1} u^a)^{\beta_{j+1,1}} (v^{\beta_{1,1}} u^b)^{\beta_{j+1,2}} \\ &\quad \times \{v^{n_1 \beta_{1,1}} u^{b n_1} (1 + \varepsilon_1 u)\}^{\beta_{j+1,3}} \{v^{e_{2,m}} u^{\rho_{2,m}} (g_2 \circ \tau_m)_{proper}\}^{\beta_{j+1,4}} \times \dots \\ &\quad \times \{v^{e_{j-1,m}} u^{\rho_{j-1,m}} (g_{j-1} \circ \tau_m)_{proper}\}^{\beta_{j+1,j+1}} \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} \{ (g_j \circ \tau_m)_{proper}^{n_{j+1}} + \varepsilon'_{j+1} v^{\Delta_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} - e_{j+1,m}} \\ &\quad \times u^{\Omega_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} - b n_1 d_{j+1}} (1 + \varepsilon_1 u)^{\beta_{j+1,3}} \dots (g_{j-1} \circ \tau_m)_{proper}^{\beta_{j+1,j+1}} \} \quad \text{by (5.4.13)} \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_{j+1} \circ \tau_m)_{proper} \quad \text{by (5.4.10),} \end{aligned}$$

where (i) $\varepsilon'_{j+1} = (\varepsilon_{j+1} \circ \tau_m)(v, u)$ is a unit in $\mathbb{C}\{v, 1 + u\}$ because $n_1 > 0$ and $\beta_{1,1} > 0$,
(ii) if we write $d_j = n_2 n_3 \dots n_j$, $e_{j,m} = n_1 \beta_{1,1} d_j$ and $\rho_{j,m} = b n_1 d_j$ for $j = 2, 3, \dots, r$, then $d_j n_{j+1} = d_{j+1}$, $e_{j,m} n_{j+1} = e_{j+1,m}$ and $\rho_{j,m} n_{j+1} = \rho_{j+1,m}$.

The proof for the representation in (5.4.12) just follows from (*) and (**) of (5.4.13):
(5.4.13) – (*) It is clear by Sublemma 5.2 that $(n_1 \beta_{j+1,1} + \beta_{1,1} \beta_{j+1,2}) + n_1 \beta_{1,1} \beta_{j+1,3} + e_{2,m} \beta_{j+1,4} + \dots + e_{j-1,m} \beta_{j+1,j+1}$
 $= \Delta_2(\beta_{j+1,1}, \beta_{j+1,2}) + n_1 \beta_{1,1} \beta_{j+1,3} + n_1 \beta_{1,1} d_2 \beta_{j+1,4} + \dots + n_1 \beta_{1,1} d_{j-1} \beta_{j+1,j+1}$
 $= \Delta_2(\beta_{j+1,1}, \beta_{j+1,2}) + n_1 \beta_{1,1} \beta_{j+1,3} + n_1 \beta_{1,1} n_2 \beta_{j+1,4} + \dots + n_1 \beta_{1,1} n_2 \dots n_{j-1} \beta_{j+1,j+1}$
 $= \Delta_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} > e_{j+1,m} = n_1 \beta_{1,1} d_{j+1}$ by (5.1.2) and by (5.4.10).

(5.4.13) – (**) It is clear by Sublemma 5.3 that $a \beta_{j+1,1} + b \beta_{j+1,2} + b n_1 \beta_{j+1,3} + \rho_{2,m} \beta_{j+1,4} + \dots + \rho_{j-1,m} \beta_{j+1,j+1}$
 $= \Omega_2(\beta_{j+1,1}, \beta_{j+1,2}) + b n_1 \beta_{j+1,3} + b n_1 d_2 \beta_{j+1,4} + \dots + b n_1 d_{j-1} \beta_{j+1,j+1}$
 $= \Omega_2(\beta_{j+1,1}, \beta_{j+1,2}) + b n_1 \beta_{j+1,3} + b n_1 n_2 \beta_{j+1,4} + \dots + b n_1 n_2 n_3 \dots n_{j-1} \beta_{j+1,j+1}$
 $= \Omega_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} \geq \rho_{j+1,m} = b n_1 d_{j+1}$ by (5.3.2) and (5.3.3) and by (5.4.10).
Thus, we can prove that (c) and so, the proof of the sublemma are finished. \square

Proof of Sublemma 5.5. First of all, let $\{Y_k : k = 1, 2, \dots, r-1\}$ with $Y_k \subset N_0$, $\{h_k : k = 1, 2, \dots, r-1\}$ with $h_k = (g_{k+1} \circ \tau_m)_{proper}$ in $\mathbb{C}\{v, 1 + u\}$ and $\{\Xi_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r-1\}$, where each Ξ_k is an integer-valued function, be three sequences satisfying the given three conditions, denoted by **The 1-th Cond**⁽¹⁾, **The 2-th Cond**⁽¹⁾, **The 3-th Cond**⁽¹⁾, in the conclusion of this sublemma.

After the proof of Sublemma 5.4 was done, it is easy to observe without any more need of the proof that the above three sequences with three conditions are well-constructed.

For the proof of this sublemma, it suffices to show that these three sequences satisfy the remaining two conditions, (i) **The (4 α)-th Cond**⁽¹⁾ with **The 4-th Cond**⁽¹⁾ and (ii)

The (5 α)-th Cond⁽¹⁾ in Conclusions of this sublemma. So, by Remark 5.5.1, it is enough to show that the equality in (5.5.4 α) of **The (4 α)-th Cond⁽¹⁾** is true.

Now, we will prove that the equality in (5.5.4 α) is true, using the following three steps:
Let ℓ and q be arbitrary positive integers such that $r - 1 \geq \ell \geq q \geq 2$.

$$\begin{aligned}
\text{Step(i)} \quad & \Xi_q(\gamma_{\ell,k})_{k=1}^q \\
& = \Delta_{q+1}(\beta_{\ell+1,k})_{k=1}^{q+1} + n_q^2 n_{q-1}^2 \cdots n_2^2 n_{1,1} \{\beta_{\ell+1,q+2} + n_{q+1} \beta_{\ell+1,q+3} \\
& \quad + n_{q+1} n_{q+2} \beta_{\ell+1,q+4} + \cdots + n_{q+1} n_{q+2} \cdots n_{\ell-1} \beta_{\ell+1,\ell+1} - n_{q+1} n_{q+2} \cdots n_{\ell+1}\}. \\
\text{Step(ii)} \quad & \text{In particular, if } \ell = q \text{ then} \\
& \Xi_q(\gamma_{q,k})_{k=1}^q = \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q^2 n_{q-1}^2 \cdots n_2^2 n_{1,1} \quad \text{from Step(i)}. \\
\text{Step(iii)} \quad & \Xi_q(\gamma_{q,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \\
& = \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,k})_{k=1}^q > 0 \quad \text{from Step(ii)}.
\end{aligned}$$

We prove Step(i), Step(ii) and Step(iii) simultaneously by induction on the integer $q \geq 2$.

So, it is enough to consider two cases, respectively:

Case(I) $q = 2$, and Case(II) $q \geq 2$.

Case(I): Let $q = 2$. Note by **The 3-th Cond⁽¹⁾** that $\Xi_2(t_1, t_2) = t_2 \Xi_1(\gamma_{1,1}) + s_1 \Xi_1(t_1) = t_2 \gamma_{1,1} + s_1 t_1$ for each $(t_1, t_2) \in N_0^2$.

$$\begin{aligned}
\text{Step(i)} \quad & \Xi_2(\gamma_{\ell,1}, \gamma_{\ell,2}) = s_1 \gamma_{\ell,1} + \gamma_{1,1} \gamma_{\ell,2} \quad \text{by definition of } \Xi_2 \\
& = n_2 \{ \Delta_{\ell+1}^\sharp(\beta_{\ell+1,k})_{k=1}^{\ell+1} - n_1 \beta_{1,1} n_2 \cdots n_{\ell+1} \} \\
& \quad + \{ \Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 \} \beta_{\ell+1,3} \quad \text{by (5.5.1)} \\
& = n_2 \{ \Delta_2(\beta_{\ell+1,1}, \beta_{\ell+1,2}) + n_1 \beta_{1,1} \beta_{\ell+1,3} + n_1 \beta_{1,1} n_2 \beta_{\ell+1,4} + \cdots \\
& \quad + n_1 \beta_{1,1} n_2 \cdots n_{\ell-1} \beta_{\ell+1,\ell+1} - n_1 \beta_{1,1} n_2 \cdots n_{\ell+1} \} \\
& \quad + \{ \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 \} \beta_{\ell+1,3} \quad \text{by (5.1.1)} \\
& = \Delta_3(\beta_{\ell+1,1}, \beta_{\ell+1,2}, \beta_{\ell+1,3}) + n_2^2 n_{1,1} \{ \beta_{\ell+1,4} + n_3 \beta_{\ell+1,5} + n_3 n_4 \beta_{\ell+1,6} \\
& \quad + \cdots + n_3 n_4 \cdots n_{\ell-1} \beta_{\ell+1,\ell+1} - n_3 n_4 \cdots n_{\ell+1} \},
\end{aligned}$$

by the definition of $\Delta_3(\beta_{\ell+1,1}, \beta_{\ell+1,2}, \beta_{\ell+1,3})$ only, which implies the proof of Step(i).

Step(ii) In particular, if $\ell = 2$ then by Step(i)

$$\Xi_2(\gamma_{2,1}, \gamma_{2,2}) = \Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) - n_3 n_2^2 n_{1,1}.$$

Thus, the proof of Step(ii) is done.

Step(iii) To prove that $\Xi_2(\gamma_{2,1}, \gamma_{2,2}) - s_2 s_1 \Xi_1(\gamma_{1,1}) > 0$, first note by (5.5.1) that

$$s_2 s_1 \Xi_1(\gamma_{1,1}) = s_2 s_1 \gamma_{1,1} = n_3 n_2 \{ \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 \}.$$

$$\begin{aligned}
\text{Then,} \quad & \Xi_2(\gamma_{2,1}, \gamma_{2,2}) - s_2 s_1 \gamma_{1,1} \\
& = \Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) - n_3 n_2^2 n_{1,1} - n_3 n_2 \{ \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 \} \\
& = \Delta_3(\beta_{3,1}, \beta_{3,2}, \beta_{3,3}) - n_3 n_2 \Delta_2(\beta_{2,1}, \beta_{2,2}) > 0,
\end{aligned}$$

by in the assumption of Theorem 5.0, which implies the proof of Step(iii).

Thus, if $q = 2$, then we proved that this sublemma is true.

Case(II): Let $q \geq 2$. By induction proof, suppose that the sublemma is true on the integer $q \leq r - 2$ with $r - 1 \geq \ell \geq q$. Then, it is enough to prove Step(i), Step(ii) and Step(iii) simultaneously on the integer $(q + 1) \leq \ell$ as follows:

$$\begin{aligned}
\text{Step(i)} \quad & \Xi_{q+1}(\gamma_{\ell,k})_{k=1}^{q+1} = \gamma_{\ell,q+1} \Xi_q(\gamma_{\ell,k})_{k=1}^q + s_q \Xi_q(\gamma_{\ell,k})_{k=1}^q \quad \text{by definition of } \Xi_{q+1} \\
& = \beta_{\ell+1,q+2} \{ \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,1} \} \\
& \quad + n_{q+1} \{ \Delta_{q+1}(\beta_{\ell+1,k})_{k=1}^{q+1} + n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,1} (\beta_{\ell+1,q+2} + n_{q+1} \beta_{\ell+1,q+3} \\
& \quad + n_{q+1} n_{q+2} \beta_{\ell+1,q+4} + \cdots + n_{q+1} n_{q+2} \cdots n_{\ell-1} \beta_{\ell+1,\ell+1} - n_{q+1} n_{q+2} \cdots n_{\ell+1}) \} \\
& \quad \text{by the induction assumption on the integer } q \\
& = \Delta_{q+2}(\beta_{\ell+1,k})_{k=1}^{q+2} + n_{q+1}^2 n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,1} \\
& \quad \times \{ \beta_{\ell+1,q+3} + n_{q+2} \beta_{\ell+1,q+4} + n_{q+2} n_{q+3} \beta_{\ell+1,q+4} + \cdots \\
& \quad + n_{q+2} n_{q+3} \cdots n_{\ell+1-2} \beta_{\ell+1,\ell+1} - n_{q+2} n_{q+3} \cdots n_{\ell+1} \},
\end{aligned}$$

by the definition of $\Delta_{q+2}(\beta_{\ell+1,k})_{k=1}^{q+2}$ only, which implies the proof of Step(i).

Step(ii) In particular, if $\ell = q + 1$, then $\ell + 1 = q + 2 < q + 3$ and so

$$\Xi_{q+1}(\gamma_{q+1,k})_{k=1}^{q+1} = \Delta_{q+2}(\beta_{q+2,k})_{k=1}^{q+2} - n_{q+2} n_{q+1}^2 n_q^2 \cdots n_2^2 n_1 \beta_{1,1},$$

by Step(i) on the integer $q + 1$, which implies the proof of Step(ii) on $q + 1$.

Step(iii) To prove that the equality in (5.5.4 α) is true, as an application of Step(ii) on two integers $\ell = q + 1$ and $\ell = q$, then we have

$$\begin{aligned}
& \Xi_{q+1}(\gamma_{q+1,k})_{k=1}^{q+1} - s_{q+1} s_q \Xi_q(\gamma_{q,k})_{k=1}^q \\
& = \{ \Delta_{q+2}(\beta_{q+2,k})_{k=1}^{q+2} - n_{q+2} n_{q+1}^2 n_q^2 \cdots n_2^2 n_1 \beta_{1,1} \} \\
& \quad - n_{q+2} n_{q+1} \{ \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,1} \} \\
& = \Delta_{q+2}(\beta_{q+2,k})_{k=1}^{q+2} - n_{q+2} n_{q+1} \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} > 0,
\end{aligned}$$

by **The 4 α -th Cond**⁽⁰⁾ in the assumption of Theorem 5.0, which implies the proof of Step(iii) on the integer $q + 1$.

Thus, we proved that the equality in (5.5.4 α) is true, and so the proof of this sublemma is finished. \square

§6.2. The proofs of Theorem 5.0 with corollaries

Proof of Theorem 5.0. For the induction proof, it suffices to consider two cases, respectively:

Case(1) $r = 1$, and Case(2) $r \geq 1$.

For each case, we shall find first, the proof of $[A]$, and after then, the the proof of $[B]$.

Case(1): Let $r = 1$. Then, there is nothing to prove for $[A]$. For the proof of $[B]$, assuming that $g_1 = z^{n_1} + y^{\beta_{1,1}}$ is irreducible in $\mathbb{C}\{y, z\}$ with $n_1 \geq 2$, then $y^\gamma g_1 \in \text{the type}[1]$ under the standard resolution by Sublemma 5.4 or Theorem 3.6, because if $\beta_{1,1} = 1$ then $\gamma = 1$ and if $\beta_{1,1} > 1$ then $\gamma = 0$. In particular, $z^\delta y g_1 \in \text{the type}[1]$ under the standard resolution by Sublemma 5.4 or Theorem 3.6 whether $\delta = 1$ or $\delta = 0$. Thus, $[B]$ can be easily proved. So, if $r = 1$, the proofs of $[A]$ and $[B]$ are done.

Case(2): Let $r \geq 1$. For each integer $r \geq 2$, we shall find first, the proof of $[A]$ and next, the the proof of $[B]$. If g_j is irreducible in $\mathbb{C}\{y, z\}$ for any $j \geq 2$, then $\gcd(n_1, \beta_{1,1}) = 1$ by (d) of Sublemma 5.2 and Theorem 3.6. So, to find the proofs of $[A]$ and $[B]$ for each $r \geq 2$, we may assume without proof that $\gcd(n_1, \beta_{1,1}) = 1$. Since $\gcd(n_1, \beta_{1,1}) = 1$, for the proof of the theorem we may follow the same notations and consequences as in Sublemma 5.4.

Therefore, as we have done in Sublemma 5.4, recall that $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ and $(g_j \circ \tau_m)_{total}$ with $(g_j \circ \tau_m)_{proper}$ satisfies all facts in (a), (b), (c) and (d) in the conclusion of Sublemma 5.4: Note that $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ is the composition of a finite number m of successive

blow-ups π_i which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ in the conclusion of Sublemma 5.4. Also, for any positive integer $r \geq 2$, observe by (c) of Sublemma 5.4 that g_r is irreducible in $\mathbb{C}\{y, z\}$ if and only if g_1 is irreducible in $\mathbb{C}\{y, z\}$ and $(g_r \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, u+1\}$, noting that g_1 is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\gcd(n_1, \beta_{1,1}) = 1$.

Now, we consider the proof of Case(2). For the proof of the theorem, by the induction assumption, suppose we have shown that [A] and [B] of the theorem are true on the integer $r \geq 1$. In order to prove the theorem on the integer $(r+1)$, first we will find the proof of [A] on the integer $(r+1)$ and next, the proof of [B] on the integer $(r+1)$.

The proof of [A]. To prove [A] on the integer $r+1$, assume that $\gcd(n_1, \beta_{1,1}) = 1$ and $g_{r+1} \in \mathbb{C}\{y, z\}$ is a semi-quasi-Puiseux convergent power series of the recursive $(r+1)$ -type.

To prove the theorem, it suffices to consider the following defining equation for g_{r+1} with the additive assumptions:

$$(5.0.1) \quad g_{r+1} = g_r^{n_{r+1}} + \varepsilon_{r+1} y^{\beta_{r+1,1}} z^{\beta_{r+1,2}} g_1^{\beta_{r+1,3}} \cdots g_{r-1}^{\beta_{r+1,r+1}},$$

where

- (i) g_1, g_2, \dots, g_r satisfies the same assumptions and conclusions as in this theorem,
- (ii) $X_{r+1} = \{n_{r+1}, \beta_{r+1,1}, \beta_{r+1,2}, \dots, \beta_{r+1,r+1}\} \subset N_0$ with $n_{r+1} \geq 2$ and $\varepsilon_{r+1} = \varepsilon_{r+1}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$,
- (iii) $\Delta_{r+1}(t_k)_{k=1}^{r+1} = t_{r+1} \Delta_r(\beta_{r,k})_{k=1}^r + n_r \Delta_r(t_k)_{k=1}^r$ for each $(t_k)_{k=1}^{r+1} \in N_0^{r+1}$,
- (iv) $\Delta_{r+1}(\beta_{r+1,k})_{k=1}^{r+1} > n_{r+1} n_r \Delta_r(\beta_{r,k})_{k=1}^r$,
- (v) $\Delta_{r+1}^\#(\beta_{r+1,k})_{k=1}^{r+1} = \Delta_2(\beta_{r+1,1}, \beta_{r+1,2}) + n_1 \beta_{1,1} \beta_{r+1,3} + n_1 \beta_{1,1} n_2 \beta_{r+1,4} + n_1 \beta_{1,1} n_2 n_3 \beta_{r+1,5} + \cdots + n_1 \beta_{1,1} n_2 \cdots n_{r-1} \beta_{r+1,r+1}$,
- (vi) $\Omega_{r+1}^\#(\beta_{r+1,k})_{k=1}^{r+1} = \Omega_2(\beta_{r+1,1}, \beta_{r+1,2}) + b n_1 \beta_{r+1,3} + b n_1 n_2 \beta_{r+1,4} + b n_1 n_2 n_3 \beta_{r+1,5} + \cdots + b n_1 n_2 \cdots n_{r-1} \beta_{r+1,r+1}$.

Remark 5.0.1.1. Note by (5.0.1) that the following inequalities in (a) and (b) can be proved by Sublemma 5.1 and Sublemma 5.3, respectively, using the same methods as we have used in the proof of Sublemma 5.1 and Sublemma 5.3.

- (a) $\Delta_{r+1}^\#(\beta_{r+1,k})_{k=1}^{j+1} > n_1 \beta_{1,1} n_2 n_3 \cdots n_r n_{r+1}$.
- (b) $\Omega_{r+1}^\#(\beta_{r+1,k})_{k=1}^{r+1} \geq b n_1 n_2 n_3 \cdots n_r n_{r+1}$.

Using the composition τ_m again of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ as in the beginning of the proof, then $h_r = (g_{r+1} \circ \tau_m)_{proper}$ can be written in the form

$$(5.0.2) \quad h_r = h_{r-1}^{s_r} + \eta_r v^{\gamma_{r,1}} (1+u)^{\gamma_{r,2}} h_1^{\gamma_{r,3}} \cdots h_{r-2}^{\gamma_{r,r}},$$

where

- (i) each $h_j = (g_{j+1} \circ \tau_m)_{proper}$ is in $\mathbb{C}\{v, 1+u\}$ for $j = 1, 2, \dots, r$, which has been already represented by Sublemma 5.5,
- (ii) $(g_{r+1} \circ \tau_m)_{total} = v^{n_1 \beta_{1,1} n_2 \cdots n_{r+1}} u^{b n_1 n_2 \cdots n_{r+1}} (g_{r+1} \circ \tau_m)_{proper}$,
- (iii) $\eta_r = \varepsilon'_{r+1} u^{\Omega_{r+1}^\#(\beta_{r+1,k})_{k=1}^{r+1} - b n_1 n_2 \cdots n_{r+1}}$ is a unit in $\mathbb{C}\{v, u+1\}$, satisfying the same notations and consequences as in Sublemma 5.5,
- (iv) $s_r = n_{r+1} \geq 2$, $\gamma_{r,1} = \Delta_{r+1}^\#(\beta_{r+1,k})_{k=1}^{r+1} - n_1 \beta_{1,1} n_2 n_3 \cdots n_{r+1} > 0$, $\gamma_{r,2} = \beta_{r+1,3}$, $\gamma_{r,3} = \beta_{r+1,4}, \dots, \gamma_{r,r} = \beta_{r+1,r+1}$.

Because blow-ups process preserves irreducibility of plane curve singularity, note by Sublemma 5.2 and Sublemma 5.4 that g_{r+1} is irreducible in $\mathbb{C}\{y, z\}$ if and only if g_1 is irreducible in $\mathbb{C}\{y, z\}$ and $h_r = (g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, u+1\}$.

Note that g_1 is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\gcd(n_1, \beta_{1,1}) = 1$. First of all, whether or not h_r is irreducible in $\mathbb{C}\{v, u+1\}$, we proved by Sublemma 5.5 that h_r satisfies the same kind of assumptions as g_r does the assumptions in this theorem, up to change of notations.

So, by the induction assumption on the integer r and by following the same notations as in Sublemma 5.5, then it is easy to get the following:

$$(5.0.3) \quad h_r \text{ is irreducible in } \mathbb{C}\{v, u+1\} \iff \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = 1 \quad \text{for } j = 1, 2, \dots, r.$$

Also, by **The (5 α)-th Cond⁽¹⁾** of Sublemma 5.5, we have the following:

$$(5.0.4) \quad \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = 1 \iff \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}) = 1. \quad \text{for } j = 1, 2, \dots, r.$$

Since it is clear by Sublemma 5.4 that g_{r+1} is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\gcd(n_1, \beta_{11}) = 1$ and $h_r = (g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, u+1\}$, then we can prove by (5.0.3) and (5.0.4) and by the induction assumption that

$$(5.0.5) \quad \begin{aligned} & g_{r+1} \text{ is irreducible in } \mathbb{C}\{y, z\} \\ \iff & \gcd(n_1, \beta_{1,1}) = 1 \text{ and } h_r \text{ is irreducible in } \mathbb{C}\{v, u+1\} \\ \iff & \gcd(n_1, \beta_{1,1}) = 1 \text{ and } \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}) = 1 \text{ for } 1 \leq j \leq r. \\ \iff & g_1, g_2, \dots, g_r \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \gcd(n_{r+1}, \Delta_{r+1}(\beta_{r+1,k})_{k=1}^{r+1}) = 1. \end{aligned}$$

Thus, we finished the proof of [A] on the integer $(r+1)$.

The proof of [B] Next to prove [B] on the integer $(r+1)$, let g_{r+1} be irreducible in $\mathbb{C}\{y, z\}$. If g_j is irreducible in $\mathbb{C}\{y, z\}$ for any $j \geq 1$, note that $\gcd(n_1, \beta_{11}) = 1$ by (d) of Sublemma 5.2 and Theorem 3.6. By Sublemma 5.4, we have the following:

$$(5.0.6) \quad \begin{aligned} & \text{(i) If } \beta_{1,1} > 1, \text{ then } g_{r+1} \in \text{the type [1] under } \tau_m. \\ & \text{(ii) If } \beta_{1,1} = 1, \text{ then } g_{r+1} \in \text{the type [0] under } \tau_m. \\ & \text{(iii) If } \beta_{1,1} \geq 1, \text{ then } z^\delta y g_{r+1} \in \text{the type [1] under } \tau_m \text{ whether } \delta \text{ is 1 or 0.} \end{aligned}$$

Let $V_{r+1} = \{(y, z) : g_{r+1}(y, z) = 0\}$ and $W_{r+1} = \{(y, z) : y g_{r+1}(y, z) = 0\}$ be analytic varieties at the origin in $\mathbb{C}\{y, z\}$, respectively. Let τ_m be again the composition of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(yg_1)$.

At $(v, 1+u) = (0, 0)$, by (5.4.1) $\tau_m^{-1}(V_{r+1})$ and $\tau_m^{-1}(W_{r+1})$ can be written as follows:

$$(5.0.7) \quad \begin{aligned} \tau_m^{-1}(V_{r+1}) &= \{(g_{r+1} \circ \tau_m)_{total} = v^{n_1 \beta_{1,1} d'_2} u^{bn_1 d'_2} (g_{r+1} \circ \tau_m)_{proper} = 0\}, \\ \tau_m^{-1}(W_{r+1}) &= \{((y g_{r+1}) \circ \tau_m)_{total} = v^{n_1 \beta_{1,1} d'_2 + n_1} u^{bn_1 d'_2 + a} (g_{r+1} \circ \tau_m)_{proper} = 0\}, \end{aligned}$$

noting by (5.0.2) that

$$(5.0.8) \quad \begin{aligned} (g_{r+1} \circ \tau_m)_{proper} &= (g_r \circ \tau_m)_{proper}^{n_{r+1}} + \eta_r v^{\Delta_{r+1}(\beta_{r+1,k})_{k=1}^{r+1} - n_1 \beta_{1,1} d'_2} \\ &\quad \times (g_1 \circ \tau_m)_{proper}^{\beta_{r+1,3}} (g_2 \circ \tau_m)_{proper}^{\beta_{r+1,4}} \cdots (g_{r-1} \circ \tau_m)_{proper}^{\beta_{r+1,r+1}}. \end{aligned}$$

where $d'_2 = n_2 n_3 \cdots n_{r+1}$ and $\eta_r = \varepsilon'_{r+1} u^{\Omega_{r+1}(\beta_{r+1,k})_{k=1}^{r+1} - bn_1 d'_2}$ is a unit in $\mathbb{C}\{v, u+1\}$.

Let $V(\psi_r) = \{(v, u+1) : \psi_r(v, u+1) = 0\}$ be an analytic variety at $(v, u+1) = (0, 0)$ defined by

$$(5.0.9) \quad \begin{aligned} \psi_r &= \psi_r(v, u+1) = v^{\gamma_1} h_r \quad \text{with} \quad h_r = (g_{r+1} \circ \tau_m)_{proper} \\ \text{such that} \quad & \begin{cases} \gamma_1 = 1, & \text{if } \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 = 1, \\ \gamma_1 = 0, & \text{if } \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 \geq 2. \end{cases} \end{aligned}$$

Let $Z_r = \{(v, u+1) : v h_r(v, u+1) = 0\}$ be an analytic variety at $(v, u+1) = (0, 0)$. Since $(g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, u+1\}$, then note that $Z_r = \tau_m^{-1}(V_{r+1}) = \tau_m^{-1}(W_{r+1})$ have the same two irreducible components at $(v, u+1) = (0, 0)$ as reduced varieties, and so they have the same standard resolution of the singular point $(v, u+1) = (0, 0)$. Let ω_s be the composition of s iterations of blow-ups which is needed to get the standard resolution of the singular point $(v, u+1) = (0, 0)$ of $V(\psi_r)$.

Since we proved by Sublemma 5.5 that $V(h_r)$ satisfies the same kind of properties and notations as $V(g_r)$ does in the assumption of this theorem, then by the induction assumption on the integer r , both $V(\psi_r)$ and Z_r belong to the type $[r]$ under ω_s . That is, both $\tau_m^{-1}(V_{r+1})$ and $\tau_m^{-1}(W_{r+1})$ belong to the type $[r]$ under ω_s at $(v, u+1) = (0, 0)$, as reduced varieties.

Now, the m -th exceptional curve of the first kind among $\tau_m^{-1}(0, 0)$, denoted by $E_m = \{v = 0\}$, can be viewed as one of two irreducible components of Z_r at $(v, u+1) = (0, 0)$. Then, both V_{r+1} and W_{r+1} can have the same standard resolution $\tau_m \circ \omega_s$ of the singular point $(y, z) = (0, 0)$, using the composition $\tau_m \circ \omega_s$ of a finite number $(m + s)$ of successive blow-ups at $(y, z) = (0, 0)$. Since $Z_r \in$ the type $[r]$ under ω_s , then by (5.0.6) or Sublemma 5.4 again, it is trivial to get the followings:

- (5.0.10) (i) If $\beta_{11} > 1$, then $g_{r+1} \in$ the type $[r+1]$ under $\tau_m \circ \omega_s$.
(ii) If $\beta_{11} = 1$, then $g_{r+1} \in$ the type $[r]$ under $\tau_m \circ \omega_s$.
(iii) If $\beta_{11} \geq 1$, then $z^\delta y g_{r+1} \in$ the type $[r+1]$ under $\tau_m \circ \omega_s$, whether δ is 1 or 0.

Thus, the proof for [B] is done. Therefore, we completed the proof of theorem. \square

The proofs of Corollary 5.6, Corollary 5.7 and Corollary 5.8 just follow from Theorem 5.0 and Sublemmas.

§7. For any two Puiseux convergent power series g_r and ϕ_ρ of the recursive types in $\mathbb{C}\{y, z\}$, how to compute the necessary and sufficient condition for $\phi_\rho \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions and their classifications

§7.0. Introduction

As we have seen in Definition 1.2 and Definition 2.4, four families with equivalence relations, Family(1), \dots , Family(4) with equivalence relations have been already defined.

Definition 7.0. Quasi-Family(1) is a family consisting of all the quasi-Puiseux convergent power series $f \in \mathbb{C}\{y, z\}$ of the recursive type with isolated singularity at the origin, denoted by $\{f \text{ is arbitrary quasi-Puiseux convergent power series of the recursive type: } f \in \text{Family}(0)\}$ by Definition 5.0.0. Note that Family(1) is a subset of Quasi-Family(1).

Problem[1]. To solve the problem is to succeed in The 1st algorithm and its application in §1.4. The solution of the problem can be divided into three small problems, denoted by Problem[1-A], Problem[1-B] and Problem[1-C] in order.

Problem[1-A]. In preparation, the first small problem is to prove that there is a one-to-one function from Family(1) into Family(j) for any $j = 2, 3, 4$. It was already proved by Theorem A([K]) that there is a one-to-one function ϕ from Family(2) onto Family(3). Then, this problem can be solvable by (i) and (ii):

(i) It will be proved by Theorem 7.3 and Theorem 7.7 in §7 that there exists a one-to-one function from Family(1) into Family(4).

(ii) It will be proved by Theorem 10.2 and Theorem 7.7 that there exists a one-to-one function from Family(1) into Family(3).

Problem[1-B]. The second small problem is to prove that we can compute a one-to-one map ϕ from Family(1) into Family(2). As an application, for given any standard Puiseux polynomial $g_r \in \mathbb{C}\{y\}[z]$ of the recursive r-type, we compute an algorithm for finding the standard Puiseux expansion $C(t)$ such that g_r and $C(t)$ have the same multiplicity sequence by Theorem 11.2 and Corollary 11.3.

Problem[1-C]. Whenever the standard Puiseux expansion $C(t)$ is chosen arbitrary, we can compute an algorithm for finding the standard Puiseux polynomial $g_r \in \mathbb{C}\{y\}[z]$ of the recursive r-type such that g_r and $C(t)$ have the same multiplicity sequence by Theorem 11.4, which implies that $\phi : \text{Family}(1) \rightarrow \text{Family}(2)$ is onto.

Now, in this section the aim is to solve (i) of Problem[1-A] by Theorem 7.3 and Theorem 7.7 completely. First of all, we are preparing Theorem 7.1.

§7.1. In preparation for solving (i) of Problem[1-A] by Theorem 7.1

In this section, the aim is to solve the following in terms of Theorem 7.1:

- (*) If $g_r \in \mathbb{C}\{y, z\}$ is any quasi-Puiseux convergent power series of the recursive r-type and $\psi_s \in \mathbb{C}\{y, z\}$ is any quasi-Puiseux convergent power series of the recursive s-type, then we can compute a necessary condition for $V(g_r) \stackrel{\text{divisor}}{\sim} V(\psi_s)$ under the standard resolutions in the sense of Definition 2.4.

For the study of Theorem 7.1, it is clear by Sublemma 5.1 and Theorem 3.6 that $f = g_r \in \mathbb{C}\{y, z\}$ with $F = y^\zeta z^\eta g_r \in \mathbb{C}\{y, z\}$ and $\phi = \psi_s \in \mathbb{C}\{y, z\}$ with $\Phi = y^{\zeta'} z^{\eta'} \psi_s \in \mathbb{C}\{y, z\}$ in (*) can be defined by the same properties and notations as in the assumption of Theorem 7.1. Then, we will solve the following in terms of Theorem 7.1:

(a) The first aim is to find a necessary condition for $V(y^\zeta z^\eta g_r) \stackrel{\text{divisor}}{\sim} V(y^{\zeta'} z^{\eta'} \psi_s)$ under the standard resolutions by Theorem 7.1 where $y^\zeta z^\eta g_r \in \mathbb{C}\{y, z\}$ and $y^{\zeta'} z^{\eta'} \psi_s \in \mathbb{C}\{y, z\}$ are defined just as above, by using Theorem 7.1.

(b) As an application of Theorem 7.1, the second aim is to find the necessary and sufficient condition for $V(y^\zeta z^\eta g_r) \stackrel{\text{divisor}}{\sim} V(y^{\zeta'} z^{\eta'} \psi_s)$ under the standard resolutions in an elementary and concrete way, by using Theorem 7.3.

Theorem 7.1. Assumptions

(a) Let $f = f(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$, and let $V(F) = \{(y, z) : F(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively in the form,

$$(7.1.1) \quad f = (z^{n_1} + \varepsilon y^{\beta_{1,1}})^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} d,$$

$$F = y^\zeta z^\eta f$$

satisfying the properties (i), (ii), \dots , (vi):

- (i) $\gcd(n_1, \beta_{1,1}) = 1$ with $1 \leq n_1 < \beta_{1,1}$ and d is a positive integer.
- (ii) ε is a unit in $\mathbb{C}\{y, z\}$.
- (iii) The $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} d$.
- (iv) ζ and η are nonnegative integers.
- (v) If $n_1 = 1$, assume that η is a positive integer.
- (vi) If $d = n_1 = 1$, note that $V(f)$ has no singular point at the origin.

(b) Let $\phi = \phi(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$, and let $V(\Phi) = \{(y, z) : \Phi(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively in the form,

$$(7.1.2) \quad \phi = (z^{\ell_1} + \bar{\varepsilon} y^{\delta_{1,1}})^{d'} + \sum_{p, q \geq 0} a_{p, q} y^p z^q \quad \text{with} \quad \ell_1 p + \delta_{1,1} q > \ell_1 \delta_{1,1} d'$$

$$\Phi = y^{\zeta'} z^{\eta'} \phi$$

satisfying the properties (i), (ii), \dots , (vi):

- (i) $\gcd(\ell_1, \delta_{1,1}) = 1$ with $1 \leq \ell_1 < \delta_{1,1}$ and d' is a positive integer.
- (ii) $\bar{\varepsilon}$ is a unit in $\mathbb{C}\{y, z\}$.
- (iii) The $a_{p, q}$ are nonzero complex numbers for some nonnegative integers p and q such that $\ell_1 p + \delta_{1,1} q > \ell_1 \delta_{1,1} d'$.
- (iv) ζ' and η' are nonnegative integers.
- (v) If $\ell_1 = 1$, assume that η' is a positive integer.
- (vi) If $d' = \ell_1 = 1$, note that $V(\phi)$ has no singular point at the origin.

Conclusions As a necessary condition for $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, we have the following fact:

$$(7.1.3) \quad \text{If } V(F) \stackrel{\text{divisor}}{\sim} V(\Phi) \text{ under the standard resolutions,}$$

$$\text{then } \zeta = \zeta', \eta = \eta', n_1 = \ell_1, \beta_{1,1} = \delta_{1,1} \text{ and } d = d'.$$

In particular, if $d = d' = 1$, then the necessary condition for $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions, being given by (7.1.3), is also sufficient. \square

Remark 7.1.0. In preparation for the proof of this theorem, first of all, we need the following lemma, that is, Lemma 7.2 with proof. After then, it will be found that there is nothing to prove for this theorem, later. \square

Lemma 7.2. Assumptions Let $V(F) = \{(y, z) : F(y, z) = 0\}$ and $V(\Phi) = \{(y, z) : \Phi(y, z) = 0\}$ be analytic varieties at $(0, 0) \in \mathbb{C}^2$, satisfying the same assumption and notation as we have seen in the assumption of Theorem 7.1.

(a)(a1) Let $V(G) = \{(y, z) : G(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by the form

$$(7.2.1) \quad G = z^\gamma g$$

$$g = z^{n_1} + y^{\beta_{1,1}} \quad \text{with} \quad \gcd(n_1, \beta_{1,1}) = 1,$$

satisfying the following properties:

- (i) $1 \leq n_1 < \beta_{1,1}$.
- (ii) If $n_1 = 1$, then $\gamma = 1$.
- (iii) If $n_1 \geq 2$, then $\gamma = 0$.

(a2) Let τ_m be the composition of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(G)$ as we have used in (3.6.2) of the assumption of Theorem 3.6. For each $t = 1, 2, \dots, m$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ is a blow-up of $M^{(i-1)}$ at a point of $M^{(i-1)}$ for $1 \leq i \leq t$, and $M^{(0)} = \mathbb{C}^2$. For brevity of notation, let $V^{(t)}(G)$ be the proper transform under τ_t for $1 \leq t \leq m$. By the conclusion of Theorem 3.6, we can use the same τ_m for the composition of the first finite number m of successive blow-ups in process of the standard resolution of the singular point $(0,0)$ of $V(F)$, as a reduced variety.

Let $E^{(m)} = \tau_m^{-1}(0,0)$, and let $E^{(m)} = \cup E_i$, $1 \leq i \leq m$, be the decomposition of $E^{(m)}$ into irreducible components where each E_i is called an exceptional curve of the first kind under τ_m . Then, $V^{(t)}(G)$ has one and only one quasisingular point along E_t for $1 \leq t \leq m-1$.

(a3) Let $(F \circ \tau_m)_{\text{divisor}}$ be a divisor of $F \circ \tau_m$ defined by

$$(7.2.2) \quad (F \circ \tau_m)_{\text{divisor}} = V^{(m)}(F) + \sum_{i=1}^m e_i E_i,$$

where each e_i is the multiplicity of $F \circ \tau_m$ along E_i for $1 \leq i \leq m$ and $V^{(m)}(F)$ is the proper transform of $V(F)$ under τ_m .

In more detail, let $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ is the i -th blow-up of τ_m at a quasisingular point of $V^{(i-1)}(F)$ for $1 \leq i \leq m-1$ in the sense of Definition 2.6 where $M^{(0)} = \mathbb{C}^2$. Then, $V^{(i)}(F)$ has one and only one quasisingular point along E_i for $1 \leq i \leq m$, if exists.

Using (3.6.3) of the conclusion of Theorem 3.6, (a1) and (a2), then the equation in (7.2.2) is well-defined, satisfying the following properties:

(a3-1) $e_i < e_{i+1}$ for $i = 1, 2, \dots, m$.

(a3-2) $V(F)$ belongs to the type[1] under τ_m in the sense of Definition 2.8 because E_m is one and only one exceptional curve of the first kind among $E^{(m)} = \cup_{i=1}^m E_i$ which has three distinct intersection points with other exceptional curves and the proper transform $V^{(m)}(F)$ under τ_m .

(b)(b1) Let $V(H) = \{(y, z) : H(y, z) = 0\}$ be an analytic variety at $(0,0)$ in \mathbb{C}^2 defined by the form

$$(7.2.3) \quad \begin{aligned} H &= z^\delta h \\ h &= z^{\ell_1} + y^{\delta_{1,1}} \quad \text{with} \quad \gcd(\ell_1, \delta_{1,1}) = 1, \end{aligned}$$

satisfying the following properties:

- (i) $1 \leq \ell_1 < \delta_{1,1}$.
- (ii) If $\ell_1 = 1$, then $\delta = 1$.
- (iii) If $\ell \geq 2$, then $\delta = 0$.

(b2) Let μ_ρ be the composition of a finite number ρ of successive blow-ups which is needed to get the standard resolution of the singular point of $V(H)$ as we have used in the assumption of Theorem 3.6. For each $t = 1, 2, \dots, \rho$, write $\mu_t = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \dots \circ \bar{\pi}_t : \bar{M}^{(t)} \rightarrow \mathbb{C}^2$ where $\bar{\pi}_i : \bar{M}^{(i)} \rightarrow \bar{M}^{(i-1)}$ is a blow-up of $\bar{M}^{(i-1)}$ at some point of $\bar{M}^{(i-1)}$ for $1 \leq i \leq t$, and $\bar{M}^{(0)} = \mathbb{C}^2$. For brevity of notation, let $V^{(t)}(H)$ be the proper transform under μ_t for $1 \leq t \leq \rho$. By the conclusion of Theorem 3.6, we can use the same μ_ρ for the composition of the first finite number ρ of successive blow-ups in process of the standard resolution of the singular point $(0,0)$ of $V(\Phi)$, as a reduced variety.

Let $\bar{E}^{(\rho)} = \mu_\rho^{-1}(0,0)$, and let $\bar{E}^{(\rho)} = \cup \bar{E}_i$, $1 \leq i \leq \rho$, be the decomposition of $\bar{E}^{(\rho)}$ into irreducible components where each \bar{E}_i is called an exceptional curve of the first kind under μ_ρ . Then, $V^{(t)}(H)$ has one and only one quasisingular point along \bar{E}_t for $1 \leq t \leq \rho-1$, which is called the t -th exceptional curve.

(b3) Let $(\Phi \circ \mu_\rho)_{\text{divisor}}$ be a divisor of $\Phi \circ \mu_\rho$ defined by

$$(7.2.4) \quad (\Phi \circ \mu_\rho)_{\text{divisor}} = V^{(\rho)}(\Phi) + \sum_{i=1}^{\rho} \bar{e}_i \bar{E}_i,$$

where each \bar{e}_i is the multiplicity of $\Phi \circ \mu_\rho$ along \bar{E}_i for $1 \leq i \leq \rho$ and $V^{(\rho)}(\Phi)$ is the proper transform of $V(\Phi)$ under μ_ρ .

In more detail, let $\bar{\pi}_i : \bar{M}^{(i)} \rightarrow \bar{M}^{(i-1)}$ is the i -th blow-up of μ_t at a quasisingular point of $V^{(t-1)}(\Phi)$ for $1 \leq t \leq \rho - 1$ in the sense of Definition 2.6 where $M^{(0)} = \mathbb{C}^2$. Then, $V^{(t)}(\Phi)$ has one and only one quasisingular point along \bar{E}_t for $1 \leq t \leq \rho$, if exists.

Using (3.6.3) of the conclusion of Theorem 3.6, (b1) and (b2), then the equation in (7.2.4) is well-defined, satisfying the following properties:

(b3-1) $\bar{e}_i < \bar{e}_{i+1}$ for $1 \leq i \leq \rho - 1$.

(b3-2) $V(\Phi)$ belongs to the type[1] under μ_ρ in the sense of Definition 2.8 because \bar{E}_ρ is one and only one exceptional curve of the first kind among $\bar{E}^{(\rho)} = \cup_{i=1}^\rho \bar{E}_i$ which has three distinct intersection points with other exceptional curves and the proper transform $V^{(\rho)}(\Phi)$ under μ_ρ .

Conclusions As a necessary condition or a sufficient condition for $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, we have the following:

Fact(1) If $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, then $\zeta = \zeta'$, $\eta = \eta'$ and $n_1 = \ell_1$, $\beta_{1,1} = \delta_{1,1}$ and $d = d'$.

Fact(2) If $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, then $m = \rho$ with $\zeta = \zeta'$ and $\eta = \eta'$, and $e_i = \bar{e}_i$ for $i = 1, 2, \dots, m = \rho$.

Fact(3) Let $d \geq 1$ and $d' \geq 1$. Then $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions, as reduced varieties, if and only if $V(v(f \circ \tau_m)_{\text{proper}}) \stackrel{\text{divisor}}{\sim} V(v(\phi \circ \mu_\rho)_{\text{proper}})$ under the standard resolutions of a quasisingular point along $E_m = \{v = 0\}$ or $\bar{E}_m = \{\bar{v} = 0\}$ with $m = \rho$, and either of Fact(1) and Fact(2) is satisfied. Note that $(F \circ \tau_m)_{\text{proper}} = (f \circ \tau_m)_{\text{proper}}$ and $(\Phi \circ \mu_\rho)_{\text{proper}} = (\phi \circ \mu_\rho)_{\text{proper}}$. \square

Remark 7.2.1. As far as the above assumptions of Lemma 7.2 are concerned, whenever there exist the statements in both (a) and (b) of the assumptions of Theorem 7.1, it was already proved by Theorem 3.6 or Sublemma 5.4 that the statements in (a1), (a2), (a3), (b1), (b2) and (b3) of the assumptions of this lemma are true. \square

§7.2. The proofs of Lemma 7.2 and Theorem 7.1

In this section, we prove Lemma 7.2, and after then, it will be found that there is nothing to prove for Theorem 7.1.

Proof of Lemma 7.2. In preparation for the proof of Fact(1), Fact(2) and Fact(3), first of all, we need to observe the following:

Whenever there are two integers n_1 and $\beta_{1,1}$ with $\gcd(n_1, \beta_{1,1}) = 1$ and $1 \leq n_1 < \beta_{1,1}$, there is a positive integer s such that $sn_1 < \beta_{1,1} \leq (s+1)n_1$. Also, whenever there are two integers ℓ_1 and $\delta_{1,1}$ with $\gcd(\ell_1, \delta_{1,1}) = 1$ and $1 \leq \ell_1 < \delta_{1,1}$, there is a positive integer \bar{s} such that $\bar{s}\ell_1 < \delta_{1,1} \leq (\bar{s}+1)\ell_1$. Then, note that $s < m$ with τ_m and $\bar{s} < \rho$ with μ_ρ .

The proof of Fact(1) Since $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, it is trivial to show that the following are true:

- (i) $V(F)$ and $V(\Phi)$ have the same number of components, that is, $1 + \zeta + \eta = 1 + \zeta' + \eta'$.
 - (ii) Then, $V^{(1)}(F) \cap E_1$ and $V^{(1)}(\Phi) \cap \bar{E}_1$ have the same number of elements as set, that is, $\zeta = \zeta'$ because $1 \leq n_1 < \beta_{1,1}$ and $1 \leq \ell_1 < \delta_{1,1}$. So, $\eta = \eta'$ by (i).
 - (iii) Note that $e_1 = \zeta + \eta + n_1 d$ and $\bar{e}_1 = \zeta' + \eta' + \ell_1 d'$ are equal, and so $n_1 d = \ell_1 d'$.
 - (iv) Note that $e_{s+1} = \zeta + (s+1)\eta + (s+1)\beta_{1,1}d$ and $e_{\bar{s}+1} = \zeta' + (\bar{s}+1)\eta' + (\bar{s}+1)\delta_{1,1}d'$ with $s = \bar{s}$ are equal, and so $\beta_{1,1}d = \delta_{1,1}d'$.
 - (v) Since $\gcd(n_1, \beta_{1,1}) = 1$ and $\gcd(\ell_1, \delta_{1,1}) = 1$, then $n_1 = \ell_1$, $\beta_{1,1} = \delta_{1,1}$, and $d = d'$.
- Thus, the proof of Fact(1) is done.

The proof of Fact(2) By (a3-2) and (b3-2) in the assumption of this lemma, as reduced varieties, E_m is one and only one exceptional curve of the first kind among $E^{(m)} = \cup_{i=1}^m E_i$ which has three distinct intersection points with other exceptional curves and the proper transform $V^{(m)}(F)$ under τ_m , and also \bar{E}_ρ is one and only one exceptional curve of the first kind among $\bar{E}^{(\rho)} = \cup_{i=1}^\rho \bar{E}_i$ which has three distinct intersection points with other exceptional curves and the proper transform $V^{(\rho)}(\Phi)$ under $\bar{\tau}_\rho$. Since it is assumed that

$V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions as reduced varieties, then it is clear by Fact(1) that $e_m = n_1\zeta + \beta_{1,1}\eta + n_1\beta_{1,1}d$ and $\bar{e}_\rho = \ell_1\zeta' + \delta_{1,1}\eta' + \ell_1\delta_{1,1}d'$ are equal, and that $G(y, z)$ of (7.2.1) and $H(y, z)$ of (7.2.3) are equal, too. So, $m = \rho$ and $e_i = \bar{e}_i$ for $i = 1, 2, \dots, m = \rho$. Thus, Fact(2) can be easily proved.

Also, it is trivial to prove by Theorem 3.6 that Fact(3) is true. \square

Proof of Theorem 7.1. If $V(F) \stackrel{\text{divisor}}{\sim} V(\Phi)$ under the standard resolutions, then it is clear by Definition 2.6 that $\zeta = \zeta'$ and $\eta = \eta'$ and $V(f)$ and $V(\phi)$ have the same multiplicity $n_1d = \ell_1d'$ at $(y, z) = (0, 0)$. For the proof of theorem, it remains to show that $n_1 = \ell_1$, $\beta_{1,1} = \delta_{1,1}$ and $d = d'$, which can be easily proved by the induction method on the multiplicity of $V(f)$ at $(y, z) = (0, 0)$, using the same kind of methods and notations as we have used in the proof of Theorem 3.6 with Lemma 4.2 and Lemma 4.3. Thus, the proof can be finished. \square

§7.3 The necessary and sufficient condition for any two quasi-Puiseux convergent power series of recursive types in $\mathbb{C}\{y, z\}$ to have the same divisor under the standard resolutions and The solution of (i) of Problem[1-A]

Theorem 7.3. *Let r and ρ be arbitrary positive integers.*

Assumptions *By the same way as in Definition 5.0.0, let $g_r \in \mathbb{C}\{y, z\}$ be the quasi-Puiseux convergent power series of the recursive r -type defined by Sequences[I], and $\phi_\rho \in \mathbb{C}\{y, z\}$ be the quasi-Puiseux convergent power series of the recursive ρ -type defined by Sequences[II] such that Sequences[I] and Sequences[II] are defined respectively, as follows: Note that Sequences[I] and Sequences[II] are the same up to the change of notations.*

Sequences[I] Let $\{X_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$ be three sequences satisfying the following five conditions:

Five conditions are denoted by **The 1st Cond⁽⁰⁾**, \dots , **The 5-th Cond⁽⁰⁾** of Sequences[I].

[I]-(1) The 1st Cond⁽⁰⁾ of Sequences[I]:

(1a) $X_1 = \{n_1, \beta_{1,1}\}$ with $n_1 \geq 2$ and $\beta_{1,1} \geq 1$.

(1b) $X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with $n_j \geq 2$ where $j = 2, \dots, r$.

If $j \geq 2$, then assume that at least one of $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}$ is nonzero.

[I]-(2) The 2nd Cond⁽⁰⁾ of Sequences[I]:

(2a) $g_1 = z^{n_1} + \varepsilon_1 y^{\beta_{1,1}}$.

(2b) $g_j = g_{j-1}^{n_j} + \varepsilon_j y^{\beta_{j,1}} z^{\beta_{j,2}} g_1^{\beta_{j,3}} \dots g_{j-2}^{\beta_{j,j}}$ where $j = 2, \dots, r$.

Note that each $\varepsilon_i = \varepsilon_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq i \leq r$.

[I]-(3) The 3rd Cond⁽⁰⁾ of Sequences[I]:

(3a) $\Delta_1(t) = t$ for each $t \in N_0$.

(3b) $\Delta_j(t_j)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, r$.

[I]-(4) The 4-th Cond⁽⁰⁾ of Sequences[I]:

(4) $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r$.

[I]-(5) The 5-th Cond⁽⁰⁾ of Sequences[I]:

(5) $\gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq r$.

Sequences[II] Let $\{W_k : k = 1, 2, \dots, \rho\}$ with $W_k \subset N_0$, $\{\phi_k : k = 1, 2, \dots, \rho\}$ with $\phi_k \in \mathbb{C}\{y, z\}$ and $\{\omega_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, \rho\}$ be three sequences satisfying the following five conditions:

Five conditions are denoted by **The 1st Cond⁽⁰⁾**, \dots , **The 5-th Cond⁽⁰⁾** of Sequences[II].

[II]-(1) The 1st Cond⁽⁰⁾ of Sequences[II]:

(1a) $W_1 = \{\ell_1, \delta_{1,1}\}$ with $\ell_1 \geq 2$ and $\delta_{1,1} \geq 1$.

(1b) $W_j = \{\ell_j, \delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}\}$ with $\ell_j \geq 2$ where $j = 2, \dots, \rho$.

If $j \geq 2$, then assume that at least one of $\delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}$ is nonzero.

[II]-(2) The 2nd Cond⁽⁰⁾ of Sequences[II]:

(2a) $\phi_1 = z^{\ell_1} + \bar{\varepsilon}_1 y^{\delta_{1,1}}$.

(2b) $\phi_j = \phi_{j-1}^{\ell_j} + \bar{\varepsilon}_j y^{\delta_{j,1}} z^{\delta_{j,2}} \phi_1^{\delta_{j,3}} \cdots \phi_{j-2}^{\delta_{j,j}}$ where $j = 2, \dots, \rho$.

Note that each $\bar{\varepsilon}_i = \bar{\varepsilon}_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq i \leq \rho$.

[II]-(3) The 3rd Cond⁽⁰⁾ of Sequences[II]:

(3a) $\omega_1(t) = t$ for each $t \in N_0$.

(3b) $\omega_j(t_k)_{k=1}^j = t_j \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1} + \ell_{j-1} \omega_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, \rho$.

[II]-(4) The 4-th Cond⁽⁰⁾ of Sequences[II]:

(4) $\omega_j(\delta_{j,k})_{k=1}^j > \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq \rho$.

[II]-(5) The 5-th Cond⁽⁰⁾ of Sequences[II]:

(5) $\gcd(\ell_j, \omega_j(\delta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq \rho$.

Conclusions

Let $V(y^\zeta g_r)$ and $V(y^\eta \phi_\rho)$ be analytic varieties at $(y, z) = (0, 0)$ with isolated singularity at the origin defined as follows:

$$(7.3.0) \quad \begin{aligned} V(y^\zeta g_r) &= \{y, z\} : y^\zeta g_r(y, z) = 0\}, \\ V(y^\eta \phi_\rho) &= \{(y, z) : y^\eta \phi_\rho(y, z) = 0\}. \end{aligned}$$

Note by assumption that g_r and ϕ_ρ are irreducible in $\mathbb{C}\{y, z\}$.

For convenience of notation, we need to assume in addition that the following hold:

- (i) if $\beta_{1,1} = 1$ from g_r , then ζ is a positive integer.
- (ii) if $\delta_{1,1} = 1$ from ϕ_ρ , then η is a positive integer.
- (iii) if $\zeta = 0$, then $\beta_{1,1} > 1$ by (i), and so we may assume without loss of generality that $2 \leq n_1 < \beta_{1,1}$, because if $n_1 > \beta_{1,1} \geq 2$ then replace z by y , and y by z .
- (iv) if $\eta = 0$, then $\delta_{1,1} > 1$ by (ii), and so we may assume without loss of generality that $2 \leq \ell_1 < \delta_{1,1}$, because if $\ell_1 > \delta_{1,1} \geq 2$ then replace z by y , and y by z .

[A] Then, we have the following:

$$(7.3.1) \quad V(y^\zeta g_r) \stackrel{\text{divisor}}{\sim} V(y^\eta \phi_\rho) \text{ under the standard resolutions as reduced varieties}$$

\iff

$$(7.3.2) \quad \begin{aligned} \zeta &= \eta, \quad r = \rho, \quad \text{and} \quad n_j = \ell_j \quad \text{for } 1 \leq j \leq r = \rho, \quad \text{and} \\ \Delta_j(\beta_{j,k})_{k=1}^j &= \omega_j(\delta_{j,k})_{k=1}^j \quad \text{for each } j = 1, 2, \dots, r. \end{aligned}$$

[B] In particular, assume that the above $g_r \in \mathbb{C}\{y, z\}$ in Sequences[I] satisfies either the property of the Puiseux convergent power series or $2 \leq n_1 < \beta_{1,1}$. Also, assume that the above $\phi_\rho \in \mathbb{C}\{y, z\}$ in Sequences[II] satisfies either the property of the Puiseux convergent power series or $2 \leq \ell_1 < \delta_{1,1}$.

Then, we have

$$(7.3.3) \quad g_r \stackrel{\text{divisor}}{\sim} \phi_\rho \text{ under the standard resolutions}$$

\iff

$$(7.3.4) \quad \begin{aligned} n_j &= \ell_j \quad \text{for each } j = 1, 2, \dots, r = \rho, \quad \text{and} \\ \Delta_j(\beta_{j,k})_{k=1}^j &= \omega_j(\delta_{j,k})_{k=1}^j \quad \text{for each } j = 1, 2, \dots, r \end{aligned}$$

\iff

$$(7.3.5) \quad \begin{aligned} n_j &= \ell_j \quad \text{for each } j = 1, 2, \dots, r = \rho, \quad \text{and} \\ \Delta_j(t_k)_{k=1}^j &= \omega_j(t_k)_{k=1}^j \quad \text{for each } (t_k)_{k=1}^j \in N_0^j, \quad \text{and} \\ \Delta_j(\beta_{j,k})_{k=1}^j &= \Delta_j(\delta_{j,k})_{k=1}^j \quad \text{for each } j = 1, 2, \dots, r. \quad \square \end{aligned}$$

Remark 7.3.1. By Definition 5.0.0, $g_r \in \mathbb{C}\{y, z\}$ of Sequences[I] is called a quasi-Puiseux convergent power series of the recursive r-type. In addition, if $2 \leq n_1 < \beta_{1,1}$ in The 1-th Cond⁽⁰⁾, then $g_r \in \mathbb{C}\{y, z\}$ of Sequences[I] is a Puiseux convergent power series of the recursive r-type. Also, by Definition 5.0.0, $\phi_\rho \in \mathbb{C}\{y, z\}$ of Sequences[II] is called a quasi-Puiseux series of the recursive ρ -type. In addition, if $2 \leq \ell_1 < \delta_{1,1}$ in The 1-th Cond⁽⁰⁾, then $\phi_\rho \in \mathbb{C}\{y, z\}$ of Sequences[II] is called a Puiseux convergent power series of the recursive ρ -type. \square

§7.4. In preparation for the proof of Theorem 7.3

For the proof of Theorem 7.3, first we will prepare two sublemmas, Sublemma 7.4 and Sublemma 7.5 in §7.4. After then, in §7.5 we will finish the proof of Theorem 7.3, using these two sublemmas. First, to construct the statement for Sublemma 7.4 with its proof, it suffices to apply Sublemma 5.1, Sublemma 5.2, Sublemma 5.3 and Sublemma 5.4 of Theorem 5.0 to the assumptions of Theorem 7.3. Next, to construct the statement for Sublemma 7.5 with its proof, it suffices to apply Sublemma 5.5 of Theorem 5.0 to Sublemma 7.4.

Sublemma 7.4. Assumptions Suppose that the same properties and notations as in Sequences[I] and Sequences[II] of the assumptions of Theorem 7.3 hold. Let r be an arbitrary integer with $r \geq 1$ in Sequences[I], and ρ be an arbitrary integer with $\rho \geq 1$ in Sequences[II].

Conclusions Then, we have the following three facts, Fact(1), Fact(2) and Fact(3).

Fact(1): By (5.2.1) in Sublemma 5.2 of Theorem 5.0, we may assume without any need of proof that two properties, Property(1) and Property(2) of Fact(1) are true:

Property(1) of Fact(1) For any $r \geq 1$, g_r of Theorem 7.3 can be written in the form

$$(7.4.1) \quad g_r = g_r(y, z) = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta$$

with $n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r$,

where $\varepsilon_1 = \varepsilon_1(y, z)$ is a unit in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β , if exist.

Property(2) of Fact(1) For any $\rho \geq 1$, ϕ_ρ of Theorem 7.3 can be written in the form

$$(7.4.2) \quad \phi_\rho = \phi_\rho(y, z) = (z^{\ell_1} + \bar{\varepsilon}_1 y^{\delta_{1,1}})^{\ell_2 \ell_3 \cdots \ell_\rho} + \sum_{\gamma, \delta \geq 0} a_{\gamma, \delta}^{(\rho)} y^\gamma z^\delta$$

with $\ell_1 \gamma + \delta_{1,1} \delta > \ell_1 \delta_{1,1} \ell_2 \ell_3 \cdots \ell_\rho$,

where $\bar{\varepsilon}_1 = \bar{\varepsilon}_1(y, z)$ is a unit in $\mathbb{C}\{y, z\}$, and the $a_{\gamma, \delta}^{(\rho)}$ are nonzero complex numbers for some nonnegative integers γ and δ , if exist.

Fact(2): Let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ and $\mu_\lambda = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \cdots \circ \bar{\pi}_\lambda : \bar{M}^{(\lambda)} \rightarrow \mathbb{C}^2$ be the composition of two finite numbers m and λ of successive blow-ups which are needed only to get the standard resolutions of the singular point $(y, z) = (0, 0)$ of $V(y^\zeta g_1)$ and $V(y^\eta \phi_1)$, respectively. For brevity of notation, write $G_j = y^\zeta g_j$ for $1 \leq j \leq r$, and let $\Phi_i = y^\eta \phi_i$ for $1 \leq i \leq \rho$, as we have seen in (7.3.0).

Then, $y^\zeta g_j \in \text{type}[1]$ under the standard resolution τ_m and $y^\eta \phi_i \in \text{type}[1]$ under the standard resolution μ_λ by Theorem 5.0.

Fact(3): As in Lemma 7.2, let $(G_j \circ \tau_m)_{\text{divisor}}$ be a divisor of $G_j \circ \tau_m$ defined by

$$(7.4.3) \quad (G_j \circ \tau_m)_{\text{divisor}} = V^{(m)}(G_j) + \sum_{i=1}^m e_i E_i,$$

where each e_i is the multiplicity of $G_j \circ \tau_m$ along E_i for $1 \leq i \leq m$ and $V^{(m)}(G_j)$ is the proper transform of $V(G_j)$ under τ_m .

Also, as in Lemma 7.2, let $(\Phi_j \circ \mu_\lambda)_{\text{divisor}}$ be a divisor of $\Phi_j \circ \mu_\lambda$ defined by

$$(7.4.4) \quad (\Phi_j \circ \mu_\lambda)_{\text{divisor}} = V^{(\lambda)}(\Phi_j) + \sum_{i=1}^{\lambda} \bar{e}_i \bar{E}_i,$$

where each \bar{e}_i is the multiplicity of $\Phi_j \circ \mu_\lambda$ along \bar{E}_i for $1 \leq i \leq \lambda$ and $V^{(\lambda)}(\Phi_j)$ is the proper transform of $V(\Phi_j)$ under μ_λ .

If $V(y^\zeta g_r) \xrightarrow{\text{divisor}} V(y^\eta \phi_\rho)$ under the standard resolutions as reduced varieties, we have the following by Lemma 7.2:

$$(7.4.5) \quad \tau_m = \mu_\lambda \text{ and } e_i = \bar{e}_i \text{ for } i = 1, 2, \dots, m = \lambda. \quad \square$$

Proof of Sublemma 7.4. Applying Sublemma 5.1, Sublemma 5.2, Sublemma 5.3 and Sublemma 5.4 of Theorem 5.0, and Lemma 7.2 to the assumptions of Theorem 7.3, then there is nothing to prove for this sublemma. \square

Sublemma 7.5. Assumptions For brevity of notation, let Sequences[I] of the $(r+1)$ -th type and Sequences[II] of the $(\rho+1)$ -th type be given sequences, each of which satisfies the same kind of properties and notations, as we have seen in Sequences[I] of the r -th type and Sequences[II] of the ρ -th type under the assumptions of Theorem 7.3, respectively. Note by Theorem 7.3 that r is an arbitrary integer with $r \geq 1$ in Sequences[I], and ρ is an arbitrary integer with $\rho \geq 1$ in Sequences[II].

Conclusions As in Sublemma 7.4, let $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ and $\mu_\lambda : \bar{M}^{(\lambda)} \rightarrow \mathbb{C}^2$ be the composition of two finite numbers m and λ of successive blow-ups which are needed only to get the standard resolutions of the singular point $(y, z) = (0, 0)$ of $V(y^\zeta g_1)$ and $V(y^\eta \phi_1)$, respectively. Then, we have two properties, Property(1) and Property(2):

Property(1) of Conclusions As we have seen in Sublemma 5.5, using $\{h_k = (g_{k+1} \circ \tau_m)_{\text{proper}} : k = 1, 2, \dots, r\}$ with $h_k \in \mathbb{C}\{v, u+1\}$, it has been already proved that we can construct another new sequences generated by Sequences[I] of the $(r+1)$ -th type, denoted by Sequences[I]⁽¹⁾ of the r -th type, which is rewritten as follows:

Sequences[I]⁽¹⁾ of the r -th type: Let $\{Y_k : k = 1, 2, \dots, r\}$ with $Y_k \subset N_0$,

$\{h_k : k = 1, 2, \dots, r\}$ with $h_k = (g_{k+1} \circ \tau_m)_{\text{proper}} \in \mathbb{C}\{v, u+1\}$, and

$\{\Xi_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$

be three sequences, satisfying the following five conditions for each k .

Such five conditions are denoted by **The 1st Cond⁽¹⁾**, \dots , **The 5-th Cond⁽¹⁾**.

[I]-(1) The 1st Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

(1a) $Y_1 = \{s_1, \gamma_{1,1}\}$ where

$$s_1 = n_2 \geq 2 \quad \text{and} \quad \gamma_{1,1} = \Delta_2^\#(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 > 0.$$

(1b) $Y_j = \{s_j, \gamma_{j,1}, \gamma_{j,2}, \dots, \gamma_{j,j}\}$ where for $j=2, 3, \dots, r$,

$$s_j = n_{j+1} \geq 2, \quad \text{and} \quad \gamma_{j,1} = \Delta_{j+1}^\#(\beta_{j+1,k})_{k=1}^{j+1} - n_1 \beta_{1,1} n_2 n_3 \cdots n_{j+1} > 0, \\ \gamma_{j,2} = \beta_{j+1,3}, \gamma_{j,3} = \beta_{j+1,4}, \dots, \gamma_{j,j} = \beta_{j+1,j+1}.$$

[I]-(2) The 2nd Cond⁽¹⁾ of Sequences[I]⁽¹⁾ : Let $2 \leq j \leq r$.

$$(7.5.1) \quad (2a) \quad h_1 = (u+1)^{s_1} + \varepsilon_{1,2} v^{\gamma_{1,1}}.$$

$$(2b) \quad h_j = h_{j-1}^{s_j} + \varepsilon_{1,j+1} v^{\gamma_{j,1}} (u+1)^{\gamma_{j,2}} h_1^{\gamma_{j,3}} h_2^{\gamma_{j,4}} \cdots h_{j-2}^{\gamma_{j,j}}.$$

Note that $\varepsilon_{1,i} = \varepsilon_{1,i}(v, u+1)$ is a unit in $\mathbb{C}\{v, u+1\}$ for $2 \leq i \leq r+1$.

[I]-(3) The 3rd Cond⁽¹⁾ of Sequences[I]⁽¹⁾ : Let $2 \leq j \leq r$.

$$(3a) \quad \Xi_1(t) = t \text{ for each } t \in N_0.$$

$$(3b) \quad \Xi_j(t_k)_{k=1}^j = t_j \Xi_{j-1}(\gamma_{j-1,k})_{k=1}^{j-1} + s_{j-1} \Xi_{j-1}(t_k)_{k=1}^{j-1} \text{ for } (t_k)_{k=1}^j \in N_0^j.$$

[I]-(4 α) The (4 α)-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

$$(7.5.2) \quad (4a) \quad \Xi_1(\gamma_{1,1}) = \gamma_{1,1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 > 0.$$

$$(4b) \quad \Xi_q(\gamma_{q,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \\ = \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,k})_{k=1}^q > 0 \text{ for } q = 2, 3, \dots, r.$$

[I]-(5 α) The (5 α)-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

$$(7.5.3) \quad \gcd(s_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}) = 1 \text{ for } j = 1, 2, \dots, r.$$

Property(2) of Conclusions *As we have seen in Sublemma 5.5, using $\{\psi_k = (\phi_{k+1} \circ \mu_\lambda)_{proper} : k = 1, 2, \dots, \rho\}$ with $\psi_k \in \mathbb{C}\{\bar{v}, \bar{u} + 1\}$, it was already proved that we can construct another new sequences generated by Sequences[II] of the $(\rho + 1)$ -th type, denoted by Sequences[II]⁽¹⁾ of the ρ -th type, which is rewritten as follows:*

Sequences[II]⁽¹⁾ of the ρ -th type: *Let $\{L_k : k = 1, 2, \dots, \rho\}$ with $L_k \subset N_0$,*

$\{\psi_k : k = 1, 2, \dots, \rho\}$ with $\psi_k = (\phi_{k+1} \circ \mu_\lambda)_{proper} \in \mathbb{C}\{\bar{v}, \bar{u} + 1\}$, and

$\{\theta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, \rho\}$

be three sequences, satisfying the following five conditions for each k :

*Such conditions are denoted by **The 1st Cond⁽¹⁾**, \dots , **The 5-th Cond⁽¹⁾**.*

[II]-(1) The 1st Cond⁽¹⁾ of Sequences[II]⁽¹⁾ :

$$(1a) \quad L_1 = \{p_1, \nu_{1,1}\} \text{ where}$$

$$p_1 = \ell_2 \geq 2 \text{ and } \nu_{1,1} = \omega_2^\#(\delta_{2,1}, \delta_{2,2}) - \ell_1 \delta_{1,1} \ell_2 > 0.$$

$$(1b) \quad L_j = \{p_j, \nu_{j,1}, \nu_{j,2}, \dots, \nu_{j,j}\} \text{ where for } j=2, 3, \dots, \rho,$$

$$p_j = \ell_{j+1} \geq 2, \text{ and } \nu_{j,1} = \omega_{j+1}^\#(\delta_{j+1,k})_{k=1}^{j+1} - \ell_1 \delta_{1,1} \ell_2 \ell_3 \cdots \ell_{j+1} > 0,$$

$$\nu_{j,2} = \delta_{j+1,3}, \nu_{j,3} = \delta_{j+1,4}, \dots, \nu_{j,j} = \delta_{j+1,j+1}.$$

[II]-(2) The 2nd Cond⁽¹⁾ of Sequences[II]⁽¹⁾ : *Let $2 \leq j \leq \rho$.*

$$(7.5.4) \quad (2a) \quad \psi_1 = (\bar{u} + 1)^{p_1} + \bar{\varepsilon}_{1,2} \bar{v}^{\nu_{1,1}}.$$

$$(2b) \quad \psi_j = \psi_{j-1}^{p_j} + \bar{\varepsilon}_{1,j+1} v^{\nu_{j,1}} (\bar{u} + 1)^{\nu_{j,2}} \psi_1^{\nu_{j,3}} \psi_2^{\nu_{j,4}} \cdots \psi_{j-2}^{\nu_{j,j}}.$$

Note that $\bar{\varepsilon}_{1,i} = \bar{\varepsilon}_{1,i}(\bar{v}, \bar{u} + 1)$ is a unit in $\mathbb{C}\{\bar{v}, \bar{u} + 1\}$ for $2 \leq i \leq \rho + 1$.

[II]-(3) The 3rd Cond⁽¹⁾ of Sequences[II]⁽¹⁾ : *Let $2 \leq j \leq \rho$.*

$$(3a) \quad \theta_1(t) = t \text{ for each } t \in N_0.$$

$$(3b) \quad \theta_j(t_k)_{k=1}^j = t_j \theta_{j-1}(\nu_{j-1,k})_{k=1}^{j-1} + p_{j-1} \theta_{j-1}(t_k)_{k=1}^{j-1} \text{ for } (t_k)_{k=1}^j \in N_0^j.$$

[II]-(4 α) The (4 α)-th Cond⁽¹⁾ of Sequences[II]⁽¹⁾ :

$$(7.5.5) \quad (4a) \quad \theta_1(\nu_{1,1}) = \nu_{1,1} = \omega_2(\delta_{2,1}, \delta_{2,2}) - \ell_1 \delta_{1,1} \ell_2 > 0.$$

$$(4b) \quad \theta_q(\nu_{q,k})_{k=1}^q - p_q p_{q-1} \theta_{q-1}(\nu_{q-1,k})_{k=1}^{q-1} \\ = \omega_{q+1}(\delta_{q+1,k})_{k=1}^{q+1} - \ell_{q+1} \ell_q \omega_q(\delta_{q,k})_{k=1}^q > 0 \text{ for } q = 2, 3, \dots, \rho.$$

[II]-(5 α) The (5 α)-th Cond⁽¹⁾ of Sequences[II]⁽¹⁾ :

$$(7.5.6) \quad \gcd(p_q, \theta_q(\nu_{q,k})_{k=1}^q) = \gcd(\ell_{q+1}, \omega_{q+1}(\delta_{q+1,k})_{k=1}^{q+1}) = 1 \text{ for } q = 1, 2, \dots, \rho. \quad \square$$

Proof of Sublemma 7.5. The proof of this sublemma just follows from Sublemma 5.5 of Theorem 5.0. \square

§7.5. For the proof of Theorem 7.3

In this section, we prove Theorem 7.3 by Sublemma 7.4 and Sublemma 7.5 in §7.4.

Proof of Theorem 7.3. The proof of the theorem is as follows:

[A] The first aim is to show that two statements in (7.3.1) and (7.3.2) are equivalent.

[B] As a consequence of [A], the second aim is to show easily that three statements in (7.3.3), (7.3.4) and (7.3.5) are equivalent.

In preparation for the proof of theorem, for notation, we write $G_j = y^\zeta g_j$ with $1 \leq j \leq r$ and $\Phi_s = y^\eta \phi_s$ with $1 \leq s \leq \rho$, as we have seen in (7.3.0). It is clear by Sublemma 5.2 or Corollary 5.6 that $y^\zeta g_r$ and $y^\eta \phi_\rho$ of (7.3.0) can be rewritten as follows:

$$\begin{aligned}
 (7.3.6) \quad & G_r = y^\zeta g_r, \\
 & g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{n_2 n_3 \cdots n_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1 \\
 & \text{and } n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r, \\
 & \Phi_\rho = y^\eta \phi_\rho, \\
 & \phi_\rho = (z^{\ell_1} + \bar{\varepsilon}_1 y^{\delta_{1,1}})^{\ell_2 \ell_3 \cdots \ell_\rho} + \sum_{\gamma, \delta \geq 0} a_{\gamma, \delta}^{(\rho)} y^\gamma z^\delta \quad \text{with } \bar{\varepsilon}_1 = 1 \\
 & \text{and } \ell_1 \gamma + \delta_{1,1} \delta > \ell_1 \delta_{1,1} \ell_2 \ell_3 \cdots \ell_\rho,
 \end{aligned}$$

where (i) $n_1 \geq 2$ and $\gcd(n_1, \beta_{1,1}) = 1$,

(ii) $\ell_1 \geq 2$ and $\gcd(\ell_1, \delta_{1,1}) = 1$,

(iii) ζ is either a positive integer or 0, and η is either a positive integer or 0,

(iv) if $\beta_{1,1} = 1$ then ζ is a positive integer, and if $\delta_{1,1} = 1$ then η is a positive integer,

(v) if $\zeta = 0$, then $\beta_{1,1} \geq 2$, and also we may assume for brevity of notation that $n_1 < \beta_{1,1}$; and if $\eta = 0$, then $\delta_{1,1} \geq 2$, and also we may assume for brevity of notation that $\ell_1 < \delta_{1,1}$,

(vi) both $\varepsilon_1 = \varepsilon_1(y, z)$ and $\bar{\varepsilon}_1 = \bar{\varepsilon}_1(y, z)$ are units in $\mathbb{C}\{y, z\}$, which may be analytically assumed to be one, if necessary,

(vii) the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β if exist, and the $a_{\gamma, \delta}^{(\rho)}$ are nonzero complex numbers for some nonnegative integers γ and δ if exist.

Now, we shall prove firstly the equivalence of the condition in [A], and secondly that of the condition in [B], respectively.

[A] For proof, we may assume without loss of generality that $1 \leq r \leq \rho$. Then, for the induction proof on $r \geq 1$, it suffices to consider two cases, respectively:

Case[I] $r = 1$, and Case[II] $r \geq 1$.

Case[I] Let $r = 1 \leq \rho$. Assume that $y^\zeta g_1 \stackrel{\text{divisor}}{\sim} y^\eta \phi_\rho$ under the standard resolutions as reduced varieties. By Theorem 7.1 and Lemma 7.2, there is nothing to prove for [A].

Case[II] Let $r \geq 1$. Now, suppose we have shown for the induction assumption on r that the equivalence of the condition in (7.3.2) for $V(y^\zeta g_r) \stackrel{\text{divisor}}{\sim} V(y^\eta \phi_\rho)$ under the standard resolutions as reduced varieties in (7.3.1) is true.

Then, it suffices to prove that [A] of the theorem is true on the positive integer $r + 1$, which can be represented as follows: We may assume that $r + 1 \leq \rho + 1$.

$$\begin{aligned}
 (7.3.7) \quad & G_{r+1} = y^\zeta g_{r+1} \stackrel{\text{divisor}}{\sim} y^\eta \phi_{\rho+1} = \Phi_{\rho+1} \text{ under the standard resolutions as reduced varieties} \\
 \iff & \zeta = \eta, r + 1 = \rho + 1, n_j = \ell_j \text{ and } \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \text{ for } 1 \leq j \leq r + 1.
 \end{aligned}$$

In order to prove (7.3.7), let $(G_{r+1} \circ \tau_m)_{\text{divisor}}$ be a divisor of $G_{r+1} \circ \tau_m$ defined by $(G_{r+1} \circ \tau_m)_{\text{divisor}} = V^{(m)}(G_{r+1}) + \sum_{i=1}^m e_i E_i$, where each e_i is the multiplicity of $G_{r+1} \circ \tau_m$ along E_i for $1 \leq i \leq m$ and $V^{(m)}(G_{r+1})$ is the proper transform of $V(G_{r+1})$ under τ_m . Also, as in Lemma 7.2, let $(\Phi_{\rho+1} \circ \mu_\lambda)_{\text{divisor}}$ be a divisor of $\Phi_{\rho+1} \circ \mu_\lambda$ defined by $(\Phi_{\rho+1} \circ \mu_\lambda)_{\text{divisor}} = V^{(\lambda)}(\Phi_{\rho+1}) + \sum_{i=1}^\lambda \bar{e}_i \bar{E}_i$, where each \bar{e}_i is the multiplicity of $\Phi_{\rho+1} \circ \mu_\lambda$ along \bar{E}_i for $1 \leq i \leq \lambda$ and $V^{(\lambda)}(\Phi_{\rho+1})$ is the proper transform of $V(\Phi_{\rho+1})$ under μ_λ .

Following the notations in Sublemma 7.5, recall that as reduced varieties,

$$(7.3.8) \quad \begin{aligned} \{vh_r = 0\} &= \{(G_{r+1} \circ \tau_m)_{total} = 0\} \text{ at } (v, u+1) = (0, 0), \\ \{\bar{v}\psi_\rho = 0\} &= \{(\Phi_{\rho+1} \circ \mu_\lambda)_{total} = 0\} \text{ at } (\bar{v}, \bar{u}+1) = (0, 0), \end{aligned}$$

where $h_r = (g_{r+1} \circ \tau_m)_{proper}$ in $\mathbb{C}\{v, u+1\}$ and $\psi_\rho = (\phi_{\rho+1} \circ \mu_\lambda)_{proper}$ in $\mathbb{C}\{\bar{v}, \bar{u}+1\}$.

It is clear by Lemma 7.2 and by Sublemma 7.4 that

$$(7.3.9) \quad y^\zeta g_{r+1} \stackrel{\text{divisor}}{\sim} y^\eta \phi_{\rho+1} \text{ under the standard resolutions as reduced varieties}$$

\iff

$$(7.3.10) \quad \begin{aligned} \zeta &= \eta, \quad n_1 = \ell_1, \quad \beta_{1,1} = \delta_{1,1}, \quad n_2 n_3 \cdots n_{r+1} = \ell_2 \ell_3 \cdots \ell_{\rho+1} \text{ and} \\ \tau_m &= \mu_\lambda \text{ and } e_i = \bar{e}_i \text{ for } 1 \leq i \leq m = \lambda, \\ v h_r &\stackrel{\text{divisor}}{\sim} \bar{v} \psi_\rho \text{ under the standard resolutions as reduced varieties} \end{aligned}$$

It is clear that $e_m = n_1 \beta_{1,1} n_2 \cdots n_{r+1} + \zeta n_1 = \ell_1 \delta_{1,1} \ell_2 \cdots \ell_{\rho+1} + \eta \ell_1 = \bar{e}_\lambda$.

So, it remains to show by (7.3.7), (7.3.9) and (7.3.10) that $r+1 = \rho+1$, $n_j = \ell_j$ and $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$ for $2 \leq j \leq r+1$.

By The 1-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ and by (7.5.1) of Sublemma 7.5, along $E_m = \{v = 0\}$, $(G_{r+1} \circ \tau_m)_{total}$ can be written in the form

$$(7.3.11) \quad \begin{aligned} (G_{r+1} \circ \tau_m)_{total} &= v^{n_1 \beta_{1,1} n_2 \cdots n_{r+1} + \zeta n_1} u^{b n_1 n_2 \cdots n_{r+1}} (g_{r+1} \circ \tau_m)_{proper}, \\ (g_{r+1} \circ \tau_m)_{proper} &= h_{r-1}^{s_r} + \varepsilon_{1,r+1} v^{\gamma_{r,1}} (u+1)^{\gamma_{r,2}} h_1^{\gamma_{r,3}} h_2^{\gamma_{r,4}} \cdots h_{r-2}^{\gamma_{r,r}} = h_r, \end{aligned}$$

where for notation, it may be assumed by Sublemma 5.4 that $(g_1 \circ \tau_m)_{proper} = (u+1)$ and that $\varepsilon_{1,r+1}$ is a unit in $\mathbb{C}\{v, u+1\}$. Note that $(G_{r+1} \circ \tau_m)_{proper} = (g_{r+1} \circ \tau_m)_{proper}$.

Also, by The 1-th Cond⁽¹⁾ of Sequences[II]⁽¹⁾ and by (7.5.4) of Sublemma 7.5, along $\bar{E}_\lambda = \{\bar{v} = 0\}$, $(\Phi_\rho \circ \mu_\lambda)_{total}$ can be written in the form

$$(7.3.12) \quad \begin{aligned} (\Phi_{\rho+1} \circ \mu_\lambda)_{total} &= \bar{v}^{\ell_1 \delta_{1,1} \ell_2 \cdots \ell_\rho + \eta \ell_1} \bar{u}^{\bar{b} \ell_1 \ell_2 \cdots \ell_\rho} (\phi_{\rho+1} \circ \mu_\lambda)_{proper}, \\ (\phi_{\rho+1} \circ \mu_\lambda)_{proper} &= \psi_{\rho-1}^{p_\rho} + \bar{\varepsilon}_{1,\rho+1} v^{\nu_{\rho,1}} (\bar{u}+1)^{\nu_{\rho,2}} \psi_1^{\nu_{\rho,3}} \psi_2^{\nu_{\rho,4}} \cdots \psi_{\rho-1}^{\nu_{\rho,\rho}} = \psi_\rho, \end{aligned}$$

where for notation, it may be assumed by Sublemma 5.4 that $(\phi_1 \circ \mu_\lambda)_{proper} = (\bar{u}+1)$ and that $\bar{\varepsilon}_{1,\rho+1}$ is a unit in $\mathbb{C}\{\bar{v}, \bar{u}+1\}$. Note that $(\Phi_{\rho+1} \circ \mu_\lambda)_{proper} = (\phi_{\rho+1} \circ \mu_\lambda)_{proper}$.

Apply the induction assumption on the integer r to two sequences in Sublemma 7.5, which are denoted by Sequences[I]⁽¹⁾ and Sequences[II]⁽¹⁾. Because these two sequences, Sequences[I]⁽¹⁾ and Sequences[II]⁽¹⁾ satisfy the same kind of five conditions as we have seen in the assumptions of Theorem 7.3, then it is clear by (7.3.11) and (7.3.12), and by induction assumption on the integer r that the following are true:

$$(7.3.13) \quad v h_r \stackrel{\text{divisor}}{\sim} \bar{v} \psi_\rho \text{ under the standard resolutions as reduced varieties}$$

\iff

$$(7.3.14) \quad \begin{aligned} (a) \quad & s_i = p_i \quad \text{for } 1 \leq i \leq r = \rho. \\ (b) \quad & \gamma_{1,1} = \nu_{1,1} \quad \text{or} \quad \Xi_1(\gamma_{1,1}) = \theta_1(\nu_{1,1}). \\ (c) \quad & \Xi_q(\gamma_{q,k})_{k=1}^q = \theta_q(\nu_{q,k})_{k=1}^q \quad \text{for each } q = 2, 3, \dots, r = \rho. \end{aligned}$$

Note that $\gamma_{1,1} = \Delta_2^\sharp(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 > 0$ and $\nu_{1,1} = \omega_2^\sharp(\delta_{2,1}, \delta_{2,2}) - \ell_1 \delta_{1,1} \ell_2 > 0$.

Noting by The 1-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ and The 1-th Cond⁽¹⁾ of Sequences[II]⁽¹⁾ that $s_i = n_{i+1}$ and $p_i = \ell_{i+1}$ for $1 \leq i \leq r = \rho$, it is clear by (a) of (7.3.14) that $n_{i+1} = \ell_{i+1}$ for $1 \leq i \leq r = \rho$.

Now, since it is clear by (c) of (7.3.14) that $\Xi_q(\gamma_{q,k})_{k=1}^q = \theta_q(\nu_{q,k})_{k=1}^q$ for each $q = 1, 2, \dots, r$, and also by (7.3.10) that $\zeta = \eta$, $n_1 = \ell_1$, $\beta_{1,1} = \delta_{1,1}$, then by Sublemma 5.5 and (7.3.14) we have the following: Note that $2 \leq q \leq r$.

$$\begin{aligned}
(7.3.15) \quad & \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1}n_q\Delta_q(\beta_{q,k})_{k=1}^q \\
&= \Xi_q(\gamma_{q,k})_{k=1}^q - s_qs_{q-1}\Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \quad \text{by (7.5.2) of Sublemma 7.5} \\
&= \theta_q(\nu_{q,k})_{k=1}^q - p_qp_{q-1}\theta_{q-1}(\nu_{q-1,k})_{k=1}^{q-1} \\
&= \omega_{q+1}(\delta_{q+1,k})_{k=1}^{q+1} - \ell_{q+1}\ell_q\omega_q(\delta_{q,k})_{k=1}^q \quad \text{by (7.5.5) of Sublemma 7.5.}
\end{aligned}$$

To finish the proof for Case[II], it remains to prove by (7.3.7) that $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$ for each $j = 2, 3, \dots, r+1$ because $\Delta_1(\beta_{1,1}) = \beta_{1,1} = \omega_1(\delta_{1,1}) = \delta_{1,1}$ and $n_j = \ell_j$ for $1 \leq j \leq r+1$ and $\zeta = \eta$ by (7.3.10) and (7.3.14). Since $\Delta_2(\beta_{2,k})_{k=1}^2 - n_2n_1\beta_{1,1} = \gamma_{1,1} = \nu_{1,1} = \omega_2(\delta_{2,k})_{k=1}^2 - \ell_2\ell_1\delta_{1,1}$ by (b) of (7.3.14), then $\Delta_2(\beta_{2,k})_{k=1}^2 = \omega_2(\delta_{2,k})_{k=1}^2$, which can be applicable to the equation of (7.3.15) by induction on the integer $r = \rho$. Since $n_j = \ell_j$ for $1 \leq j \leq r+1$ by (7.3.14), then it is clear by (7.3.15) that $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$ for each $j = 1, 2, \dots, r+1$. Thus, we finished the proof for Case[II]. So, the proof of [A] is done by Case[I] and Case[II].

[B] As an application of [A], it is clear that three statements in (7.3.3), (7.3.4) and (7.3.5) are equivalent, and so the proof of [B] is done.

Therefore, we have completed the proof of the theorem by [A] and [B]. \square

§7.6. In preparation for construction of the standard Puiseux series $\phi_\rho \in \mathbb{C}\{y, z\}$ such that $\phi_\rho \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions for given any Puiseux series $g_r \in \mathbb{C}\{y, z\}$

In preparation for finding the solution of a unique standard Puiseux polynomial $\phi_\rho \in \mathbb{C}[y, z]$ such that $\phi_\rho \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions for given any Puiseux series $g_r \in \mathbb{C}\{y, z\}$, first of all, we need to find Corollary 7.6, which can be proved by Theorem 7.6(The Euclidean Algorithm). Note by Definition 5.0.0 and Corollary 5.7 that ϕ_ρ is a Weierstrass polynomial.

After then, we will solve the above problem by Theorem 7.7, using Corollary 7.6 only.

Theorem (A well-known theorem). *Let A and B be positive integers.*

(1) *It is well-known by Definition 1.14 of the Euclidean algorithm that there are two integers γ and δ such that $\gcd(A, B) = \gamma A + \delta B$ where $\gcd(A, B)$ is a greatest common divisor of A and B .*

(2) *In addition, we have the following:*

(i) *Let γ and δ be two integers such that $\gcd(A, B) = \gamma A + \delta B$. Whether $\gamma\delta \neq 0$ or not, we may assume without loss of generality that δ is chosen negative.*

(ii) *For example, if A and B are relatively prime, then $1 = \gamma A + \delta B$ for some integers γ and δ where $\gamma > 0$ and $\delta < 0$. \square*

Corollary 7.6. *Let A and B be positive integers which are relatively prime.*

(1) *If p is a positive integer such that $p > AB$, then there are two positive integers s and t such that $p = sA + tB$.*

(2) *In particular, under the same assumption as in (1), there is a unique pair of two nonnegative integers s_1 and t_1 such that $p = s_1A + t_1B$ with $0 \leq s_1 < B$ and $t_1 > 0$.*

(3) *For example, let p be a positive integer such that $p > nAB$ for some integer $n \geq 2$.*

(3a) *Then it is clear by (2) that there is a unique pair of two nonnegative integers s_1 and t_1 such that $p = s_1A + t_1B$ with $0 \leq s_1 < B$ and $t_1 > A$.*

(3b) *There is a finitely different pairs of two nonnegative integers $(s_1 + kB) \geq 0$ and $(t_1 - kA) \geq 0$ with $1 \leq k < n$ such that $p = (s_1 + kB)A + (t_1 - kA)B$ with $0 \leq (s_1 + kB)$ and $0 \leq (t_1 - kA)$. \square*

The proof just follows from Theorem(A well-known theorem).

§7.7. An algorithm for finding the standard Puiseux polynomial ϕ_ρ such that $\phi_\rho \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions for given any Puiseux convergent power series g_r

Theorem 7.7. Assumptions By the same way as in Definition 1.1, define arbitrary Puiseux convergent power series $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type by Sequences[I], which consists of three sequences with the following five conditions(or, by the same way as in the assumption of Theorem 7.3, define arbitrary quasi-Puiseux convergent power series $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type by Sequences[I] with an additional inequality $2 \leq n_1 < \beta_{11}$ in The 1-th Cond⁽⁰⁾, which consists of three sequences with the following five conditions):

Sequences[I] Let $\{X_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$ be three different sequences satisfying the following five conditions:

Five conditions are denoted by **The 1st Cond⁽⁰⁾**, \dots , **The 5-th Cond⁽⁰⁾** of Sequences[I].

[I]-(1) The 1st Cond⁽⁰⁾ of Sequences[I]:

(1a) $X_1 = \{n_1, \beta_{1,1}\}$ with $2 \leq n_1 < \beta_{1,1}$.

(1b) $X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with $n_j \geq 2$ where $j = 2, 3, \dots, r$.

For each $j \geq 2$, assume that at least one of $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}$ is nonzero.

[I]-(2) The 2nd Cond⁽⁰⁾ of Sequences[I]:

(2a) $g_1 = z^{n_1} + y^{\beta_{1,1}}$.

(2b) $g_j = g_{j-1}^{n_j} + y^{\beta_{j,1}} z^{\beta_{j,2}} g_1^{\beta_{j,3}} \dots g_{j-2}^{\beta_{j,j}}$ where $j = 2, 3, \dots, r$.

[I]-(3) The 3rd Cond⁽⁰⁾ of Sequences[I]:

(3a) $\Delta_1(t) = t$ for each $t \in N_0$.

(3b) $\Delta_j(t_j)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, 3, \dots, r$.

[I]-(4) The 4-th Cond⁽⁰⁾ of Sequences[I]:

(4a) $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r$.

[I]-(5) The 5-th Cond⁽⁰⁾ of Sequences[I]:

(5a) $\gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = 1$ for $1 \leq j \leq r$.

Conclusions We have two statements, denoted by Fact[I] and Fact[II]. It is clear that Fact[I] and Fact[II] are equivalent, and so it remains to show that Fact[II] is true.

Fact[I]: We can find the computation algorithm of a unique standard Puiseux polynomial ϕ_r in $\mathbb{C}[y, z]$ of the recursive r -type with $\phi_r \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions, satisfying the same properties and notations in Sequences[II] of the assumption of Theorem 7.3, and also the following two additional conditions. Such two additional conditions are denoted by **The 6-th Cond⁽⁰⁾ of Sequences[II]** and **The 7-th Cond⁽⁰⁾ of Sequences[II]** for brevity of notation.

[II]-(6) The 6-th Cond⁽⁰⁾ of Sequences[II]:

(6a) $2 \leq \ell_1 < \delta_{1,1}$.

(6b) $\ell_j \geq 2$, $\delta_{j,1} > 0$, and $0 \leq \delta_{j,k} < \ell_{k-1}$ for $2 \leq j \leq r$ and $2 \leq k \leq j$.

[II]-(7) The 7-th Cond⁽⁰⁾ (for $\phi_r \stackrel{\text{divisor}}{\sim} g_r$ under the standard resolutions):

(7a) $n_j = \ell_j$ for each $j = 1, 2, \dots, r$.

(7b) $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$ for each $j = 1, 2, \dots, r$.

Fact[II]: To find the computation algorithm of $\phi_r(y, z)$ in Fact[I], it remains to apply (3) of Corollary 7.6 only to the following step by induction on the positive integer, and then we can find the desired algorithm in terms of Sublemma 7.9.

Sublemma 7.8. It is clear that $X_1 = W_1$ with $2 \leq n_1 = \ell_1 < \beta_{1,1} = \delta_{1,1}$. Whenever each given set X_j satisfies the same conditions in Sequences[I] of Theorem 7.3 with $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for each $j = 2, 3, \dots, r$, then by (3) of Corollary 7.6 we can find an algorithm for computing one and only one solution set $W_j = \{\ell_j, \delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}\}$ with $\omega_j(\delta_{j,k})_{k=1}^j > \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$ for W_j such that the following conditions are true:

(7.8.1) (a) $n_j = \ell_j \geq 2$ for $1 \leq j \leq r$.

- (b) $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$ for each $j = 1, 2, \dots, r$.
- (c) $\delta_{j,1} > 0$, and $0 \leq \delta_{j,k} < \ell_{k-1}$ for $1 \leq j \leq r$ and $2 \leq k \leq j$.

Note In order to apply (3) of Corollary 7.6 to Subemma 7.8, we use the following:

For $j \geq 2$, $p = \Delta_j(\beta_{j,k})_{k=1}^j > n_j AB$ where $A = n_{j-1}$ and $B = \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ with $\gcd(A, B) = 1$, and $n_j \geq 2$. In general, p may be chosen arbitrary such that $p > n_j AB$.

Subemma 7.9(The computation algorithm for finding $\phi_r(y, z)$ in Fact[I]).
Problem on The 1st step: Let $\omega_1 : N_0 \rightarrow N_0$ be a function defined by

$$(7.9.1) \quad \omega_1(t) = t.$$

For a given set $X_1 = \{n_1, \beta_{1,1}\}$ with $2 \leq n_1 < \beta_{1,1}$, we can compute a unique solution set $W_1 = \{\ell_1, \delta_{1,1}\}$ such that

$$(7.9.2) \quad \ell_1 = n_1 \text{ and } \Delta_1(\beta_{1,1}) = \omega_1(\delta_{1,1}) \text{ with } \gcd(\ell_1, \delta_{1,1}) = 1.$$

The computation algorithm for the 1st step: It is clear.

Problem on The 2nd step: For a solution set $W_1 = \{\ell_1, \delta_{1,1}\}$ in (7.9.2) of the 1st step, let $\omega_2 : N_0^2 \rightarrow N_0$ be a function defined by

$$(7.9.3) \quad \omega_2(t_1, t_2) = t_2 \delta_{1,1} + \ell_1 t_1 = \Delta_2(t_1, t_2).$$

For a given set $X_2 = \{n_2, \beta_{2,1}, \beta_{2,2}\}$ with $n_2 \geq 2$, by the computation algorithm we can find a unique solution set $W_2 = \{\ell_2, \delta_{2,1}, \delta_{2,2}\}$ with $\ell_2 \geq 2$ such that

$$(7.9.4) \quad \begin{aligned} \ell_2 = n_2 \text{ and } \Delta_2(\beta_{2,k})_{k=1}^2 = \omega_2(\delta_{2,k})_{k=1}^2 \text{ with } \gcd(\ell_2, \omega_2(\delta_{2,1}, \delta_{2,2})) = 1, \\ \ell_2 \geq 2, \delta_{2,1} > 0, \text{ and } 0 \leq \delta_{2,2} < \ell_1. \end{aligned}$$

The computation algorithm for the 2nd step: Let $n_2 = \ell_2$. Then, $\Delta_2(\beta_{2,k})_{k=1}^2 > 2\ell_1\omega_1(\delta_{1,1}) = 2\ell_1\delta_{1,1}$ with $\gcd(\ell_1, \delta_{1,1}) = 1$, because of the following:

$$\Delta_2(\beta_{2,k})_{k=1}^2 > n_2 n_1 \Delta_1(\beta_{1,1}) = n_2 n_1 \beta_{1,1} \text{ and } n_2 \geq 2, n_1 = \ell_1, \beta_{1,1} = \delta_{1,1} \text{ and } \gcd(n_1, \beta_{1,1}) = 1.$$

By (3) of Corollary 7.6, we can find a unique solution $\{(\delta_{2,1}, \delta_{2,2})\} \subset N_0^2$ such that

$$(7.9.5) \quad \omega_2(\delta_{2,k})_{k=1}^2 = \Delta_2(\beta_{2,k})_{k=1}^2 \text{ with } \delta_{2,1} > \delta_{1,1} > 0 \text{ and } 0 \leq \delta_{2,2} < \ell_1,$$

where $\omega_2(\delta_{2,k})_{k=1}^2 = \delta_{2,2}\delta_{1,1} + \ell_1\delta_{2,1}$.

The general case is proved by induction on the positive integer $j < r$. By a finite number $\frac{j(j-1)}{2}$ iterations of (3) of Corollary 7.6, suppose we have shown by induction on the integer $j < r$ that we can compute a unique set $W_j = \{\ell_j, \delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}\}$ with $\ell_j \geq 2$ as a sequence such that the following conditions are satisfied:

- (7.9.6) (a) $n_j = \ell_j \geq 2$.
- (b) $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$.
- (c) $\ell_j \geq 2$, $\delta_{j,1} > 0$, and $0 \leq \delta_{j,k} < \ell_{k-1}$ for $2 \leq k \leq j$.

Therefore, whenever p is any integer such that $p > \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$, then we may assume without loss of generality that by a finite number $\frac{j(j-1)}{2}$ iteration of (3) of Corollary 7.6, we can compute $\{t_k : k = 1, 2, \dots, j\} \subset N$ such that $t_1 > 0$, $t_k < \ell_{k-1}$ for $k = 2, 3, \dots, j$ and $p = \omega_j(t_k)_{k=1}^j$, using the same method as we have done in the j -th step.

Then, it remain to prove that we can find the algorithm for the problem on $(j+1)$ -th step, as follows:

Problem on The $(j+1)$ -th step: From a unique solution set $W_j = \{\ell_j, \delta_{j,1}, \delta_{j,2}, \dots, \delta_{j,j}\}$ in the j -th step, let $\omega_{j+1} : N_0^{j+1} \rightarrow N_0$ be a function defined by

$$(7.9.7) \quad \omega_{j+1}(t_k)_{k=1}^{j+1} = t_{j+1} \omega_j(\delta_{j,k})_{k=1}^j + \ell_j \omega_j(t_k)_{k=1}^j.$$

For a given set $X_{j+1} = \{n_{j+1}, \beta_{j+1,1}, \beta_{j+1,2}, \dots, \beta_{j+1,j+1}\}$ with $n_{j+1} \geq 2$, we can compute a unique solution set $W_{j+1} = \{\ell_{j+1}, \delta_{j+1,1}, \delta_{j+1,2}, \dots, \delta_{j+1,j+1}\}$ with $\ell_{j+1} \geq 2$ such that

(7.9.8)

$$\begin{aligned} \ell_{j+1} &= n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} = \omega_{j+1}(\delta_{j+1,k})_{k=1}^{j+1} \text{ with } \gcd(\ell_{j+1}, \omega_{j+1}(\delta_{j+1,k})_{k=1}^{j+1}) = 1, \\ \ell_{j+1} &\geq 2, \delta_{j+1,1} > 0 \text{ and } 0 \leq \delta_{j+1,k} < \ell_{k-1} \text{ for } 2 \leq k < j+1. \end{aligned}$$

The computation algorithm for the (j+1)-th step: Let $n_{j+1} = \ell_{j+1}$.

Then, $\Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} > 2\ell_j \omega_j(\delta_{j,k})_{k=1}^j$ with $\gcd(\ell_j, \omega_j(\delta_{j,k})_{k=1}^j) = 1$, because of the following:

$$(7.9.9) \quad \begin{aligned} \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} &> n_{j+1} n_j \Delta_j(\beta_{j,k})_{k=1}^j = \ell_{j+1} \ell_j \omega_j(\delta_{j,k})_{k=1}^j, \quad n_{j+1} = \ell_{j+1} \geq 2, \\ n_j &= \ell_j, \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \text{ and } \gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = 1. \end{aligned}$$

Applying (3) of Corollary 7.6 to (7.9.9) once, we can find a unique solution $\{p, \delta_{j+1,j+1}\} \subset N_0^2$ such that

$$(7.9.10) \quad \begin{aligned} \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} &= \delta_{j+1,j+1} \omega_j(\delta_{j,k})_{k=1}^j + p \ell_j \quad \text{with} \\ p &> \omega_j(\delta_{j,k})_{k=1}^j, \quad 0 \leq \delta_{j+1,j+1} < \ell_j. \end{aligned}$$

Also, note by (7.9.10) that

$$(7.9.11) \quad \begin{aligned} p &> \omega_j(\delta_{j,k})_{k=1}^j = \Delta_j(\beta_{j,k})_{k=1}^j \\ &> n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} = \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}. \end{aligned}$$

Applying (3) of Corollary 7.6 to (7.9.11), then we assume by induction method on the positive integer j -th step that by a finite number iterations $\frac{j(j-1)}{2}$ of (3) of Corollary 7.6, we can find a unique solution $(t_1, t_2, \dots, t_j) \in N_0^j$, denoted by $(\delta_{j+1,1}, \delta_{j+1,2}, \dots, \delta_{j+1,j})$, such that

$$(7.9.12) \quad p = \omega_j(\delta_{j+1,k})_{k=1}^j \quad \text{with } \delta_{j+1,1} > 0, \text{ and } 0 \leq \delta_{j+1,k} < \ell_{k-1} \text{ for } 2 \leq k \leq j.$$

By (7.9.10) and (7.9.12), we get the following:

$$(7.9.13) \quad \begin{aligned} \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} &= \delta_{j+1,j+1} \omega_j(\delta_{j,k})_{k=1}^j + p \ell_j \\ &= \delta_{j+1,j+1} \omega_j(\delta_{j,k})_{k=1}^j + \ell_j \omega_j(\delta_{j+1,k})_{k=1}^j = \omega_{j+1}(\delta_{j+1,k})_{k=1}^{j+1}. \end{aligned}$$

Thus, with a finite number $\frac{(j+1)j}{2}$ iteration of (3) of Corollary 7.6, the computation algorithm for finding $\phi_{j+1}(y, z)$ or W_{j+1} on the (j+1)-th step can be completely finished. \square

Proof of Theorem 7.7. It is clear that Fact[I] and Fact[II] in the conclusion of Theorem 7.7 are equivalent. So, as far as the proof of Theorem 7.7 are concerned, it suffices to show that Fact[II] is true. But, the proof of Fact[II] just follows from (3) of Corollary 7.6. Therefore, there is nothing to prove for the theorem. \square

Chapter V: Review on the definition of the Puiseux pairs and the multiplicity sequences

§8. The definition of the multiplicity and Puiseux exponents(the Puiseux pairs) and the review of theorems about an equivalence of the multiplicity and Puiseux exponents and the multiplicity sequence for irreducible curves

§8.0. Introduction

In this section, in preparation for a good success of the main algorithm in this paper, first of all, we will review the complete proof of the following theorem(Theorem A) only, without using the well-known theorem(Theorem B):

Theorem A: Whenever any two irreducible parametrization have the same Puiseux pairs by a nonsingular change of the parametrization(equivalently, the same multiplicity and Puiseux exponents in the sense of Definition 8.1, later), then they have the same multiplicity sequences, and conversely.

Theorem B: As far as arbitrary Puiseux expansion of irreducible plane curve singularities is concerned, any two irreducible plane curve singularities have the same topological types if and only if they have the same Puiseux pairs.

So, the first part is to find an equivalence of irreducible parametrization for irreducible plane curve singularities by Theorem 8.8. After then, as an application, the second part is to review the proof of Theorem A(Theorem 8.10) in an elementary way, without using the well-known theorem(Theorem B or Theorem 8.5).

For example, assume that the standard Puiseux expansion of an irreducible curve C_1 is given by $y(t) = t^n$ and $z(t) = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_r}$ where $2 \leq n < \alpha_1 < \cdots < \alpha_r$ and $n > d_1 > \cdots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$, and also that the parametrization of another irreducible plane curve C_2 is given by $y(t) = t^n(1 + H(t))$ and $z(t) = t^{\alpha_1}$ where $1 + H(t) = 1 + t^{\alpha_2 - \alpha_1} + \cdots + t^{\alpha_r - \alpha_1}$. Then, it is easily shown by Theorem A that two irreducible curves C_1 and C_2 have the same multiplicity sequence, and also the same Puiseux pairs by a nonsingular change of a parameter, without using Theorem B. But, without using Theorem A, it has been not yet proved by Theorem B only that these two irreducible curves C_1 and C_2 have the same multiplicity sequences.

§8.1. The definition and known preliminaries

Now, in order to avoid the complexity of the terminology in this section, first of all, we can rewrite the statement of the definition of the Puiseux pairs.

Definition 8.1. Let the parametrization for arbitrary irreducible curve C be defined by

$$(8.1.1) \quad y(t) = t^n \text{ and } z(t) = c_1 t^{k_1} + c_2 t^{k_2} + \cdots = c_1 t^{k_1} (1 + H(t)),$$

where $1 < n$, $1 < k_1 < k_2 < \cdots$, and the c_i are nonzero complex numbers and $H(t)$ is just the substitution.

Moreover, note that the curve C is irreducible in $\mathbb{C}\{y, z\} \iff n \geq \gcd(n, k_1) \geq \gcd(n, k_1, k_2) \geq \cdots \geq \gcd(n, k_1, k_2, \dots) = 1$.

Now, consider two cases, respectively.

Case[I] Let $n \leq k_1$. Then, the parametrization for the curve C of (8.1.1) is called the Puiseux expansion.

Case[II] Let $n > k_1$. Then, the parametrization for the curve C of (8.1.1) is not called the Puiseux expansion.

Case[I] Assume that $n \leq k_1$. Now, we can define the sequence $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ from the set $\{k_i : i = 1, 2, \dots\}$, consisting of the exponents of the above parameter t , as follows: Note that n is the multiplicity of the curve C at the origin.

(*) γ_1 is the smallest positive integer among the exponents k_i such that $n > \gcd(n, k_i)$; γ_2 is the smallest positive integer among the exponents k_i such that $n > \gcd(n, \gamma_1) > \gcd(n, \gamma_1, k_i)$; \dots ; γ_p is the smallest positive integer among the exponents k_i such that $n > \gcd(n, \gamma_1) > \gcd(n, \gamma_1, \gamma_2) > \gcd(n, \gamma_1, \gamma_2, \gamma_3) > \cdots > \gcd(n, \gamma_1, \gamma_2, \dots, \gamma_p) = 1$.

(1) By the uniqueness of construction of the set $\{\gamma_i : 1 \leq i \leq p\}$, γ_i is called i -th Puiseux exponent in this paper.

(2) By (1), let S be the set defined by $\{n, \gamma_1, \gamma_2, \dots, \gamma_p\}$. Whenever the Puiseux expansion for the curve C is given, the set S is uniquely determined by the curve C .

(2a) S is called the multiplicity and Puiseux exponents for the curve C , that is, a new terminology.

(2b) As in (2a), the parametrization defined by $y = t^n$ and $z = t^{\gamma_1} + t^{\gamma_2} + \dots + t^{\gamma_p}$ is called the standard Puiseux expansion for the curve C . Note that $2 \leq n < \gamma_1 < \gamma_2 < \dots < \gamma_p$ and $n > \gcd(n, \gamma_1) > \gcd(n, \gamma_1, \gamma_2) > \dots > \gcd(n, \gamma_1, \gamma_2, \dots, \gamma_p) = 1$.

(3) By (2), let $d_i = \gcd(n, \gamma_1, \dots, \gamma_i)$ for $1 \leq i \leq p$, and write $d_0 = n$ for brevity of notation.

Define λ_i and μ_i by $\lambda_i = \frac{\gamma_i}{d_i}$ and $\mu_i = \frac{d_{i-1}}{d_i}$ for $1 \leq i \leq p$, and let (λ_i, μ_i) be defined by the Puiseux pair for each i . Then, $\{(\lambda_i, \mu_i) : i = 1, 2, \dots, p\}$ is called a finite sequence of Puiseux pairs for the curve C .

Case[II] Assume that $n > k_1$. The generalization of definitions for the Puiseux pairs and the standard Puiseux expansion for irreducible curves is as follows.

For the convenience of the notation, we may begin without loss of generality that the parametrization of the pair $(y(t), z(t))$ for the curve C of (8.1.1) is rewritten in the following:

$$(8.1.2) \quad y(t) = t^m, \quad z(t) = b_1 t^{\beta_1} + b_2 t^{\beta_2} + \dots, \quad \text{with } m > \beta_1$$

where the b_i are nonzero complex numbers, and $m > 1$ and $1 < \beta_1 < \beta_2 < \dots$, and $m \geq \gcd(m, \beta_1) \geq \gcd(m, \beta_1, \beta_2) \geq \dots \geq \gcd(n, \beta_1, \beta_2, \dots) = 1$.

By (8.1.2), let s be the new parameter defined by a conformal mapping

$$(8.1.3) \quad s(t) = t(b_1 + \sum_{i \geq 2} b_i t^{\beta_i - \beta_1})^{\frac{1}{\beta_1}}$$

of t at the origin such that $z(t) = (s(t))^{\beta_1}$ and $s(0) = 0$, and let $t = \phi(s)$ be its inverse.

Then, the Puiseux expansion defined by $y_1(s) = y(\phi(s))$ and $z_1(s) = z(\phi(s))$, which is equivalent to the parametrization of the pair $(y(t), z(t))$ in (8.1.2), can be written as follows:

$$(8.1.4) \quad z_1(s) = s^{\beta_1}, \quad y_1(s) = c_1 s^{\ell_1} + c_2 s^{\ell_2} + \dots, \quad \text{with } \beta_1 < \ell_1$$

where $1 < m = \ell_1 < \ell_2 < \dots$, and $\beta_1 < \ell_1$, and the c_i are nonzero complex numbers.

Therefore, if $m = \ell_1$ is greater than β_1 , first we will find the inverse $t = \phi(s)$ of a conformal mapping $s = s(t)$ in (8.1.3) by using (8.8.3) of Theorem 8.8 in this section, and next compute an algorithm for the construction of the Puiseux expansion in (8.1.4) of Theorem 8.8, that is, an equivalent parametrization for the above curve C .

Next, by the same way as we have used in Case[I] for the definition of the words in (8.1.5), and by Definition 8.9 and Theorem 8.10, we can naturally generalize the definition of the words in (8.1.5) for this curve C of (8.1.2) in Case[II], respectively:

(8.1.5) The multiplicity and Puiseux exponents; the standard Puiseux expansion; a finite sequence of the Puiseux pairs. \square

Remark 8.1.1 for Case[I]. (i) It can be easily shown that there is a one-to-one correspondence between the set of the multiplicity and Puiseux exponents, and the set of Puiseux pairs because (2) and (3) have the same type of definitions arithmetically.

(ii) If the parametrization defined by $(y(t), z(t))$ in (8.1.1) is the Puiseux expansion, then it is said that this Puiseux expansion have either the multiplicity and Puiseux exponents $\{n, \gamma_1, \gamma_2, \dots, \gamma_p\}$ or the Puiseux pairs $\{(\lambda_i, \mu_i) : i = 1, 2, \dots, p\}$ where each λ_i and μ_i is defined as we have seen in (3).

(iii) By (i) of the above remark, throughout this paper, we prefer to choose the terminology in (2) rather than that in (3), if necessary. \square

Definition 8.2. Definition 8.2 has the same statement as Definition 1.2 of §1 does in Family(3) with respect to the definition of the multiplicity sequences of irreducible plane curves with isolated singularity under the standard resolution. \square

Theorem 8.3(Enriques-Chisini).

(i) For an irreducible curve with Puiseux expansion

$$(8.3.1) \quad \begin{aligned} x &= t^m \\ y &= a_1 t^{k_1} + a_2 t^{k_2} + \cdots + a_q t^{k_q}, \end{aligned}$$

in which only essential (characteristic) term appear, the multiplicity sequence is determined by the following chain of g Euclidean algorithms: Let $i = 1, 2, \dots$

$$\begin{aligned} \lambda_i &= \mu_{i,1} r_{i,1} + r_{i,2}, \\ r_{i,1} &= \mu_{i,2} r_{i,2} + r_{i,3}, \\ &\dots\dots\dots \\ r_{i,w(i)-1} &= \mu_{i,w(i)} r_{i,w(i)} \quad \text{with} \quad 0 \leq r_{i,j+1} < r_{i,j}, \\ \lambda_i &= k_i - k_{i-1} \quad \text{for } 1 \leq i \leq g, \quad \text{and} \quad k_0 = 0, \\ r_{i,1} &= r_{i-1,w(i-1)} \quad \text{for } i > 1, \quad \text{and} \quad r_{1,1} = m. \end{aligned}$$

In the multiplicity sequence, the multiplicity r_{ij} then appears μ_{ij} times, where $i = 1, \dots, g; j = 1, \dots, w(i)$. (If a certain multiplicity arises from several successive algorithms, then it is also counted multiply.)

(ii) For an arbitrary irreducible curve one obtains the multiplicity sequences by omitting all non-characteristic terms from the Puiseux expansion and then applying the algorithm above.

(iii) Conversely, one can reconstruct the exponents of the characteristic terms of the Puiseux expansion of an irreducible curve, i.e. the Puiseux pairs of the curve, from the multiplicity sequence, by the chain of Euclidean algorithms. \square

Proof of Theorem 8.3. See [Bri-Kn].

Definition 8.4. Let $V(f) = \{(y, z) : f(y, z) = 0\}$ and $V(g) = \{(y, z) : g(y, z) = 0\}$ be analytic varieties at the origin in \mathbb{C}^2 where f and g are analytically irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin \mathbb{C}^2 .

(i) f and g are said to have the same topological type of the singularity at the origin if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \rightarrow (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0) = 0$ where U_1 and U_2 are open subsets in \mathbb{C}^2 . In this case, denote this relation by $f \sim g$ or $V \sim W$. Otherwise, we write $f \not\sim g$ or $V \not\sim W$.

(ii) f and g are said to have the same analytic type of the singularity at the origin if there is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \rightarrow (U_2, 0)$ such that $\psi(V) = W$ and $\psi(0) = 0$ where U_1 and U_2 are open subsets in \mathbb{C}^2 , that is, $f \circ \psi = ug$ where u is a unit in $2\mathcal{O}$, the ring of germs of holomorphic functions at the origin in \mathbb{C}^2 . In this case, denote this relation by $f \approx g$ or $V \approx W$. Otherwise, we write $f \not\approx g$ or $V \not\approx W$.

(iii) Following Definition 8.2, for notation, $\text{Multiseq}(V(f))$ or $\{[\text{Mult}(V(f))]\}$ is called the multiplicity sequence of $V(f)$. If $V(f)$ and $V(g)$ have the same multiplicity sequences, we write either $\text{Multiseq}(V(f)) \equiv \text{Multiseq}(V(g))$ as sequence or $f \sim g$ (Multiseq). Otherwise, we write either $\text{Multiseq}(V(f)) \not\equiv \text{Multiseq}(V(g))$ as sequence or $f \not\sim g$ (Multiseq). \square

Theorem 8.5([Br],[Bu],[Z1]). Let $f(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin in \mathbb{C}^2 . Then the curve defined by f at the origin can be described topologically by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_p}$ where $n < \alpha_1 < \cdots < \alpha_p$ and $n > \gcd(n, \alpha_1) > \cdots > \gcd(n, \alpha_1, \dots, \alpha_p) = 1$. If for a given f there is another homeomorphic parametrization defined by $y = t^m$ and $z = t^{\beta_1} + \cdots + t^{\beta_q}$ where $m < \beta_1 < \cdots < \beta_q$ and $m > \gcd(m, \beta_1) > \cdots > \gcd(m, \beta_1, \dots, \beta_q) = 1$, then $n = m$, and $p = q$ and $\alpha_i = \beta_i$ for $1 \leq i \leq p$.

Note by Definition 8.1 and Remark 8.1.1 that the multiplicity and Puiseux exponents for the standard Puiseux expansion determine the topological types of the irreducible plane curve singularities, and conversely. \square

§8.2. How to get an equivalent parametrization from given any irreducible parametrization by the inverse mapping theorem of one complex variable

Definition 8.6. Let $\phi(t)$ be an analytic function in a neighborhood of zero such that $\phi(0) = \phi'(0) = \dots = \phi^{(k)}(0) = 0$, but $\phi^{(k+1)}(0) \neq 0$. Then, it is said that $\phi(t)$ has a multiplicity k at $t = 0$ and write $\text{Mult}(\phi(t), 0) = k$ for notation. Let $f(y, z)$ be in $\mathbb{C}\{y, z\}$. It is said that $f(y, z)$ has a multiplicity ν at $(y, z) = (0, 0)$, denoted by $\text{Mult}(f(y, z), (0, 0)) = \nu$, if there is the least integer ν such that some partial derivative of f of order ν is nonzero at the origin. \square

Lemma 8.7(The rearrangement of an irreducible parametrization).

Assumptions Let the curve $V = \{f(y, z) = 0\}$ with $f(y, z) \in \mathbb{C}\{y, z\}$ have an irreducible parametrization at the origin, which is defined by

$$(8.7.1) \quad y(t) = t^n \text{ and } z(t) = c_1 t^{k_1} + c_2 t^{k_2} + \dots = c_1 t^{k_1} (1 + H(t)),$$

where the c_i are nonzero complex numbers and $1 \leq n, 1 \leq k_1 < k_2 < \dots$, and $n \geq \gcd(n, k_1) \geq \gcd(n, k_1, k_2) \geq \dots \geq \gcd(n, k_1, k_2, \dots) = 1$.

To get a desired rearrangement of $y = t^n$ and $z = \sum_{i=1}^{\infty} c_i t^{k_i}$ in the conclusion of this lemma, first we can define a finite sequence $\{\alpha_1, \alpha_2, \dots, \alpha_{r+1}\}$ from the sequence $\{k_i : i = 1, 2, \dots\}$ consisting of the exponents k_i in (8.7.1) as follows:

(1) Let $\alpha_1 = k_1$, and then note that $n \geq \gcd(n, \alpha_1)$. That is, either $n = \gcd(n, \alpha_1)$ or $n > \gcd(n, \alpha_1)$.

(2) Let α_2 be the smallest positive integer among the exponents k_i such that $n \geq \gcd(n, \alpha_1) > \gcd(n, \alpha_1, k_i)$.

.....
(r+1) Let α_{r+1} be the smallest positive integer among the exponents k_i such that $n \geq \gcd(n, \alpha_1) > \gcd(n, \alpha_1, \alpha_2) > \dots > \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_r) > \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_r, k_i) = 1$.

Let d and k be arbitrary positive integers. For brevity of notation, if k is divisible by d , then we write $d|k$. Otherwise, we write $d \nmid k$.

Now, let $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$ for $1 \leq i \leq r+1$, and then $n \geq d_1 > d_2 > \dots > d_{r+1}$. Note that $d_i | (\alpha_i - \alpha_1)$, $d_i \nmid (\alpha_{i+1} - \alpha_1)$, and $d_{i+1} | d_i$.

Conclusions The irreducible parametrization of V can be rearranged in t as follows:

$$(8.7.2) \quad y = t^n \quad \text{and} \\ z = ct^{\alpha_1} \{(1 + D_1(t)) + t^{\alpha_2 - \alpha_1} (c_{2,0} + D_2(t)) + \dots \\ + t^{\alpha_r - \alpha_1} (c_{r,0} + D_r(t)) + t^{\alpha_{r+1} - \alpha_1} (c_{r+1,0} + D_{r+1}(t))\}$$

satisfying the properties (i), (ii), ..., (v).

(i) $1 \leq n$ and $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{r+1}$.

(ii) $n \geq d_1 > d_2 > \dots > d_{r+1} = 1$ with $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_i) = d_i$ for $1 \leq i \leq r+1$.

(iii) p_1, p_2, \dots, p_r are nonnegative integers such that $p_1 d_1 < \alpha_2 - \alpha_1 < (p_1 + 1) d_1$, $p_2 d_2 < \alpha_3 - \alpha_2 < (p_2 + 1) d_2$, ..., $p_r d_r < \alpha_{r+1} - \alpha_r < (p_r + 1) d_r$.

(iv) If $p_j \neq 0$ for some $j \leq r$, write $D_j(t) = \sum_{i=1}^{p_j} c_{j,i} t^{i d_j} \in \mathbb{C}[t]$, and if $p_j = 0$ for some $j \leq r$, write $D_j(t) = 0$, and also $D_{r+1}(t) = \sum_{k=1}^{\infty} c_{r+1,k} t^k \in \mathbb{C}\{t\}$.

(v) $c, c_{1,0} = 1, c_{2,0}, c_{3,0}, \dots, c_{r+1,0}$ are all nonzero complex numbers. \square

For the proof of Lemma 8.7, see Lemma 3.3([K2]).

Theorem 8.8(An equivalence of irreducible parametrization).

Assumptions We may assume without loss of generality that the curve $V = \{f(y, z) = 0\}$ with $f(y, z) \in \mathbb{C}\{y, z\}$ at the origin has an irreducible parametrization as follows:

$$(8.8.1) \quad y = t^n \quad \text{and} \\ z = ct^{\alpha_1} \{(1 + D_1(t)) + t^{\alpha_2 - \alpha_1} (c_{2,0} + D_2(t)) + \dots \\ + t^{\alpha_r - \alpha_1} (c_{r,0} + D_r(t)) + t^{\alpha_{r+1} - \alpha_1} (c_{r+1,0} + D_{r+1}(t))\} = ct^{\alpha_1} (1 + H(t)),$$

satisfying the properties (i), (ii), ..., (v).

(i) $1 \leq n$ and $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{r+1}$.

(ii) $n \geq d_1 > d_2 > \dots > d_{r+1} = 1$ with $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_i) = d_i$ for $1 \leq i \leq r+1$.

(iii) p_1, p_2, \dots, p_r are nonnegative integers such that $\alpha_1 + p_1 d_1 < \alpha_2 < \alpha_1 + (p_1 + 1)d_1$, $\alpha_2 + p_2 d_2 < \alpha_3 < \alpha_2 + (p_2 + 1)d_2, \dots, \alpha_r + p_r d_r < \alpha_{r+1} < \alpha_r + (p_r + 1)d_r$.

(iv) If $p_j \neq 0$ for some $j \leq r$, write $D_j(t) = \sum_{i=1}^{p_j} c_{j,i} t^{id_j} \in \mathbb{C}[t]$, and if $p_j = 0$ for some $j \leq r$, write $D_j(t) = 0$, and also $D_{r+1}(t) = \sum_{k=1}^{\infty} c_{r+1,k} t^k \in \mathbb{C}\{t\}$. Note that $1 + H(t) = 1 + D_1(t) + t^{\alpha_2 - \alpha_1}(c_{2,0} + D_2(t)) + \dots + t^{\alpha_r - \alpha_1}(c_{r,0} + D_r(t)) + t^{\alpha_{r+1} - \alpha_1}(c_{r+1,0} + D_{r+1}(t))$.

(v) $c, c_{1,0} = 1, c_{2,0}, c_{3,0}, \dots, c_{r+1,0}$ are all nonzero complex numbers.

Conclusions In preparation for the construction of an equivalent irreducible parametrization of V , let s be the new parameter defined by

$$(8.8.2) \quad s(t) = c^{\frac{1}{\alpha_1}} t (1 + H(t))^{\frac{1}{\alpha_1}}$$

where (i) $c^{\frac{1}{\alpha_1}}$ is a complex root such that $\omega^{\alpha_1} = c$,

(ii) $s = s(t)$ is a conformal mapping of t at the origin,

(iii) $z = s^{\alpha_1}$.

Then, we have the following (I) and (II):

(I) $t = c^{-\frac{1}{\alpha_1}} s (1 + H(t))^{-\frac{1}{\alpha_1}}$, as $t = \phi(s) \in \mathbb{C}\{s\}$, can be written as follows: Note that $y = (\phi(s))^n$.

$$(8.8.3) \quad \begin{aligned} t &= \phi(s) \\ &= c^{-\frac{1}{\alpha_1}} s \{1 + Q_1(s) + s^{\alpha_2 - \alpha_1} (B_{2,0} + Q_2(s)) \\ &\quad + \dots + s^{\alpha_r - \alpha_1} (B_{r,0} + Q_r(s)) + s^{\alpha_{r+1} - \alpha_1} (B_{r+1,0} + Q_{r+1}(s))\} \end{aligned}$$

satisfying the properties (i) and (ii).

(i) $B_{2,0} = \frac{c_{2,0}}{-\alpha_1} (c^{-\frac{1}{\alpha_1}})^{\alpha_2 - \alpha_1}$, $B_{3,0} = \frac{c_{3,0}}{-\alpha_1} (c^{-\frac{1}{\alpha_1}})^{\alpha_3 - \alpha_1}$, \dots , $B_{r+1,0} = \frac{c_{r+1,0}}{-\alpha_1} (c^{-\frac{1}{\alpha_1}})^{\alpha_{r+1} - \alpha_1}$.
(ii) $Q_1(s) = B_{1,1}s^{d_1} + B_{1,2}s^{2d_1} + \dots + B_{1,p_1}s^{p_1 d_1} \in \mathbb{C}[s]$, $Q_2(s) = B_{2,1}s^{d_2} + B_{2,2}s^{2d_2} + \dots + B_{2,p_2}s^{p_2 d_2} \in \mathbb{C}[s]$, \dots , $Q_r(s) = B_{r,1}s^{d_r} + B_{r,2}s^{2d_r} + \dots + B_{r,p_r}s^{p_r d_r} \in \mathbb{C}[s]$, $Q_{r+1}(s) = \sum_{k=1}^{\infty} B_{r+1,k}s^k \in \mathbb{C}\{s\}$ such that all the $B_{i,j}$ are complex numbers and that in particular the $B_{i,0}$ are nonzero for $2 \leq i \leq r+1$. Note that $Q_i(0) = 0$ for $1 \leq i \leq r+1$.

(II) Then, the equivalent parametrization with the new parameter s for V can be analytically written in the following form:

$$(8.8.4) \quad \begin{aligned} z &= s^{\alpha_1}, \\ y &= c^{-\frac{n}{\alpha_1}} s^n \{1 + Q_1^*(s) + s^{\alpha_2 - \alpha_1} (b_{2,0} + Q_2^*(s)) \\ &\quad + s^{\alpha_3 - \alpha_1} (b_{3,0} + Q_3^*(s)) + \dots + s^{\alpha_{r+1} - \alpha_1} (b_{r+1,0} + Q_{r+1}^*(s))\} \end{aligned}$$

satisfying the properties (i) and (ii).

(i) $b_{2,0} = \frac{n}{-\alpha_1} c_{2,0} (c^{-\frac{1}{\alpha_1}})^{\alpha_2 - \alpha_1}$, $b_{3,0} = \frac{n}{-\alpha_1} c_{3,0} (c^{-\frac{1}{\alpha_1}})^{\alpha_3 - \alpha_1}$, \dots , $b_{r+1,0} = \frac{n}{-\alpha_1} c_{r+1,0} (c^{-\frac{1}{\alpha_1}})^{\alpha_{r+1} - \alpha_1}$.
(ii) $Q_1^*(s) = b_{1,1}s^{d_1} + b_{1,2}s^{2d_1} + \dots + b_{1,p_1}s^{p_1 d_1} \in \mathbb{C}[s]$, $Q_2^*(s) = b_{2,1}s^{d_2} + b_{2,2}s^{2d_2} + \dots + b_{2,p_2}s^{p_2 d_2} \in \mathbb{C}[s]$, \dots , $Q_{r+1}^*(s) = \sum_{k=1}^{\infty} b_{r+1,k}s^k \in \mathbb{C}\{s\}$, such that all the $b_{i,j}$ are complex numbers and that in particular the $b_{i,0}$ are nonzero for $2 \leq i \leq r+1$. Note that $Q_i^*(0) = 0$ for all $i = 1, 2, \dots, r+1$. \square

Remark 8.8.1. Observe by (8.8.3) and (8.8.4) that $b_{20} = nB_{20}, b_{30} = nB_{30}, \dots, b_{r+1,0} = nB_{r+1,0}$. For the proof of Theorem 8.8, see Theorem 3.4([K2]). \square

§8.3. The generalization of definitions for the Puiseux pairs and the standard Puiseux expansion for irreducible curves

Now, as an application of Theorem 8.8, we are going to generalize the words in (8.1.5) of Definition 8.1 by the following.

Definition 8.9. Let $f(y, z)$ be analytically irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin in \mathbb{C}^2 . By Lemma 8.7, we may assume without loss of generality that the curve $V(f)$ defined by the above f at the origin has an irreducible parametrization as follows:

$$(8.9.1) \quad V(f) := \begin{cases} y = t^n \\ z = a_1 t^{\alpha_1} (1 + D_1(t)) + a_2 t^{\alpha_2} (1 + D_2(t)) + \cdots + a_{r+1} t^{\alpha_{r+1}} (1 + D_{r+1}(t)), \end{cases}$$

satisfying the properties, (1a), (1b), \dots , (1e).

(1a) $2 \leq n$ and $2 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{r+1}$.

(1b) $n \geq d_1 > d_2 > \cdots > d_{r+1} = 1$ with $d_i = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_i)$ for $1 \leq i \leq r+1$.

(1c) The a_i are all nonzero numbers for $i = 1, 2, \dots, r+1$.

(1d) Define p_1, p_2, \dots, p_r to be nonnegative integers such that $p_1 d_1 < \alpha_2 - \alpha_1 < (p_1 + 1) d_1$, $p_2 d_2 < \alpha_3 - \alpha_2 < (p_2 + 1) d_2$, \dots , $p_r d_r < \alpha_{r+1} - \alpha_r < (p_r + 1) d_r$.

(1e) If $p_j \neq 0$ for some $j \leq r$, write $D_j(t) = \sum_{i=1}^{p_j} c_{j,i} t^{i d_j} \in \mathbb{C}[t]$, and if $p_j = 0$ for some $j \leq r$, write $D_j(t) = 0$, and also $D_{r+1}(t) = \sum_{k=1}^{\infty} c_{r+1,k} t^k \in \mathbb{C}\{t\}$.

(1f) $c, c_{10} = 1, c_{20}, c_{30}, \dots, c_{r+1,0}$ are all nonzero complex numbers.

Then, the multiplicity and Puiseux exponents for the curve $V(f)$ are defined as follows:

(A) If $n \leq \alpha_1$ and n is not a divisor of α_1 , then note that the parametrization defined by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_{r+1}}$ is called the standard Puiseux expansion. Then, it is said that the set $\{n, \alpha_1, \alpha_2, \dots, \alpha_{r+1}\}$ is a finite sequence of the multiplicity and Puiseux exponents for the Puiseux expansion of $V(f)$.

(B) If $n \leq \alpha_1$ and n is a divisor of α_1 , then note that the parametrization defined by $y = t^n$ and $z = t^{\alpha_2} + t^{\alpha_3} + \cdots + t^{\alpha_{r+1}}$ is called the standard Puiseux expansion. Then, it is said that the set $\{n, \alpha_2, \alpha_3, \dots, \alpha_{r+1}\}$ is a finite sequence of the multiplicity and Puiseux exponents for the Puiseux expansion of $V(f)$.

In case $n > \alpha_1$, using the equation of (8.8.4) in the conclusion of Theorem 8.8, we can compute the Puiseux expansion which is equivalent to the parametrization of $V(f)$, as follows:

$$(8.9.2) \quad V(f) \approx \begin{cases} z = s^{\alpha_1} \\ y = c_1^{-\frac{n}{\alpha_1}} s^n \{ (1 + Q_1^*(s)) + s^{\alpha_2 - \alpha_1} (b_{20} + Q_2^*(s)) \\ \quad + s^{\alpha_3 - \alpha_1} (b_{30} + Q_3^*(s)) + \cdots + s^{\alpha_{r+1} - \alpha_1} (b_{r+1,0} + Q_{r+1}^*(s)) \} \\ \quad = c_1^{-\frac{n}{\alpha_1}} s^n \{ 1 + L(s) \}, \end{cases}$$

where

- (i) $\gcd(n, \alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_i - \alpha_1) = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_i) = d_i$ for $1 \leq i \leq r+1$,
- (ii) $Q_j^*(s) = \sum_{i=1}^{p_j} b_{j,i} s^{i d_j} \in \mathbb{C}[s]$ for $1 \leq j \leq r$ and $Q_{r+1}^*(s) = \sum_{i=1}^{\infty} b_{r+1,i} s^i \in \mathbb{C}\{s\}$,
- (iii) all the $b_{j,i(j)}$ are complex numbers with $1 \leq j \leq r+1$ and $1 \leq i(j) \leq p_j$, noting that p_{r+1} may be infinite,
- (iv*) the $b_{j,0}$ are all nonzero complex numbers for $2 \leq j \leq r+1$, noting by (8.8.3) that $b_{j,0} = \frac{n}{-\alpha_1} c_{j0} c^{-\frac{1}{-\alpha_1}(\alpha_j - \alpha_1)}$ for $2 \leq j \leq r+1$,
- (v) $L(s)$ is just the substitution.

By the same method as we have done in two cases (A) and (B), then it is enough to consider the following cases:

- (C) If $n > \alpha_1$ and $\alpha_1 > \gcd(n, \alpha_1)$, then it is said that $y = t^{\alpha_1}$ and $z = t^n + t^{n+\alpha_2-\alpha_1} + t^{n+\alpha_3-\alpha_1} + \cdots + t^{n+\alpha_{r+1}-\alpha_1}$ is the standard Puiseux expansion. Then, it is said that the set $\{\alpha_1, n, n + \alpha_2 - \alpha_1, n + \alpha_3 - \alpha_1, \dots, n + \alpha_{r+1} - \alpha_1\}$ is a finite sequence of the multiplicity and Puiseux exponents for the curve $V(f)$.
- (D) If $n > \alpha_1$ and α_1 is a divisor of n , then it is said that $y = t^{\alpha_1}$ and $z = t^{n+\alpha_2-\alpha_1} + t^{n+\alpha_3-\alpha_1} + \cdots + t^{n+\alpha_{r+1}-\alpha_1}$ is the standard Puiseux expansion. Then, it is said that the set $\{\alpha_1, n + \alpha_2 - \alpha_1, n + \alpha_3 - \alpha_1, \dots, n + \alpha_{r+1} - \alpha_1\}$ is a finite sequence of the multiplicity and Puiseux exponents for the curve $V(f)$. \square

§8.4. The review of theorems about an equivalence of the Puiseux pairs and the multiplicity sequence for irreducible curves

Finally, as mentioned in the beginning of this section, we will review the statement of Theorem 5.1[K] without its proof, recalling that Theorem 5.1[K] is equivalent to Theorem A. So, to finish this section, we just rewrite Theorem 5.1[K] by the following theorem (Theorem 8.10), which was already proved by Theorem 8.8 (an equivalence of irreducible plane curve singularities) and σ -process only, without using the well-known theorem (Theorem B).

Theorem 8.10(Theorem A, An equivalence of the Puiseux expansions with the multiplicity and Puiseux exponents in the sense of Definition 8.1 and the multiplicity sequences for irreducible parametrization by Theorem 5.1[K]).

Assumptions Let $f(y, z)$, $g(y, z)$ and $h(y, z)$ be analytically irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin in \mathbb{C}^2 . Assume that three curves $V(f)$, $V(g)$ and $V(h)$ defined by the above analytic functions f , g and h at the origin have irreducible parametrization, respectively as follows:

(1) Let the parametrization of $V(f)$ be the Puiseux expansion with the multiplicity and Puiseux exponents $\{n, \alpha_1, \alpha_2, \dots, \alpha_{r+1}\}$, defined by the following:

$$(8.10.1) \quad V(f) := \begin{cases} y = t^n \\ z = a_1 t^{\alpha_1} (1 + D_1(t)) + a_2 t^{\alpha_2} (1 + D_2(t)) + \dots + a_{r+1} t^{\alpha_{r+1}} (1 + D_{r+1}(t)), \end{cases}$$

satisfying the properties, (1a), (1b), \dots , (1e).

- (1a) $2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_{r+1}$.
- (1b*) $n > d_1 > d_2 > \dots > d_{r+1} = 1$ with $d_i = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_i)$ for $1 \leq i \leq r+1$.
- (1c) The a_i are all nonzero numbers for $i = 1, 2, \dots, r+1$.
- (1d) Define p_1, p_2, \dots, p_r to be nonnegative integers such that $p_1 d_1 < \alpha_2 - \alpha_1 < (p_1 + 1) d_1$, $p_2 d_2 < \alpha_3 - \alpha_2 < (p_2 + 1) d_2$, \dots , $p_r d_r < \alpha_{r+1} - \alpha_r < (p_r + 1) d_r$.
- (1e) If $p_j \neq 0$ for some $j \leq r$, write $D_j(t) = \sum_{i=1}^{p_j} a_{j,i} t^{i d_j} \in \mathbb{C}[t]$, and if $p_j = 0$ for some $j \leq r$, write $D_j(t) = 0$, and also $D_{r+1}(t) = \sum_{k=1}^{\infty} a_{r+1,k} t^k \in \mathbb{C}\{t\}$.

Remark: In the above condition (1b*) of (8.10.1), if $n \geq \gcd(n, \alpha_1)$ and n is a divisor of α_1 , then it is clear that the singularity of $V(f)$ is analytically invariant at the origin, whether or not a_1 is zero, and so from the beginning we may assume without loss of generality that $n > \gcd(n, \alpha_1)$ and $a_1 \neq 0$.

(2) Let the parametrization of $V(g)$ be the Puiseux expansion with the multiplicity and Puiseux exponents $\{m, \beta_1, \beta_2, \dots, \beta_{u+1}\}$, defined by the following:

$$(8.10.2) \quad V(g) := \begin{cases} y = t^m \\ z = b_1 t^{\beta_1} (1 + L_1(t)) + b_2 t^{\beta_2} (1 + L_2(t)) + \dots + b_{u+1} t^{\beta_{u+1}} (1 + L_{u+1}(t)), \end{cases}$$

satisfying the properties, (2a), (2b), \dots , (2e).

- (2a) $2 \leq m < \beta_1 < \beta_2 < \dots < \beta_{u+1}$,
- (2b) $m > e_1 > e_2 > \dots > e_{u+1} = 1$ with $e_i = \gcd(m, \beta_1, \beta_2, \dots, \beta_i)$ for $1 \leq i \leq u+1$.
- (2c) The b_i are all nonzero numbers for $i = 1, 2, \dots, u+1$.
- (2d) Define q_1, q_2, \dots, q_u to be nonnegative integers such that $q_1 e_1 < \beta_2 - \beta_1 < (q_1 + 1) e_1$, $q_2 e_2 < \beta_3 - \beta_2 < (q_2 + 1) e_2$, \dots , $q_u e_u < \beta_{u+1} - \beta_u < (q_u + 1) e_u$.
- (2e) If $q_j \neq 0$ for some $j \leq u$, write $L_j(t) = \sum_{i=1}^{q_j} b_{j,i} t^{i e_j} \in \mathbb{C}[t]$, and if $q_j = 0$ for some $j \leq u$, write $L_j(t) = 0$, and also $L_{u+1}(t) = \sum_{i=1}^{\infty} b_{u+1,i} t^i \in \mathbb{C}\{t\}$.

(3) Let the parametrization of $V(h)$ be defined by the following:

$$(8.10.3) \quad V(h) := \begin{cases} y = c_1 t^{\ell_1} (1 + R_1(t)) + c_2 t^{\ell_2} (1 + R_2(t)) + \dots + c_{v+1} t^{\ell_{v+1}} (1 + R_{v+1}(t)) \\ z = t^\gamma, \end{cases}$$

satisfying properties, (3a), (3b), \dots , (3e).

- (3a) $2 \leq \ell_1 < \gamma$ and $\ell_1 < \ell_2 < \dots < \ell_{v+1}$.
- (3b) $\ell_1 \geq \tau_1 > \tau_2 > \dots > \tau_{v+1} = 1$ with $\tau_i = \gcd(\gamma, \ell_1, \ell_2, \dots, \ell_i)$ for $1 \leq i \leq v+1$.
- (3c) the c_i are all nonzero numbers for $i = 1, 2, \dots, v+1$.
- (3d) Define $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_v$ to be nonnegative integers such that $\varepsilon_1 \tau_1 < \ell_2 - \ell_1 < (\varepsilon_1 + 1) \tau_1$, $\varepsilon_2 \tau_2 < \ell_3 - \ell_2 < (\varepsilon_2 + 1) \tau_2$, \dots , $\varepsilon_v \tau_v < \ell_{v+1} - \ell_v < (\varepsilon_v + 1) \tau_v$.
- (3e) If $\varepsilon_j \neq 0$ for some $j \leq v$, write $R_j(t) = \sum_{i=1}^{\varepsilon_j} c_{j,i} t^{i \tau_j} \in \mathbb{C}[t]$, and if $\varepsilon_j = 0$ for some $j \leq v$, write $R_j(t) = 0$, and also $R_{v+1}(t) = \sum_{i=1}^{\infty} c_{v+1,i} t^i \in \mathbb{C}\{t\}$.

Conclusions We get the following:

(I) Note that $n > \gcd(n, \alpha_1)$ and $m > \gcd(m, \beta_1)$.

(8.10.4) Multisec(V(f)) = Multisec(V(g)) as sequence

\iff the multiplicity and Puiseux exponents are the same, by Definition 8.1,
that is, $n = m$, $r + 1 = u + 1$, and $\alpha_i = \beta_i$ for all $i = 1, 2, \dots, r + 1$

\iff the Puiseux pairs for both $V(f)$ and $V(g)$ are the same.

(II) Let $\gamma > \ell_1 \geq 2$. Then, it is enough to consider two cases:

(IIa) $\ell_1 > \gcd(\gamma, \ell_1)$. (IIb) $\ell_1 = \gcd(\gamma, \ell_1)$, that is, ℓ_1 is a divisor of γ .

(IIa) Let $\ell_1 > \gcd(\gamma, \ell_1)$.

(8.10.5) Multisec(V(f)) = Multisec(V(h)) as sequence

\iff $n = \ell_1, \alpha_1 = \gamma, r + 1 = v + 1$ and $\alpha_i = \gamma + \ell_i - \ell_1$ for $1 \leq i \leq r + 1$

\iff the Puiseux pairs for both $V(f)$ and $V(h)$ are the same.

(IIb) Let $\ell_1 = \gcd(\gamma, \ell_1)$, that is, ℓ_1 is a divisor of γ .

(8.10.6) Multisec(V(f)) = Multisec(V(h)) as sequence

\iff $n = \ell_1, \alpha_1 = \gamma + \ell_2 - \ell_1, r + 1 = v$ and $\alpha_i = \gamma + \ell_{i+1} - \ell_1$ for $2 \leq i \leq r + 1$

\iff the Puiseux pairs for both $V(f)$ and $V(h)$ are the same. \square

Proof of Theorem 8.10. See Theorem 5.1([K2]).

§9 New definition of the join of subsequences of a finite sequence and its application to the representation of the multiplicity sequences for irreducible plane curve singularities

§9.0. Introduction

In preparation for finding a computation algorithm for the multiplicity sequences of all the irreducible plane curve singularities, using the Euclidean algorithm of an integer ring Z , first we are going to construct two new terminology in Definition 9.0 and Definition 9.1.

§9.1. New Definitions

Definition 9.1. Let $S = \{e_i \in N : i = 1, 2, \dots, q\}$ be a finite sequence of positive integers. Then, it is said that S is the join of r subsequences of S in order, denoted by $S = \text{Join}\{B_1, B_2, \dots, B_r\}$ of r subsequences in order, where each B_i is a subsequence of S for $i = 1, 2, \dots, r$, if the following properties are satisfied:

(a) Let $q = \lambda_r$. For each $j = 1, 2, \dots, r$, define the number of elements of B_j by $\lambda_j - \lambda_{j-1}$, which is positive. Note that $\lambda_0 = 0$.

(b) Each subsequence B_i of S can be written as follows: Let $q = \lambda_r$.

$$(9.1.1) \quad \begin{aligned} B_1 &= \{b_{0,i} = e_i : i = 1, 2, \dots, \lambda_1\}, \\ B_j &= \{b_{j-1,i} = e_{\lambda_{j-1}+i} : i = 1, 2, \dots, (\lambda_j - \lambda_{j-1})\} \quad \text{for } j = 2, 3, \dots, r. \end{aligned}$$

Definition 9.2. (1) Let β and m be two arbitrary positive integers such that $\beta \geq m$. To find $\gcd(\beta, m)$, it suffices to consider the following case except for that $\beta = m = 1$:

Let $\beta \geq m \geq 1$ with $\beta m > 1$. To find $\gcd(\beta, m)$ with $\beta m > 1$, then the Euclidean algorithms for $\beta = \mu_0$ and $m = \mu_1$ can be written as follows:

$$(9.2.1) \quad \begin{aligned} \beta &= q_1 \mu_1 + \mu_2 \quad \text{with } 0 \leq \mu_2 < \mu_1, \\ \mu_1 &= q_2 \mu_2 + \mu_3 \quad \text{with } 0 \leq \mu_3 < \mu_2, \\ \mu_2 &= q_3 \mu_3 + \mu_4 \quad \text{with } 0 \leq \mu_4 < \mu_3, \\ &\dots \\ \mu_{w-2} &= q_{w-1} \mu_{w-1} + \mu_w \quad \text{with } 0 \leq \mu_w < \mu_{w-1}, \\ \mu_{w-1} &= q_w \mu_w + 0 \quad \text{with } \gcd(\beta, m) = \mu_w, \end{aligned}$$

where $S = \{\mu_i : i = 1, 2, \dots, w\}$ is a strictly decreasing finite sequence.

Then, it is said that μ_w is the greatest common divisor of β and m , denoted by $\gcd(\beta, m)$.

(2) Let m and β be any two positive integers such that $\beta \geq m \geq 1$ and $\beta m > 1$. By (9.2.1), **the Euclidean multiplicity sequence for two positive integers β and m** with $\gcd(\beta, m)$, denoted by either one of four notations $\text{Ems}[\beta : m]$, $\{[\beta : m]\}$, $\text{Ems}[m : \beta]$ and $\{[m : \beta]\}$, is defined by the following:

$$(9.2.2) \quad \begin{aligned} \{[\beta : m]\} &= \{\mu_1, \mu_1, \dots, \mu_1; \mu_2, \mu_2, \dots, \mu_2; \dots; \mu_w, \mu_w, \dots, \mu_w\} \\ &= \text{Join}\{S_1, S_2, \dots, S_w\} \quad \text{by Definition 9.1,} \end{aligned}$$

where the sequence $S_i = \{\mu_i : i = 1, 2, \dots, w\}$ is a subsequence of $\{[\beta : m]\}$ for each $i = 1, 2, \dots, w$.

For example, we use the same kind of notations as in (9.2.3), as follows:

Let $\gcd(\beta, m) = 1$ with $\beta \geq m \geq 1$ and $\beta m > 1$ and $\{[\beta, m]\} = \{c_1, c_2, \dots, c_t\}$ by (9.2.2). For brevity of notation, the following may be rewritten by

$$(9.2.3) \quad \{[d\beta, dm]\} = \{d[\beta, m]\} = \{dc_1, dc_2, \dots, dc_t\} \quad \text{for any positive integer } d.$$

(3) Let $V(g) = \{(y, z) : g(y, z) = z^m + y^\beta = 0\}$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $0 \in \mathbb{C}^2$ where $m \geq 2$ and $\beta \geq 2$ are positive integers and $\gcd(m, \beta) = 1$. In this case, it is said by (9.2.1) and (9.2.2) that $\text{Multiseq}(V(g))$, called the multiplicity sequence of $V(g)$, can be rewritten by either $\{[\beta : m]\}$ or $\{[m : \beta]\}$, as a sequence.

Remark 9.2.1.

(i) For any positive integers β and m such that $\beta \geq m$, it is well-known by (1) of Definition 9.2 that there are two integers γ and δ such that $\gcd(\beta, m) = \gamma\beta + \delta m$.

(ii) Assuming that $\beta > m \geq 2$ for given any Euclidean multiplicity sequence $\text{Ems}[\beta : m]$ and that m is not a divisor of β , to count the number σ of $d = \min\{\text{Ems}[\beta : m]\}$ in $\text{Ems}[\beta : m]$ as a sequence, then it is enough to compute $\sigma = \frac{\min\{a_i \in E : a_i > d\}}{d} = \frac{\mu_{w-1, q_{w-1}}}{\mu_{w, q_w}} = \frac{\mu_{w-1}}{\mu_w}$ from (9.2.1) and (9.2.2), because if m is a divisor of β then $\text{Ems}[\beta : m]$ is equal to a one-point set.

(iii) Let $\text{Eds}[\beta : m]$ and $\text{Eds}[\beta' : m']$ be two arbitrary Euclidean multiplicity sequences where $m \leq \beta$ and $m' \leq \beta'$. Then, $\text{Eds}[\beta : m]$ and $\text{Eds}[\beta' : m']$ are the same Euclidean multiplicity sequences if and only if $m = m'$ and $\beta = \beta'$. Note by definition that $\text{Eds}[m : \beta]$ and $\text{Eds}[\beta : m]$ are the same Euclidean multiplicity sequences.

§9.2 Some examples for the representation of the multiplicity sequence for any irreducible plane curve singularity in terms of a collection of subsequences of D with the subdivision

Definition 9.3. Let $V(f) = \{(y, z) : f(y, z) = 0\}$ be an analytic variety at $(y, z) = (0, 0)$ in \mathbb{C}^2 where $f(y, z)$ is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin in \mathbb{C}^2 .

(i) Following Definition 8.2, brevity for notation, either $\text{Multiseq}(V(f))$ or $\{[\text{Mult}(V(f))]\}$ is said to be the multiplicity sequence of $V(f)$.

(ii) As we have seen in the definitions just before Theorem 2.2, let $V^{(\sigma)}(f)$ be the σ -th proper transform of $V(f)$ at the origin, whenever any finite suitable number σ iterations of blow-ups in process of the standard resolution of the singular point $(0, 0)$ of $V(f)$ are chosen arbitrary. Following the notation in (i), $\text{Multiseq}(V^{(\sigma)}(f))$ or $\{[\text{Mult}(V^{(\sigma)}(f))]\}$ is said to be the multiplicity sequence of $V^{(\sigma)}(f)$.

In this section, in preparation for finding a solution of Problem[2], by using Definition 9.2 and Definition 9.3, observe the following proposition and corollaries in order.

Proposition 9.4. Assumptions Let $V(f) = \{(y, z) : f(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by $f(y, z) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ in $\mathbb{C}\{y, z\}$, where each a_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers.

Let $1 \leq n < k = \alpha_n$ with $d = \gcd(n, k)$, and write $n = n_1 d$ and $k = k_1 d$.

(1)(1a) By Theorem 3.6 of §3, we may assume without any need of proof that if f is irreducible in $\mathbb{C}\{y, z\}$, then f can be represented as follows:

$$(9.4.1) \quad f = A(z^{n_1} + \xi y^{k_1})^d + \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d,$$

where the $c_{\alpha\beta}$ are nonzero complex numbers for some nonnegative integers α and β , and A and ξ are the unique nonzero complex numbers.

(1b) If $\gcd(n, \alpha_n) = 1$, the necessary condition for f to be irreducible in $\mathbb{C}\{y, z\}$ in (9.4.1) is sufficient. Note that $n = n_1$ and $k = k_1$ with $d = 1$.

(2) As in the assumption of Theorem 3.7 of §3, let $V(G) = \{(y, z) : G(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by the form

$$(9.4.2) \quad \begin{aligned} G &= z^\gamma g, \\ g_1 &= z^{n_1} + y^{k_1} \quad \text{with} \quad \gcd(n_1, k_1) = 1, \end{aligned}$$

satisfying the following properties:

- (i) $1 \leq n_1 < k_1$.
- (ii) If $n_1 = 1$, then $\gamma = 1$.
- (iii) If $n_1 \geq 2$, then $\gamma = 0$.

Let τ_m be the composition of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(G)$ in (9.4.2) by the same way as we have used in the assumption of Theorem 3.7.

For brevity, let $V^{(t)}(G)$ be the proper transform under τ_t for $1 \leq t \leq m$. For each $t = 1, 2, \dots, m$, suppose that $\tau_t : M^{(t)} \rightarrow \mathbb{C}^2$ satisfies the same properties and notations as in the assumption of Theorem 3.7.

(3) As we have seen in Definition 9.3, let $V^{(\sigma)}(f)$ be the σ -th proper transform of $V(f)$ at the origin, and $\{[\text{Mult}(V^{(\sigma)}(f))]\}$ be the multiplicity sequence of $V^{(\sigma)}(f)$, whenever any finite suitable number σ iterations of blow-ups in process of the standard resolution of the singular point $(0, 0)$ of $V(f)$ are chosen arbitrary.

Conclusions Assuming that f is analytically irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at the origin \mathbb{C}^2 , then we have the following:

$$(9.4.3) \quad \begin{aligned} \text{If } 1 \leq n_1 < k_1, \text{ then } \{[\text{Mult}(V(f))]\} &= \text{Join}(\{[dn_1 : dk_1]\}, \{[\text{Mult}(V^{(m)}(f))]\}). \\ \text{If } 2 \leq n_1 < k_1, \text{ then } \{[\text{Mult}(V(G))]\} &= \{[\text{Mult}(V(g_1))]\} = \{[n_1 : k_1]\}. \end{aligned}$$

For brevity of notation, we write $\{[dn_1 : dk_1]\} = \{d[n_1 : k_1]\}$, if necessary. \square

Proof of Proposition 9.4. By Theorem 3.7 or Corollary 3.8, the proof is done. \square

Remark 9.4.1. (a) Let $f(y, z) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \dots + a_n y^{\alpha_n}$ be in $\mathbb{C}\{y, z\}$ where $2 \leq n < \alpha_n$, each a_i is a unit in $\mathbb{C}\{y, z\}$ if exists, and the α_i are positive integers.

If $\gcd(n, \alpha_n) = 1$, then one of the necessary and sufficient condition for f to be irreducible in $\mathbb{C}\{y, z\}$ is that $f = 0$ and $z^n + y^{\alpha_n} = 0$ have the same multiplicity sequence, denoted by $\{[n, \alpha_n]\}$, by Proposition 9.4 or Corollary 3.3. \square

By Proposition 9.4, it is easy to get the next corollary.

Corollary 9.5. Assumptions *Let $g_r \in \mathbb{C}\{y, z\}$ be a semi-quasi-Puiseux series of the recursive r -type, as either in [A] of Definition 5.0.0 or in the assumption of Theorem 5.0. If g_r is irreducible in $\mathbb{C}\{y, z\}$, for each $r \geq 2$ g_r can be written in the form*

$$(9.5.1) \quad g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^d + \sum_{\alpha, \beta \geq 0} c_{\alpha\beta}^{(r)} y^\alpha z^\beta \quad \text{with } d = n_2 n_3 \cdots n_r,$$

by Sublemma 5.2 of §5 where ε_1 is a unit in $\mathbb{C}\{y, z\}$, and the $c_{\alpha\beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} n_2 n_3 \cdots n_r$.

Conclusions Using the same properties and notations in Proposition 9.4, we have the following:

$$(9.5.2) \quad \{[\text{Mult}(V(g_r))]\} = \text{Join}\{[dn_1 : dk_1]; [\text{Mult}(V^{(m)}(g_r))]\}$$

where $k_1 = \beta_{1,1}$, $\{[dn_1 : dk_1]\} = \{d[n_1 : k_1]\}$ and $d = n_2 n_3 \cdots n_r$. \square

Chapter VI: In preparation for the proof of The 1st Algorithm

§10. To find the necessary and sufficient condition for any two Puiseux convergent power series of recursive types in $\mathbb{C}\{y, z\}$ to have the same multiplicity sequence and their classifications

§10.0. Introduction

In this section, the problem is to prove by Theorem 10.2 that we can compute a one-to-one function from Family(1) into Family(2), using Theorem 7.7, which gives a solution of Problem[1-B] in §7. Therefore, it suffices to prove by Theorem 7.7 and Theorem 10.2 that if any two Puiseux convergent power series of recursive types in $\mathbb{C}\{y, z\}$ have the same multiplicity sequence then they have the same divisor under two standard resolutions in the sense of Definition 2.4, and conversely. Note by Definition 7.0 in §7 that Family(1) is the subset of Quasi-Family(1) where Quasi-Family(1) = $\{f \text{ is arbitrary quasi-Puiseux convergent power series of the recursive type: } f \in \text{Family}(0)\}$ in the sense of Definition 5.0.0.

§10.1. In preparation for computing a one-to-one function from Family(1) into Family(2)

Theorem 10.1. Assumptions By the same way as in Theorem 7.3 or Definition 5.0.0, define a quasi-Puiseux convergent power series g_r of recursive r -type in $\mathbb{C}\{y, z\}$ by Sequences[I] in either the assumptions of Theorem 7.3 or Definition 5.0.0. Suppose that the same properties and notations as either in the assumptions of Theorem 7.3 or in Definition 5.0.0 hold. Let r be an arbitrary positive integer.

Conclusions By the same notations in Definition 9.1 and Definition 9.2, $\text{Multiseq}(V(g_r))$, called the multiplicity sequence of $V(g_r)$, can be represented as follows:

$$(10.1.1) \quad \begin{aligned} \text{Multiseq}(V(g_r)) = & \text{Join}(\{[n : \alpha_1]\}, \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \dots, \\ & \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-1}) : \alpha_r - \alpha_{r-1}]\}), \\ & \text{such that} \quad n = n_1 n_2 \cdots n_r, \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_r \quad \text{and} \\ & \alpha_j = \alpha_{j-1} + \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1} n_{j+2} \cdots n_r \quad \text{for } 2 \leq j \leq r, \end{aligned}$$

where $\widehat{\Delta}_j(\beta_{j,k})_{k=1}^j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r$. \square

Remark 10.1.1. (a) $\text{Multiseq}(V(g_r))$ of (10.1.1) can be rewritten as follows:

$$(10.1.1^*) \quad \begin{aligned} \text{Multiseq}(V(g_r)) = & \text{Join}(\{[n : \alpha_1]\}, \{[d_1 : \alpha_2 - \alpha_1]\}, \dots, \{[d_{r-1} : \alpha_r - \alpha_{r-1}]\}), \\ & \text{such that} \quad n = n_1 d_1, \quad \alpha_1 = \beta_{1,1} d_1 \quad \text{and} \\ & \alpha_j - \alpha_{j-1} = \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j d_j \quad \text{for } 2 \leq j \leq r, \end{aligned}$$

where $d_j = n_{j+1} n_{j+2} \cdots n_r$ for $1 \leq j \leq r-1$ and $d_r = 1$, and $d_1 = \gcd(n, \alpha_1)$ and $d_j = \gcd(n, \alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_j - \alpha_{j-1}) = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1})$ for $2 \leq j \leq r$ because $d_j = \gcd(n, \alpha_1, \dots, \alpha_j)$.

(b) Suppose that g_r is irreducible in $\mathbb{C}\{y, z\}$. Then, it was already proved by Theorem 5.0 that the following are true:

$$(10.1.1.0) \quad \begin{aligned} g_r \text{ is irreducible in } \mathbb{C}\{y, z\} \\ \iff g_1, g_2, \dots, g_{r-1} \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \gcd(n_r, \Delta_r(\beta_{r,k})_{k=1}^r) = 1 \\ \iff \gcd(n_1, \beta_{1,1}) = 1, \gcd(n_2, \widehat{\Delta}_2(\beta_{2,1}, \beta_{2,2})) = 1, \dots, \gcd(n_r, \widehat{\Delta}_r(\beta_{r,k})_{k=1}^r) = 1. \end{aligned}$$

For each $j = 1, 2, \dots, r$, note that $(0, 0)$ is an isolated singular point of an analytic variety $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ except the case that $V(g_1)$ with $n_1 = 1$. \square

§10.2. The proof of Theorem 10.1

Note that the assumptions of Theorem 10.1 satisfies the same assumptions and notations as in Theorem 5.0. For the proof of Theorem 10.1, we can use five sublemmas, that is, Sublemma 5.1, Sublemma 5.2, ..., Sublemma 5.5 of Theorem 5.0, respectively.

Proof of Theorem 10.1. Now, the proof of the equality in (10.1.1) will be by induction on the positive integer $r \geq 1$. Note that $n_1 \geq 2$ and $\beta_{1,1} \geq 1$ with $\gcd(n_1, \beta_{1,1}) = 1$.

Then, it is enough to consider two cases, respectively.

Case(I) $r = 1$, and Case(II) $r \geq 2$.

Case(I) Let $r = 1$. To prove the equality in (10.1.1), if $n_1 \geq 2$ and $\beta_{1,1} = 1$, then there is nothing to prove. For the proof of theorem, if $r = 1$ then it may be assumed that $n_1 \geq 2$ and $\beta_{1,1} > 1$ with $\gcd(n_1, \beta_{1,1}) = 1$. So, it suffices to prove the following:

$$(10.1.2) \quad \{[\text{Multiseq}(V(g_1))]\} = \{[n : \alpha_1]\},$$

where $n = n_1$ and $\alpha_1 = \beta_{1,1}$. Then, there is nothing to prove by Proposition 9.4.

Case(II) Let $r \geq 2$. To prove the equality in (10.1.1), suppose we have shown on the positive integer $(r-1)$ by the induction method that for a given $V(g_{r-1}) = \{(y, z) : g_{r-1}(y, z) = 0\}$, $\text{Multiseq}(V(g_{r-1})) = \{[\text{Mult}(V(g_{r-1}))]\}$, called the multiplicity sequence of $V(g_{r-1})$, can be represented as follows:

$$(10.1.3) \quad \begin{aligned} \text{Multiseq}(V(g_{r-1})) &= \text{Join}(\{[n : \alpha_1]\}, \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \dots, \\ &\quad \{[\gcd(n, \alpha_1, \dots, \alpha_{r-3}) : \alpha_{r-2} - \alpha_{r-3}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}), \\ &\text{such that } n = n_1 n_2 \cdots n_{r-1}, \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_{r-1} \quad \text{and} \\ &\quad \alpha_j = \alpha_{j-1} + \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1} n_{j+2} \cdots n_{r-1} \quad \text{for } 2 \leq j \leq r-1, \end{aligned}$$

where $\widehat{\Delta}_j(\beta_{j,k})_{k=1}^j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r-1$ and $\Delta_1(t) = t$.

In preparation for the proof of the equality in (10.1.1), it was already proved by Proposition 9.4(Corollary 9.5) or Sublemma 5.4 that the following equality is true:

$$(10.1.4) \quad \begin{aligned} \{[\text{Mult}(V(g_r))]\} &= \text{Join}(\{[n : \alpha_1]\}, \{[\text{Mult}(V^{(\lambda_1)}(g_r))]\}), \\ &\text{where } n = n_1 n_2 \cdots n_r \quad \text{and} \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_r. \end{aligned}$$

So, for the proof of the equality in (10.1.1), it suffices to show by (10.1.3) and (10.1.4) that the following equality is true:

$$(10.1.5) \quad \begin{aligned} \{[\text{Mult}(V^{(\lambda_1)}(g_r))]\} &= \text{Join}(\{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \{[\gcd(n, \alpha_1, \alpha_2) : \alpha_3 - \alpha_2]\}, \\ &\quad \dots, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-1}) : \alpha_r - \alpha_{r-1}]\}). \end{aligned}$$

For the proof of the equality in (10.1.5), it remains to compute $\{[\text{Mult}(V^{(\lambda_1)}(g_r))]\}$.

First of all, note by Sublemma 5.5 that $V^{(\lambda_1)}(g_r) = V(h_{r-1})$ is well-defined where $h_{r-1} = (g_r \circ \tau_m)_{\text{proper}} = h_{r-2}^{s_{r-1}} + \eta_{r-1} v^{\gamma_{r-1,1}} (u+1)^{\gamma_{r-1,2}} h_1^{\gamma_{r-1,3}} \cdots h_{r-3}^{\gamma_{r-1,r-1}}$ with $\lambda_1 = m$. That is, it was already shown by Sublemma 5.5 that the local defining equation $h_{r-1} \in \mathbb{C}\{v, u+1\}$ of $V(h_{r-1})$ satisfies the same kind of assumptions and notations as the local defining equation of $V(g_{r-1})$ in (10.1.3) does, which is a necessary and sufficient condition for the use of the induction proof on the positive integer $(r-1)$.

So, by the induction assumption on the positive integer $(r-1)$, we can apply the same results of (10.1.3) to $V(h_{r-1})$, up to the same kind of notations and properties below.

Therefore, we can prove by (10.1.3) and by the same kind of notations and properties as in Sublemma 5.5 that for a given $V(h_{r-1}) = \{(u+1, v) : h_{r-1}(u+1, v) = 0\}$, $\text{Multiseq}(V(h_{r-1}))$, called the multiplicity sequence of $V(h_{r-1})$, is represented as follows:

$$(10.1.6) \quad \begin{aligned} \{[\text{Mult}(V^{(\lambda_1)}(g_r))]\} &= \{[\text{Mult}(V(h_{r-1}))]\} \\ &= \text{Join}(\{[b : \delta_1]; [\gcd(b, \delta_1) : \delta_2 - \delta_1]; [\gcd(b, \delta_1, \delta_2) : \delta_3 - \delta_2]; \dots; \\ &\quad [\gcd(b, \delta_1, \delta_2, \dots, \delta_{r-3}) : \delta_{r-2} - \delta_{r-3}]; [\gcd(b, \delta_1, \delta_2, \dots, \delta_{r-2}) : \delta_{r-1} - \delta_{r-2}]\}), \\ &\text{such that } b = s_1 s_2 \cdots s_{r-1}, \quad \delta_1 = \gamma_{1,1} s_2 \cdots s_{r-1} \quad \text{and} \\ &\quad \delta_j = \delta_{j-1} + \widehat{\Xi}_j(\gamma_{j,k})_{k=1}^j s_{j+1} s_{j+2} \cdots s_{r-1} \quad \text{for } 2 \leq j \leq r-1, \end{aligned}$$

where $\widehat{\Xi}_j(\gamma_{j,k})_{k=1}^j = \Xi_j(\gamma_{j,k})_{k=1}^j - n_j n_{j-1} \Xi_{j-1}(\gamma_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r-1$ and $\Xi_1(t) = t$. Here, note by Sublemma 5.5 that

$$(10.1.7) \quad \begin{aligned} s_1 = n_2 \geq 2, \quad \gamma_{1,1} &= \Delta_2^\#(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 > 0, \\ s_{j-1} = n_j \geq 2, \quad \gamma_{j-1,1} &= \Delta_j^\#(\beta_{j,k})_{k=1}^j - n_1 \beta_{1,1} n_2 n_3 \cdots n_j > 0, \\ \gamma_{j-1,2} = \beta_{j,3}, \gamma_{j-1,3} &= \beta_{j,4}, \dots, \gamma_{j-1,j-1} = \beta_{j,j} \quad \text{for } 2 \leq j \leq r, \end{aligned}$$

where $\gamma_{1,1}, \gamma_{2,1}, \dots, \gamma_{r-1,1}$ are positive by Sublemma 5.1, noting that $\gamma_{1,1} = \Delta_2^\#(\beta_{2,1}, \beta_{2,2}) - n_1 \beta_{1,1} n_2 = \widehat{\Delta}_2(\beta_{2,1}, \beta_{2,2})$.

Moreover, as we have seen in a conclusion of Sublemma 5.5, we have the following representation, too: Let $q = 2, 3, \dots, r-1$.

$$(10.1.8) \quad \begin{aligned} \widehat{\Xi}_q(\gamma_{q,k})_{k=1}^q &= \Xi_q(\gamma_{q,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,k})_{k=1}^{q-1} \\ &= \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,k})_{k=1}^q = \widehat{\Delta}_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} > 0. \end{aligned} \quad (\text{Sublemma 5.5})$$

Now, for the complete proof of this theorem, comparing (10.1.5) with (10.1.6), it suffices to show the following equations are true:

$$(10.1.9) \quad \begin{aligned} \text{(i)} \quad [\gcd(n, \alpha_1) : \alpha_2 - \alpha_1] &= [b : \delta_1]. \\ \text{(ii)} \quad [\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{j-1}) : \alpha_j - \alpha_{j-1}] &= [\gcd(b, \delta_1, \delta_2, \dots, \delta_{j-2}) : \delta_{j-1} - \delta_{j-2}] \\ &\quad \text{for each } j = 3, 4, \dots, r. \end{aligned}$$

Recall by (10.1.1) that

$$(10.1.10) \quad \begin{aligned} n &= n_1 n_2 \cdots n_r, \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_r \quad \text{and} \\ \alpha_j &= \alpha_{j-1} + \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1} n_{j+2} \cdots n_r \quad \text{for } 2 \leq j \leq r. \end{aligned}$$

In preparation for the proof of the equality in (10.1.9), first of all, we can get easily the following equations from (10.1.6), (10.1.7), (10.1.8) and (10.1.10):

$$(10.1.11) \quad \begin{aligned} b &= s_1 s_2 \cdots s_{r-1} = n_2 n_3 \cdots n_r = \gcd(n, \alpha_1), \\ \delta_1 &= \gamma_{1,1} s_2 \cdots s_{r-1} = \widehat{\Delta}_2(\beta_{2,1}, \beta_{2,2}) n_3 n_4 \cdots n_r = \alpha_2 - \alpha_1 > 0, \\ \delta_j - \delta_{j-1} &= \widehat{\Xi}_j(\gamma_{j,k})_{k=1}^j s_{j+1} s_{j+2} \cdots s_{r-1} \quad \text{for } 2 \leq j \leq r-1 \\ &= \widehat{\Delta}_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} n_{j+2} n_{j+3} \cdots n_r = \alpha_{j+1} - \alpha_j > 0, \end{aligned}$$

noting by assumption that $\gcd(n, \alpha_1) = \gcd(n_1 n_2 \cdots n_r, \beta_{1,1} n_2 \cdots n_r) = n_2 n_3 \cdots n_r = b$ because $\gcd(n_1, \beta_{1,1}) = 1$.

In order to finish the proof of the equality of (10.1.9), compare (10.1.9) with (10.1.11), and then it remains to prove the following:

$$(10.1.12) \quad \begin{aligned} \text{(i)} \quad \gcd(n, \alpha_1) &= b. \\ \text{(ii)} \quad \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) &= \gcd(b, \delta_1, \delta_2, \dots, \delta_{j-1}) \quad \text{for each } j = 2, 3, \dots, r-1. \end{aligned}$$

Using (10.1.11) again, the proof of (10.1.12) is as follows:

(a) Since it is clear by (10.1.11) that $\gcd(n, \alpha_1) = b$, then there is nothing for the proof of (i) of (10.1.12).

(b) If $j = 2$, then it is clear by (a) and (10.1.11) that $\gcd(n, \alpha_1, \alpha_2) = \gcd(n, \alpha_1, \alpha_2 - \alpha_1) = \gcd(b, \delta_1)$. So, if $j = 2$, then the proof of (ii) of (10.1.12) is done.

The general case in (ii) of (10.1.12) will be proved by induction. Suppose we have shown on the positive integer $j < r-1$ that $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) = \gcd(b, \delta_1, \delta_2, \dots, \delta_{j-1})$. Then, $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{j+1}) = \gcd(n, \alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_{j+1} - \alpha_j) = \gcd(b, \delta_1, \delta_2 - \delta_1, \dots, \delta_j - \delta_{j-1}) = \gcd(b, \delta_1, \delta_2, \dots, \delta_j)$ by (10.1.11) and (a). Thus, the proof of (ii) of (10.1.12) is done, and so the proof of (10.1.12) is finished.

Therefore, the proof of the theorem can be completely finished. \square

§10.3. How to compute a 1-1 function from Family(1) into Family(2)

Theorem 10.2. Assumptions Let r and ρ be arbitrary positive integers. By the same way as in the assumption of Theorem 7.3, define arbitrary quasi-Puiseux series $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type by Sequences[I] and arbitrary quasi-Puiseux series $\phi_\rho \in \mathbb{C}\{y, z\}$ of the recursive ρ -type by Sequences[II] in Theorem 7.3.

We may assume that the same properties and notations as in the assumptions of Theorem 7.3 hold. Note that Sequences[I] and Sequences[II] are the same up to the change of notations. Assume in addition that $2 \leq n_1 < \beta_{1,1}$ and $2 \leq \ell_1 < \delta_{1,1}$.

Conclusions Note by either Definition 1.1 or Definition 5.0.0 that g_r and ϕ_ρ are called the Puiseux series in $\mathbb{C}\{y, z\}$ because $2 \leq n_1 < \beta_{1,1}$ and $2 \leq \ell_1 < \delta_{1,1}$ by assumptions. Then, we have the following:

$$(10.2.1) \quad g_r \text{ and } \phi_\rho \text{ have the same multiplicity sequence.}$$

$$\iff$$

$$(10.2.2) \quad n_j = \ell_j \text{ and } \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \text{ for each } j = 1, 2, \dots, r = \rho,$$

Moreover, it can be easily proved by Theorem 7.3 that the following holds:

$$(10.2.3) \quad g_r \text{ and } \phi_\rho \text{ have the same multiplicity sequence}$$

$$\iff$$

$$(10.2.4) \quad g_r \overset{\text{divisor}}{\sim} \phi_\rho \text{ under the standard resolutions.} \quad \square$$

Remark 10.2.1. Without assuming that both $2 \leq n_1 < \beta_{1,1}$ and $2 \leq \ell_1 < \delta_{1,1}$, it can be easily proved that the conclusion of the theorem may not be true by the following example:

$$(10.2.1.1) \quad \begin{aligned} g_1 &= z^3 + y^8 & \text{and} \\ \phi_2 &= \phi_1^3 + y^2 z^3 & \text{with } \phi_1 = z + y^2, \end{aligned}$$

because g_1 and ϕ_2 have the same multiplicity sequence, and also they have the same divisor under two standard resolutions, but the condition in (10.2.2) does not hold.

§10.4. The proof of Theorem 10.2

Proof of Theorem 10.2. First of all, the first half for the proof of Theorem 10.2 is to show that two statements in (10.2.1) and (10.2.2) are equivalent, and then the second half for the proof of Theorem 10.2 is to show that two statements in (10.2.3) and (10.2.4) are equivalent. After the proof of the first half is done, there is nothing to prove for the second half, because it was already proved by (7.3.4) of Theorem 7.3 that the condition in (10.2.2) is necessary and sufficient for $g_r \overset{\text{divisor}}{\sim} \phi_\rho$ under the standard resolutions.

Now, for the proof of the first half, it suffices to consider two cases:

Fact[I]. We prove the sufficiency of the condition in (10.2.2) for $V(g_r)$ and $V(\phi_\rho)$ to have the same multiplicity sequence at $(0, 0) \in \mathbb{C}^2$.

Fact[II]. We prove the necessity of the condition in (10.2.2) for $V(g_r)$ and $V(\phi_\rho)$ to have the same multiplicity sequence at $(0, 0) \in \mathbb{C}^2$.

In preparation for solving the first half, since each of Sequences[I] and Sequences[II] in Theorem 10.2 satisfies the same kind of assumption as in Sequences[I] of Theorem 10.1, then we can apply Theorem 10.1 to each of Sequences[I] and Sequences[II] in Theorem 10.2, respectively. So, we can easily get the following sublemma, which will be applicable for the proofs of Fact(I) and Fact(II).

Sublemma 10.2.2. Assumptions Suppose that the same properties and notations as in the assumption of Theorem 10.2 are true. Note that g_r and ϕ_ρ are irreducible in $\mathbb{C}\{y, z\}$.

Conclusions Then, the multiplicity sequences of $V(g_r)$ and $V(\phi_\rho)$ are as follows:

$$(10.2.2.1) \quad \text{Multiseq}(V(g_r)) = \text{Join}(\{[n : \alpha_1]\}, \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \dots, \\ \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-1}) : \alpha_r - \alpha_{r-1}]\}), \\ \text{such that} \quad n = n_1 n_2 \cdots n_r, \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_r \quad \text{and} \\ \alpha_j = \alpha_{j-1} + \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1} n_{j+2} \cdots n_r \quad \text{for } 2 \leq j \leq r-1, \\ \alpha_r = \alpha_{r-1} + \widehat{\Delta}_r(\beta_{r,k})_{k=1}^r,$$

where $\widehat{\Delta}_j(\beta_{j,k})_{k=1}^j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r$ and $\Delta_1(t) = t$.

$$(10.2.2.2) \quad \text{Multiseq}(V(\phi_\rho)) = \text{Join}(\{[b : \chi_1]\}, \{[\gcd(b, \chi_1) : \chi_2 - \chi_1]\}, \dots, \\ \{[\gcd(b, \chi_1, \chi_2, \dots, \chi_{\rho-2}) : \chi_{\rho-1} - \chi_{\rho-2}]\}, \{[\gcd(n, \chi_1, \chi_2, \dots, \chi_{\rho-1}) : \chi_\rho - \chi_{\rho-1}]\}), \\ \text{such that} \quad b = \ell_1 \ell_2 \cdots \ell_\rho, \quad \chi_1 = \delta_{1,1} \ell_2 \ell_3 \cdots \ell_\rho \quad \text{and} \\ \chi_j = \chi_{j-1} + \widehat{\omega}_j(\delta_{j,k})_{k=1}^j \ell_{j+1} \ell_{j+2} \cdots \ell_\rho \quad \text{for } 2 \leq j \leq \rho-1, \\ \chi_\rho = \chi_{\rho-1} + \widehat{\omega}_\rho(\delta_{\rho,k})_{k=1}^\rho,$$

where $\widehat{\omega}_j(\delta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j - \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq \rho$ and $\omega_1(t) = t$. \square

The proof of this sublemma just follows from Theorem 10.1.

Fact[I] For the proof of the sufficiency of the condition in (10.2.2), suppose that the following are true:

$$(10.2.5) \quad n_j = \ell_j \text{ and } \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \text{ for each } j = 1, 2, \dots, r = \rho.$$

Since $n_j = \ell_j$ for each $j = 1, 2, \dots, r = \rho$ and $\beta_{1,1} = \Delta_1(\beta_{1,1}) = \omega_1(\delta_{1,1}) = \delta_{1,1}$ by (10.2.5), then $\widehat{\Delta}_j(\beta_{j,k})_{k=1}^j = \widehat{\omega}_j(\delta_{j,k})_{k=1}^j$ for each $j = 1, 2, \dots, r$ by (10.2.5), again.

So, it is clear by (10.2.5) and Sublemma 10.2.2 that the following equalities are true:

$$(10.2.6) \quad n = b \text{ and } \alpha_1 = \chi_1, \quad \text{and} \\ \alpha_j - \alpha_{j-1} = \chi_j - \chi_{j-1} \text{ for each } j = 2, \dots, r = \rho.$$

Noting by (10.2.6) that $\alpha_j = \chi_j$ for each $j = 1, 2, \dots, r = \rho$, then it is clear by (10.2.2.1) and (10.2.2.2) of Sublemma 10.2.2 and by (10.2.6) that g_r and ϕ_ρ have the same multiplicity sequence.

Fact[II] To prove the necessity of the condition in (10.2.2), suppose that g_r and ϕ_ρ have the same multiplicity sequence and by assumption that $2 \leq n_1 < \beta_{1,1}$ and $2 \leq \ell_1 < \delta_{1,1}$. For the proof, we may assume that $1 \leq r \leq \rho$.

Case(I) $r = 1 \leq \rho$, and Case(II) $2 \leq r \leq \rho$.

Case(I) of Fact[II] Let $r = 1 \leq \rho$. By the definition of $\text{Multiseq}(V(g_r))$ and $\text{Multiseq}(V(\phi_\rho))$, observe (i) and (ii):

(i) The largest element of $\{[\text{Mult}(V(g_1))]\}$ is n_1 , and the largest element of $\{[\text{Mult}(V(\phi_\rho))]\}$ is $b = \ell_1 \ell_2 \cdots \ell_\rho$, too. So, $n_1 = b$.

(ii) Let s be a positive integer such that $sn_1 < \beta_{1,1} < (s+1)n_1$. Then, we can define the second largest element K_2 of $\{[\text{Mult}(V(g_1))]\}$ with $n_1 > K_2$. So, it is clear that $K_2 = \beta_{1,1} - sn_1$, because $K_2 = \beta_{1,1} - sn_1$ is the $(s+1)$ -th element of $\{[\text{Mult}(V(g_1))]\}$, viewed as a finite sequence.

Applying the same method to (10.2.2.2) of Sublemma 10.2.2 just as above, let s' be a positive integer such that $s'b < \chi_1 < (s'+1)b$. Then, we can define the second largest element L_2 of $\{[\text{Mult}(V(\phi_\rho))]\}$ with $b > L_2$, noting that $L_2 = \chi_1 - s'b$ is the $(s'+1)$ -th element of $\{[\text{Mult}(V(\phi_\rho))]\}$, as a finite sequence.

Since any multiplicity sequence is monotonically decreasing, then it is trivial by (i) and (ii) that $s = s'$ and $\beta_{1,1} - sn_1 = K_2 = L_2 = \chi_1 - s'b$ by the definition of the multiplicity sequence for two positive integers in Definition 9.1. So, $\beta_{1,1} = \chi_1 = \delta_{1,1}\ell_2\ell_3\cdots\ell_\rho$ because $n_1 = b$. So, $\frac{\beta_{1,1}}{n_1} = \frac{\chi_1}{b} = \frac{\delta_{1,1}}{\ell_1}$ by (10.2.2.2) of Sublemma 10.2.2. Since $\gcd(n_1, \beta_{1,1}) = 1$ and $\gcd(\ell_1, \delta_{1,1}) = 1$, then $n_1 = \ell_1$ and $\beta_{1,1} = \delta_{1,1}$. Since $n_1 = b$, then $\delta_{1,1} = \chi_1$ with $\rho = 1$. Thus, the proof for Case(I) is done.

Case(II) of Fact[II] Let $2 \leq r \leq \rho$. For the proof for Case(II), we prove first that the assertion in (10.2.7) is true, and after then, it suffices to show by (10.2.7) that the assertion in (10.2.8) is true:

$$(10.2.7) \quad \begin{aligned} &\alpha_j = \chi_j \quad \text{for each } j = 1, 2, \dots, r \leq \rho, \\ &\text{and so } \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) = \gcd(b, \chi_1, \chi_2, \dots, \chi_j), \\ &\text{which is the smallest element of } \{\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{j-1}) : \alpha_j - \alpha_{j-1}\}. \end{aligned}$$

$$(10.2.8) \quad n_j = \ell_j \text{ and } \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \quad \text{for each } j = 1, 2, \dots, r = \rho.$$

The assertion in (10.2.7) The assertion in (10.2.7) will be proved by induction on the positive integer r where $1 \leq r \leq \rho$, using the following steps:

Step(1). $\alpha_1 = \chi_1$.

Step(j). $\alpha_j = \chi_j$ for each $j = 2, 3, \dots, r$.

Step(1) for the proof of the assertion in (10.2.7) As in Case(I), note from the definition of $\{\text{[Mult}(V(g_r))]\}$ and $\{\text{[Mult}(V(\phi_\rho))]\}$ that the largest element of $\{\text{[Mult}(V(g_r))]\}$ is $n = n_1n_2\cdots n_r$ and the largest element of $\{\text{[Mult}(V(\phi_\rho))]\}$ is $b = \ell_1\ell_2\cdots\ell_\rho$. So, $n = b$ by the necessity of the condition in (10.2.2).

Let s be a positive integer such that $sn < \alpha_1 < (s+1)n$ because $2 \leq n_1 < \beta_{1,1}$ and $\gcd(n_1, \beta_{1,1}) = 1$ and $\frac{\alpha_1}{n} = \frac{\beta_{1,1}}{n_1}$. Then, we can define the second largest integer K_2 of $\{\text{[Mult}(V(g_r))]\}$ with $n > K_2$. So, it is clear that $K_2 = \alpha_1 - sn$, noting that $K_2 = \alpha_1 - sn$ is the $(s+1)$ -th element of $\{\text{[Mult}(V(g_r))]\}$, as a sequence.

By the same way as above, let s' be a positive integer such that $s'b < \chi_1 < (s'+1)b$. Then, we can define the second largest integer L_2 of $\{\text{[Mult}(V(\phi_\rho))]\}$ with $b > L_2$, noting that $L_2 = \chi_1 - s'b$ is the $(s'+1)$ -th element of $\{\text{[Mult}(V(\phi_\rho))]\}$, as a sequence. Since any multiplicity sequence is monotonically decreasing, then it is trivial that $s+1 = s'+1$ and $\alpha_1 - sn = K_2 = L_2 = \chi_1 - s'b$ by the definition of the multiplicity sequence for two positive integers in Definition 9.1, and therefore $\alpha_1 = \chi_1$. Thus, the proof of Step(1) is done.

Step(j+1) for the proof of the assertion in (10.2.7) By induction assumption on the positive integer r , suppose we have shown by Step(1), Step(2), ..., Step(j) that $\alpha_i = \chi_i$ for $i = 2, 3, \dots, j$ where j is an arbitrary integer such that $2 \leq j < r \leq \rho$. Then, it remains to prove that $\alpha_{j+1} = \chi_{j+1}$. Since $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) > \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{j+1})$ by assumption, and then $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) \neq \alpha_{j+1} - \alpha_j$, for the proof, it suffices to consider two subcases, respectively.

Subcase(A) $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) < \alpha_{j+1} - \alpha_j$, and

Subcase(B) $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) > \alpha_{j+1} - \alpha_j$.

Subcase(A): Let $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) < \alpha_{j+1} - \alpha_j$. Note that the largest element of $\{\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) : \alpha_{j+1} - \alpha_j\}$ is $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j)$. Since $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) > \gcd(b, \chi_1, \chi_2, \dots, \chi_{j+1})$ by assumption, then $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) \neq \chi_{j+1} - \chi_j$. Then, for this case, it is enough to consider two possibilities, respectively:

(A1) $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) < \chi_{j+1} - \chi_j$, and (A2) $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) > \chi_{j+1} - \chi_j$.

(A1) Let $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) < \chi_{j+1} - \chi_j$. By the definition of the multiplicity sequence for the greatest common divisor of two positive integers, the largest element of $\{\gcd(b, \chi_1, \dots, \chi_j) : \chi_{j+1} - \chi_j\}$ is $\gcd(b, \chi_1, \dots, \chi_j)$. Since $\gcd(n, \alpha_1, \dots, \alpha_j) < \alpha_{j+1} - \alpha_j$ and $\gcd(b, \chi_1, \dots, \chi_j) < \chi_{j+1} - \chi_j$, then $\gcd(n, \alpha_1, \dots, \alpha_j) = \gcd(b, \chi_1, \dots, \chi_j)$ implies that $\alpha_{j+1} - \alpha_j = \chi_{j+1} - \chi_j$ by the same techniques as we have used for the proof of Case(I). So, $\alpha_{j+1} = \chi_{j+1}$.

(A2) Let $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) > \chi_{j+1} - \chi_j$. By the definition of the multiplicity sequence for the greatest common divisor of two positive integers, the largest element of

$\{\gcd(b, \chi_1, \chi_2, \dots, \chi_j) : \chi_{j+1} - \chi_j\}$ is $\chi_{j+1} - \chi_j$. So, $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) = \chi_{j+1} - \chi_j < \gcd(b, \chi_1, \chi_2, \dots, \chi_j) = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j)$, which would be impossible.

Subcase(B): Let $\gcd(n, \alpha_1, \alpha_2, \dots, \alpha_j) > \alpha_{j+1} - \alpha_j$. By the definition of the multiplicity sequence for the greatest common divisor of two positive integers, note that the largest element of $\{\gcd(n, \alpha_1, \dots, \alpha_j) : \alpha_{j+1} - \alpha_j\}$ is $\alpha_{j+1} - \alpha_j$. Since $\gcd(b, \chi_1, \dots, \chi_j) > \gcd(b, \chi_1, \chi_2, \dots, \chi_{j+1})$ by assumption, then $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) \neq \chi_{j+1} - \chi_j$. Then, for this case, it is enough to consider two possibilities, respectively:

(B1) $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) > \chi_{j+1} - \chi_j$, and (B2) $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) < \chi_{j+1} - \chi_j$.

(B1) Let $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) > \chi_{j+1} - \chi_j$. By the definition of the multiplicity sequence for the greatest common divisor of two positive integers, the largest element of $\{\gcd(b, \chi_1, \chi_2, \dots, \chi_j) : \chi_{j+1} - \chi_j\}$ is $\chi_{j+1} - \chi_j$. So, $\chi_{j+1} - \chi_j = \alpha_{j+1} - \alpha_j$. So, $\alpha_{j+1} = \chi_{j+1}$ because $\alpha_j = \chi_j$ by the induction assumption.

(B2) Let $\gcd(b, \chi_1, \chi_2, \dots, \chi_j) < \chi_{j+1} - \chi_j$. By definition of the multiplicity sequence for the greatest common divisor of two positive integers, the largest element of $\{\gcd(b, \chi_1, \chi_2, \dots, \chi_j) : \chi_{j+1} - \chi_j\}$ is $\gcd(b, \chi_1, \chi_2, \dots, \chi_j)$. So, $\alpha_{j+1} - \alpha_j = \gcd(b, \chi_1, \dots, \chi_j) = \gcd(n, \alpha_1, \dots, \alpha_j)$, which would be impossible because $\gcd(n, \alpha_1, \dots, \alpha_j) = n_{j+1}n_{j+2} \cdots n_r > n_{j+2}n_{j+3} \cdots n_r = \gcd(n, \alpha_1, \dots, \alpha_{j+1})$.

By summarizing two subcases, that is, Subcase(A) and Subcase(B), we proved that $\alpha_{j+1} = \chi_{j+1}$, and so, the proof of Step(j+1) is done. Thus, the proof of (10.2.7) is finished.

The assertion in (10.2.8) In preparation for the proof of the assertion in (10.2.8), first we prove that the following are true:

$$(10.2.9) \quad n_j = \ell_j, \quad \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \quad \text{and} \\ n_{j+1}n_{j+2} \cdots n_r = \ell_{j+1}\ell_{j+2} \cdots \ell_\rho \quad \text{for } j = 1, 2, \dots, r \leq \rho.$$

After the proof of (10.2.9) is done, then for the proof of (10.2.8) it remains to show that $r = \rho$.

For the proof of the assertion in (10.2.8) Now, in order to prove the equalities in (10.2.9), we will use (10.2.7) together with (10.2.2.1) and (10.2.2.2) of Sublemma 10.2.2.

Now, the proof of (10.2.9) will be by induction on $j = 1, 2, \dots, r$, as follows:

(1) First, note by Sublemma 10.2.2 and (10.2.7) that $\frac{\beta_{1,1}}{n_1} = \frac{\alpha_1}{n} = \frac{\chi_1}{b} = \frac{\delta_{1,1}}{\ell_1}$ and that $\gcd(n_1, \beta_{1,1}) = \gcd(\ell_1, \delta_{1,1}) = 1$. So, $n_1 = \ell_1$ and $\Delta_1(\beta_{1,1}) = \beta_{1,1} = \delta_{1,1} = \omega_1(\delta_{1,1})$. Also, $n_1n_2n_3 \cdots n_r = n = b = \ell_1\ell_2\ell_3 \cdots \ell_\rho$ imply that $n_2n_3 \cdots n_r = \ell_2\ell_3 \cdots \ell_\rho$.

(2) Suppose we have shown by induction on the integer r that for any integer $j < r$, the following are true:

$$(10.2.10) \quad n_{j-1} = \ell_{j-1} \quad \text{and} \quad \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} = \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}, \\ n_j n_{j+1} \cdots n_r = \ell_j \ell_{j+1} \cdots \ell_\rho \quad \text{for } 2 \leq j \leq r.$$

Then, by (10.2.7) and Sublemma 10.2.2,

$$\frac{\hat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1}n_{j+2} \cdots n_r}{n_j n_{j+1} \cdots n_r} = \frac{\alpha_j - \alpha_{j-1}}{n_j n_{j+1} \cdots n_r} = \frac{\chi_j - \chi_{j-1}}{\ell_j \ell_{j+1} \cdots \ell_\rho} = \frac{\hat{\omega}_j(\delta_{j,k})_{k=1}^j \ell_{j+1} \ell_{j+2} \cdots \ell_\rho}{\ell_j \ell_{j+1} \cdots \ell_\rho},$$

$$\text{which implies that } \frac{\hat{\Delta}_j(\beta_{j,k})_{k=1}^j}{n_j} = \frac{\hat{\omega}_j(\delta_{j,k})_{k=1}^j}{\ell_j}.$$

$$\text{That is, } \frac{\Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}}{n_j} = \frac{\omega_j(\delta_{j,k})_{k=1}^j - \ell_j \ell_{j-1} \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}}{\ell_j}.$$

Since $\gcd(n_j, \Delta_j(\beta_{j,k})_{k=1}^j) = \gcd(\ell_j, \omega_j(\delta_{j,k})_{k=1}^j) = 1$ by Sublemma 10.2.2, and $n_{j-1} = \ell_{j-1}$ and $\Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} = \omega_{j-1}(\delta_{j-1,k})_{k=1}^{j-1}$ by (10.2.10), then $n_j = \ell_j$ and $\Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j$. Noting by (10.2.10) that $n_j n_{j+1} \cdots n_r = \ell_j \ell_{j+1} \cdots \ell_\rho$, it is clear that $n_{j+1}n_{j+2} \cdots n_r = \ell_{j+1}\ell_{j+2} \cdots \ell_\rho$. Thus, the proof of (10.2.9) is done.

Now, to finish the proof of (10.2.8), it remains to prove that $r = \rho$. Assume the contrary. Since it was assumed that $1 \leq j \leq r \leq \rho$, then $r < \rho$. Then, $n_r = \ell_r$ and $\Delta_r(\beta_{r,k})_{k=1}^r = \omega_r(\delta_{r,k})_{k=1}^r$. Also, $n_r = \ell_r \ell_{r+1} \cdots \ell_\rho$ implies that $1 = \ell_{r+1} \ell_{r+2} \cdots \ell_\rho$, and so $\ell_{r+1} = \ell_{r+2} = \cdots = \ell_\rho = 1$, which would be impossible. So, we proved that $r = \rho$.

Thus, the proof of (10.2.8) is done, and then the proof for Case(II) is done. So, the proof for Fact[II] is done by Case(I) and Case(II), and therefore the proof of this theorem is completely finished by Fact[I] and Fact[II]. \square

§ 10.5. The difference between quasi-Puiseux convergent power series of the recursive r-type and the Puiseux convergent power series of the recursive r-type by Theorem 10.3

Theorem 10.3. Let ρ and r be arbitrary positive integers.

Assumptions

Assumptions[I] By the same way as in either Definition 5.0.0 or Theorem 7.3, define a semi-quasi-Puiseux convergent power series g_r by Sequences[I], as follows:

Sequences[I] Let $\{X_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k = g_k(y, z) \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0\}$ is an integer-valued function for $1 \leq k \leq r$ be three sequences satisfying the following four conditions:

Four conditions are denoted by **The 1st Cond⁽⁰⁾**, \dots , **The 4-th Cond⁽⁰⁾** of Sequences[I].

[I]-(1) The 1st Cond⁽⁰⁾ of Sequences[I]:

(1a) $X_1 = \{n_1, \beta_{1,1}\}$ with $n_1 = 1 < \beta_{1,1}$.

(1b) $X_j = \{n_j, \beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}\}$ with $n_j \geq 2$ where $j = 2, \dots, r$.

If $j \geq 2$, then assume that at least one of $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,j}$ is nonzero.

[I]-(2) The 2nd Cond⁽⁰⁾ of Sequences[I]:

(2a) $g_1 = z^{n_1} + y^{\beta_{1,1}}$.

(2b) $g_j = g_{j-1}^{n_j} + y^{\beta_{j,1}} z^{\beta_{j,2}} g_1^{\beta_{j,3}} \dots g_{j-2}^{\beta_{j,j}}$ where $j = 2, \dots, r$.

[I]-(3) The 3rd Cond⁽⁰⁾ of Sequences[I]:

(3a) $\Delta_1(t) = t$ for each $t \in N_0$.

(3b) $\Delta_j(t_j)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$ where $j = 2, \dots, r$.

[I]-(4) The 4-th Cond⁽⁰⁾ of Sequences[I]:

(4) $\Delta_j(\beta_{j,k})_{k=1}^j > n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r$.

Assumptions[II] By using Sequences[I] as above, it is clear to define another convergent power series h_{r-1} by Sequences[II], as follows:

Sequences[II] Using the nonsingular analytic mapping defined by $(g_1, y) \rightarrow (z, y)$ from **[I]-(2) of Sequences[I]** where $g_1 = z + y^{\beta_{1,1}}$, construct **Sequences[II]** consisting of three sequences, that is, $\{W_k : k = 1, 2, \dots, r-1\}$ with $W_k \subset N_0$, $\{h_k : k = 1, 2, \dots, r-1\}$ with $h_k = h_k(y, z) \in \mathbb{C}\{y, z\}$ and $\{\omega_k : N_0^k \rightarrow N_0\}$ is an integer-valued function for $k = 1, 2, \dots, r-1$, satisfying the following three conditions:

Three conditions are denoted by **The 1-th Cond⁽⁰⁾**, \dots , **The 3-th Cond⁽⁰⁾** of Sequences[II].

[II]-(1) The 1st Cond⁽⁰⁾ of Sequences[II]:

(1a) $W_1 = \{c_1, \sigma_{1,1}\}$ with $c_1 \geq 2$,

(1b) $W_j = \{c_j, \sigma_{j,1}, \sigma_{j,2}, \dots, \sigma_{j,j}\}$ with $c_j \geq 2$, noting that $2 \leq j \leq r-1$, satisfying the following inequalities:

(1a-a) $c_1 = n_2$, $\sigma_{1,1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) > n_2 n_1 \beta_{1,1} = n_2 \beta_{1,1} > n_2$,

(1b-a) $c_j = n_{j+1}$, $\sigma_{j,1} = \Delta_2(\beta_{j+1,1}, \beta_{j+1,2})$, $\sigma_{j,2} = \beta_{j+1,3}$, $\sigma_{j,3} = \beta_{j+1,4}$, \dots , $\sigma_{j,j} = \beta_{j+1,j+1}$.

[II]-(2) The 2nd Cond⁽⁰⁾ of Sequences[II]: Note that $2 \leq j \leq r-1$.

(2a) $h_1 = z^{c_1} + \eta_1 y^{\sigma_{1,1}}$.

(2b) $h_j = h_{j-1}^{c_j} + \eta_j y^{\sigma_{j,1}} z^{\sigma_{j,2}} h_1^{\sigma_{j,3}} \dots h_{j-2}^{\sigma_{j,j}}$ for $2 \leq j \leq r-1$.

Note that $\eta_i = \eta_i(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq i \leq r-1$.

[II]-(3) The 3rd Cond⁽⁰⁾ of Sequences[II]: Note that $2 \leq j \leq r-1$.

(3a) $\omega_1(t) = t$ for each $t \in N_0$.

(3b) $\omega_j(t_k)_{k=1}^j = t_j \omega_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1} + c_{j-1} \omega_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$.

In the above assumptions, note that **Sequences[I]** and **Sequences[II]** have the same kind of three conditions except for **The 4-th Cond⁽⁰⁾** of Sequences[I].

Conclusions Then, we have the following three facts:

Fact(1) Then, h_{r-1} of **Sequences[II]** either is a semi-quasi-Puiseux convergent power series or satisfies the following equations.

$$(10.3.1) \quad \omega_j(\sigma_{j,k})_{k=1}^j = \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} \quad \text{for each } j = 1, 2, \dots, r-1,$$

$$(10.3.2) \quad \omega_j(\sigma_{j,k})_{k=1}^j > c_j c_{j-1} \omega_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1} \quad \text{for each } j = 2, 3, \dots, r-1.$$

In particular, the inequality in (10.3.2) is equivalent to the 4-th condition of **Sequences[II]**, denoted by **The 4-th Cond⁽⁰⁾ of Sequences[II]**, noting that $\omega_1(t) = t$ for each $t \in N_0$.

Fact(2) g_r is quasi-Puiseux convergent power series by Sequence(I) $\iff h_{r-1}$ is quasi-Puiseux convergent power series.

Fact(3) Let g_r be irreducible in $\mathbb{C}\{y, z\}$. Then, we have the following:

$$(10.3.3) \quad g_r \text{ and } h_{r-1} \text{ have the same multiplicity sequence.} \quad \square$$

Proof of Theorem 10.3. We prove Fact(1), Fact(2) and Fact(3), respectively.

Fact(1) If the equation in (10.3.1) is true, there is nothing to prove for the inequality in (10.3.2) because $\omega_j(\sigma_{j,k})_{k=1}^j = \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} > n_{j+1}n_j\Delta_j(\beta_{j,k})_{k=1}^j = c_jc_{j-1}\omega_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1}$ for each $j = 2, 3, \dots, r-1$, by using the equation in (10.3.1), The 4-th Cond⁽⁰⁾ of Sequences[I] and The 1st Cond⁽⁰⁾ of Sequences[II]. So, for the proof of Fact(1), it suffices to show that the equality in (10.3.1) is true.

In preparation for the proof of the equality in (10.3.1), using $c_j = n_{j+1}$ and $\sigma_{j,1} = \Delta_2(\beta_{j+1,1}, \beta_{j+1,2})$ for $1 \leq j \leq r-1$ in The 1st Cond⁽⁰⁾ of Sequences[II], first it suffices to show by (10.3.5) that the following is true: Let $(t_k)_{k=1}^j \in N_0^j$ be chosen arbitrary.

$$(10.3.4) \quad \text{Whenever } t_1 \text{ satisfies the equation } t_1 = \Delta_2(\delta_1, \delta_2) \text{ for any } (\delta_1, \delta_2) \in N_0^2, \\ \omega_j(t_1, t_2, t_3, \dots, t_j) = \Delta_{j+1}(\delta_1, \delta_2, t_2, t_3, \dots, t_j) \text{ for all } j = 1, 2, \dots, r-1.$$

As an application of (10.3.4), if $t_1 = \sigma_{j,1} = \Delta_2(\beta_{j+1,1}, \beta_{j+1,2})$, $t_2 = \sigma_{j,2}$, $t_3 = \sigma_{j,3}$, \dots , $t_j = \sigma_{j,j}$, then there is nothing to prove for the equality in (10.3.1), because

$$(10.3.5) \quad \begin{aligned} \omega_j(\sigma_{j,1}, \sigma_{j,2}, \dots, \sigma_{j,j}) &= \Delta_{j+1}(\beta_{j+1,1}, \beta_{j+1,2}, \sigma_{j,2}, \sigma_{j,3}, \dots, \sigma_{j,j}) \\ &= \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}, \end{aligned}$$

where $\sigma_{j,1} = \Delta_2(\beta_{j+1,1}, \beta_{j+1,2})$, $\sigma_{j,k} = \beta_{j+1,k}$ with $2 \leq k \leq j$, by using the equation in The 1st Cond⁽⁰⁾ of Sequences[I].

Now, the proof of (10.3.4) will be by induction on the positive integer $j \leq r-1$.

If $j = 1$, then there is nothing to prove because $\omega_1(t) = t$ for each $t \in N_0$.

By the induction proof, suppose we have shown that (10.3.4) is true for any positive integer j where $1 \leq j \leq r-2$.

Then, it is enough to prove the following: Let $(t_k)_{k=1}^{j+1} \in N_0^{j+1}$ be chosen arbitrary.

$$(10.3.6) \quad \begin{aligned} \text{Whenever } t_1 \text{ satisfies the equation } t_1 = \Delta_2(\delta_1, \delta_2) \text{ for any } (\delta_1, \delta_2) \in N_0^2, \\ \omega_{j+1}(t_1, t_2, t_3, \dots, t_{j+1}) = \Delta_{j+2}(\delta_1, \delta_2, t_2, t_3, \dots, t_{j+1}), \end{aligned}$$

for any $j+1$ where $2 \leq j \leq r-2$.

For the proof of (10.3.6), whenever t_1 satisfies the equation $t_1 = \Delta_2(\delta_1, \delta_2)$ for any $(\delta_1, \delta_2) \in N_0^2$, then we may suppose by the induction assumption on $j < r-1$ that $\omega_j(t_1, t_2, \dots, t_j) = \Delta_{j+1}(\delta_1, \delta_2, t_2, \dots, t_j)$ with $\omega_j(\sigma_{j,k})_{k=1}^j = \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}$.

Then, it is trivial to show that the following equality is true:

$$(10.3.7) \quad \begin{aligned} \omega_{j+1}(t_k)_{k=1}^{j+1} &= c_j\omega_j(t_1, t_2, \dots, t_j) + t_{j+1}\omega_j(\sigma_{j,k})_{k=1}^j \\ &= n_{j+1}\Delta_{j+1}(\delta_1, \delta_2, t_2, \dots, t_j) + t_{j+1}\Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} \\ &= \Delta_{j+2}(\delta_1, \delta_2, t_2, \dots, t_j, t_{j+1}), \end{aligned}$$

by [II]-(3) of Sequences[II], by the induction assumption on $j < r$ and [I]-(3) of Sequences[I], which gives the proof for (10.3.4) on the integer $(j+1)$.

Thus, the proof of (10.3.4) is done, and so the proof of (10.3.1) is done, too. Therefore, the proof of Fact(1) is finished.

Fact(2) Since Sequences[II] has the same kind of four conditions as we have seen in the assumption of Theorem 5.0 by Fact(1), then by Theorem 5.0 we have the following:

$$(10.3.8) \quad h_r \text{ is irreducible in } \mathbb{C}\{y, z\} \\ \iff \gcd(c_1, \sigma_{1,1}) = \gcd(c_2, \omega_2(\sigma_{2,1}, \sigma_{2,2})) = \cdots = \gcd(c_r, \omega_r(\sigma_{r,k})_{k=1}^r) = 1.$$

Since for each $j = 1, 2, \dots, r-1$, $\omega_j(\sigma_{j,k})_{k=1}^j = \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1}$ by (10.3.1) and $c_j = n_{j+1}$ by (1) of Sequences(II), then $\gcd(c_j, \omega_j(\sigma_{j,k})_{k=1}^j) = \gcd(n_{j+1}, \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1})$. Therefore, by (10.3.8) and Theorem 5.0, we have the following:

$$(10.3.9) \quad h_{r-1} \text{ is irreducible in } \mathbb{C}\{y, z\} \iff g_r \text{ is irreducible in } \mathbb{C}\{y, z\}.$$

Thus, the proof of Fact(2) is done.

Fact(3) Let g_r be irreducible in $\mathbb{C}\{y, z\}$. By (10.1.1) of Theorem 10.1, $\{\text{Mult}(V(g_r))\}$, called the multiplicity sequence of $V(g_r)$, can be represented as follows:

$$(10.3.10) \quad \text{Multiseq}(V(g_r)) = \text{Join}(\{[n : \alpha_1]\}, \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \dots, \\ \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-1}) : \alpha_r - \alpha_{r-1}]\}), \\ \text{such that} \quad n = n_1 n_2 \cdots n_r, \quad \alpha_1 = \beta_{1,1} n_2 \cdots n_r \quad \text{and} \\ \alpha_j = \alpha_{j-1} + \widehat{\Delta}_j n_{j+1} n_{j+2} \cdots n_r \quad \text{for } 2 \leq j \leq r,$$

where $\widehat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} > 0$ for $2 \leq j \leq r$ and $\Delta_1(t) = t$.

Note by Fact(2) that h_{r-1} is irreducible in $\mathbb{C}\{y, z\}$. So, as above, $\text{Multiseq}(V(h_{r-1}))$ can be represented as follows:

$$(10.3.11) \quad \text{Multiseq}(V(h_{r-1})) = \text{Join}(\{[c : \xi_1]\}, \{[\gcd(c, \xi_1) : \xi_2 - \xi_1]\}, \dots, \\ \{[\gcd(c, \xi_1, \xi_2, \dots, \xi_{r-3}) : \xi_{r-2} - \xi_{r-3}]\}, \{[\gcd(c, \xi_1, \xi_2, \dots, \xi_{r-2}) : \xi_{r-1} - \xi_{r-2}]\}), \\ \text{such that} \quad c = c_1 c_2 \cdots c_{r-1}, \quad \xi_1 = \sigma_{1,1} c_2 \cdots c_{r-1} \quad \text{and} \\ \xi_j = \xi_{j-1} + \widehat{\omega}_j(\sigma_{j,k})_{k=1}^j c_{j+1} c_{j+2} \cdots c_{r-1} \quad \text{for } 2 \leq j \leq r-1,$$

where $\widehat{\omega}_j = \omega_j(\sigma_{j,k})_{k=1}^j - c_j c_{j-1} \omega_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1}$ for $2 \leq j \leq r-1$ and $\omega_1(t) = t$.

Then, by Fact(1), (10.3.10) and (10.3.11), we have the following properties:

$$(10.3.12) \quad \xi_j - \xi_{j-1} = \widehat{\omega}_j c_{j+1} c_{j+2} \cdots c_{r-1} \quad \text{for any } j = 2, 3, \dots, r-1, \\ = \widehat{\Delta}_{j+1} n_{j+2} n_{j+3} \cdots n_r = \alpha_{j+1} - \alpha_j > 0.$$

Since $c = c_1 c_2 \cdots c_{r-1}$ and $\xi_1 = \sigma_{1,1} c_2 \cdots c_{r-1}$ by (10.3.11) and $\alpha_1 = \beta_{1,1} n_2 \cdots n_r$ by (10.3.10), then

$$(10.3.13) \quad \xi_1 - \beta_{1,1} c = (\sigma_{1,1} - \beta_{1,1} c_1) c_2 \cdots c_{r-1} \\ = (\Delta_2(\beta_{2,k})_{k=1}^2 - n_1 n_2 \beta_{1,1}) n_3 \cdots n_r = \alpha_2 - \alpha_1 > 0$$

by (10.3.10) where $n_1 = 1$ was in Sequences[I], and $\sigma_{1,1} = \Delta_2(\beta_{2,k})_{k=1}^2$ and $c_j = n_{j+1}$ for $1 \leq j \leq r-1$ were defined by Sequences[II].

Since $n = n_1 n_2 \cdots n_r$ and $\alpha_1 = \beta_{1,1} n_2 n_3 \cdots n_r$ with $n_1 = 1$ is defined by (10.3.10), then $\gcd(n, \alpha_1) = n_2 \cdots n_r = c_1 c_2 \cdots c_{r-1} = c$, and so we have the following:

$$(10.3.14) \quad \gcd(c, \xi_1, \xi_2, \dots, \xi_j) \quad \text{for any } j = 2, 3, \dots, r-1, \\ = \gcd(c, \xi_1 - \beta_{1,1} c, \xi_2 - \xi_1, \dots, \xi_j - \xi_{j-1}) \quad \text{by (10.3.13)} \\ = \gcd(\gcd(n, \alpha_1), \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_{j+1} - \alpha_j) \quad \text{by (10.3.12)} \\ = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{j+1}).$$

By (10.3.12) and (10.3.14), it can be proved that the following equality is true in the sense of Definition 9.2: For any $j = 2, 3, \dots, r-1$,

$$(10.3.15) \quad \{\gcd(n, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{j+1}), \alpha_{j+1} - \alpha_j\} = \{\gcd(c, \xi_1, \xi_2, \dots, \xi_j), \xi_j - \xi_{j-1}\}.$$

To prove that (10.3.10) and (10.3.11) are the same, it remains to show by (10.3.15) that the following equality is true:

$$(10.3.16) \quad \text{Join}(\{[n : \alpha_1]\}, \{\gcd(n, \alpha_1) : \alpha_2 - \alpha_1\}) = \{[c : \xi_1]\}.$$

Now, the proof of (10.3.16) is as follows:

$$\begin{aligned} (10.3.17) \quad \{[c : \xi_1]\} &= \{[c_1 c_2 \cdots c_{r-1} : \sigma_{1,1} c_2 c_3 \cdots c_{r-1}]\} \quad \text{by (10.3.11)} \\ &= \{[n_2 n_3 n_4 \cdots n_r : \sigma_{1,1} n_3 n_4 \cdots n_r]\} \quad \text{by Sequences[II]} \\ &= \{[n_1 n_2 n_3 \cdots n_r : \Delta_2(\beta_{2,k})_{k=1}^2 n_3 n_4 \cdots n_r]\} \quad \text{with } 1 = n_1 < \beta_{1,1} \\ &= \text{Join}(\{[n_1 n_2 \cdots n_r : \beta_{1,1} n_2 \cdots n_r]\}, \{[n_2 n_3 \cdots n_r : (\widehat{\Delta}_2 n_3 \cdots n_r)]\}) \\ &= \text{Join}(\{[n : \alpha_1]\}; \{\gcd(n, \alpha_1) : \alpha_2 - \alpha_1\}), \end{aligned}$$

by Definition 9.2 (the definition of the multiplicity sequence for two positive integers) because $\beta_{1,1} > n_1 = 1$ and $\widehat{\Delta}_2 = \Delta_2(\beta_{2,k})_{k=1}^2 - \beta_{1,1} n_1 n_2 > 0$ by Sequences[II]. Thus, the proof of Fact(3) is finished.

Therefore, the proof of theorem is completely finished by the proofs of Fact(1), Fact(2) and Fact(3). \square

Chapter VII: The 1st Algorithm with proofs

§11. The 1st Algorithm for computing a one-to-one function from Family(1) onto Family(2) with proofs

§11.0. Introduction

In this section, the aim is to find the 1st Algorithm for compute a one-to-one function between Family(1) and Family(2). Then, the first half of The 1st Algorithm is given by Theorem 11.2 of §11.2, and the second half of The 1st Algorithm is given by Theorem 11.4 of §11.4.

§11.1. Some definitions

Definition 11.1. Let $f(y, z)$ be irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $0 \in \mathbb{C}^2$. By Definition 1.2, let the standard Puiseux expansion $C_r(t)$ of the r -type for any irreducible curve $C(t)$ be given as follows:

$$(11.1.1) \quad C_r(t) = \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_r}, \end{cases}$$

where $2 \leq n < \alpha_1 < \cdots < \alpha_r$ and
 $n > d_1 > \cdots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

If the above curve $C(t)$ and $V(f)$ have the same multiplicity sequence at $(y, z) = (0, 0)$, it is said by Definition 1.2 that

$$(11.1.2) \quad \text{either } C(t) \equiv V(f) \text{ (Multiseq) or Multiseq}(C(t)) \equiv \text{Multiseq}(V(f)) \text{ as sequence.}$$

Remark 11.1.1 for Definition 11.1. As in Definition 8.1, let the parametrization $C(t)$ for the above irreducible curve C of (11.1.2) be defined by

$$(11.1.3) \quad y(t) = t^n \text{ and } z(t) = c_1 t^{k_1} + c_2 t^{k_2} + \cdots = c_1 t^{k_1} (1 + H(t)),$$

where $1 < n, 1 < k_1 < k_2 < \cdots$, and the c_i are nonzero complex numbers and $H(t)$ is just the substitution. Also, let the parametrization of another irreducible plane curve C' be given by $y(t) = t^n(1 + H(t))$ and $z(t) = t^{k_1}$. Note that either $n \leq k_1$ or $k_1 < n$.

Then, it was already directly shown by Theorem A(Theorem 8.10) that two irreducible curves C and C' have the same multiplicity sequence, and also the same Puiseux pairs by a nonsingular change of a parameter, without using Theorem B. But, without using Theorem A, it has been not yet proved by Theorem B only that these two irreducible curves C and C' have the same multiplicity sequences.

So, it was proved by Theorem 8.10 that an equivalence relation in (11.1.2) is well-defined.

§11.2. The first half of The 1st Algorithm(Theorem 11.2)

Theorem 11.2(Theorem 1.4:Algorithm for finding a one-to-one function from Family(1) into Family(2)).

Assumptions Let r be arbitrary positive integer. By the same way as in Definition 5.0.0 or Theorem 7.3, define a quasi-Puiseux convergent power series g_r of recursive r -type in $\mathbb{C}\{y, z\}$ by Sequences[I] in the assumptions of Theorem 7.3. For notation, assume in addition that $1 \leq n_1 < \beta_{1,1}$ in The 1-th Cond⁽⁰⁾ of Sequences[I] of Theorem 7.3. Now, we may use the same conditions and notations as in the assumptions of Theorem 7.3.

Conclusions It is very elementary to compute the standard Puiseux expansion $C^*(t)$ of an irreducible curve $C(t)$ such that $V(g_r) \equiv C^*(t)$ (Multiseq) with desired algorithms in Fact(A) and Fact(B).

Fact(A): By explicit algorithm(Algorithm 11.2.1), we can compute the Puiseux expansion $C(g_r : t)$ for the curve $C(t)$ such that $V(g_r) \equiv C(g_r : t)$ (Multiseq).

(Algorithm 11.2.1 for Theorem 11.2)

$$(11.2.1) \quad C(g_r : t) := \begin{cases} y = t^n \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases}$$

such that $n = n_1 n_2 \dots n_r$ and $\alpha_1 = \beta_{1,1} n_2 \dots n_r$,

$$\alpha_j = \alpha_{j-1} + \widehat{\Delta}_j n_{j+1} n_{j+2} \dots n_r,$$

where $\widehat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1}$ is a positive integer for $2 \leq j \leq r$ and $\Delta_1(t) = t$.

Remark for Fact(A). Note that the parametrization in (11.2.1) satisfies the following:

$$(11.2.2) \quad \begin{aligned} \text{(i)} \quad & n < \alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_r. \\ \text{(ii)} \quad & n \geq d_1 > d_2 > \dots > d_r = 1 \text{ with } d_i = \gcd(n, \alpha_1, \dots, \alpha_i), \\ & \text{where } d_i = n_{i+1} n_{i+2} \dots n_r \text{ for } i = 1, 2, \dots, r-1. \end{aligned}$$

Fact(B): By explicit algorithm(Algorithm 11.2.2), we can compute the standard Puiseux expansion $C^*(t)$ for the curve $C(g_r : t)$ in Fact(A) such that $V(g_r) \equiv C^*(t)$ (Multiseq).

(Algorithm 11.2.2 for Theorem 11.2)

(a) If $n > \gcd(n, \alpha_1)$, $C^*(t) = C(g_r : t)$ is the standard Puiseux expansion of the r -th type where the curve $C(g_r : t)$ was already in Fact(A).

(b) If $n = \gcd(n, \alpha_1)$, $C^*(t)$ can be parametrized by the standard Puiseux expansion $C_{r-1}(t)$ of the $(r-1)$ -th type:

$$(11.2.3) \quad C_{r-1}(t) := \begin{cases} y = t^n \\ z = t^{\alpha_2} + t^{\alpha_3} + \dots + t^{\alpha_r}. \end{cases}$$

Note that $C(g_r : t) \equiv C_{r-1}(t)$ (Multiseq) where the curve $C(g_r : t)$ is in (11.2.1).

Fact(C): If for any g_r in Family(1) define a function $\Psi : \text{Family}(1) \rightarrow \text{Family}(2)$ by $\Psi(g_r) = C^*(t)$, then Ψ is a one-to-one function from Family(1) into Family(2). \square

Remark 11.2.1. (a) $n_1 = 1$ if and only if $n = \gcd(n, \alpha_1) = d_1$.

(b) For each $j = 1, 2, \dots, r$, note that $(0, 0)$ is an isolated singularity of an analytic variety $V(g_j)$ except for $V(g_1)$ with $n_1 = 1$.

(c) Whenever $V(g_r)$ is chosen arbitrary as in the assumption of Theorem 11.2, then it was already proved by (10.1.1) of Theorem 10.1 that the multiplicity sequence of $V(g_r)$, that is, $\text{Multiseq}(V(g_r))$, can be represented as follows:

$$(11.2.1-1) \quad \text{Multiseq}(V(g_r)) = \text{Join}(\{[n : \alpha_1]\}, \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}, \dots, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-2}) : \alpha_{r-1} - \alpha_{r-2}]\}, \{[\gcd(n, \alpha_1, \dots, \alpha_{r-1}) : \alpha_r - \alpha_{r-1}]\}). \quad \square$$

Example 11.2.2 for the first half of The 1st algorithm in Theorem 11.2: Example 11.2.2 and Example 1.4.1 of §1 are the same. \square

Rigorously, the statement for Fact(C) of Theorem 11.2 can be rewritten by Corollary 11.3.

Corollary 11.3(The uniqueness of the solution for **Fact(B)** of Theorem 11.2).

Assumptions First, suppose that the same assumption as in Sequences[I] of Theorem 7.3 or Theorem 11.2 is satisfied with an additional condition that $2 \leq n_1 < \beta_{1,1}$. Next, as we have seen in Sequences[I] of the assumption of Theorem 7.3, we define another quasi-Puiseux convergent power series ϕ_ρ of recursive ρ -type in $\mathbb{C}\{y, z\}$ by Sequences[II] in the assumptions of Theorem 7.3, with an additional condition that $2 \leq \ell_1 < \delta_{1,1}$.

Conclusions Note that $V(g_r) = \{(y, z) : g_r(y, z) = 0\}$ and $V(\phi_\rho) = \{(y, z) : \phi_\rho(y, z) = 0\}$ be analytic varieties at $(y, z) = (0, 0)$ with isolated singularity at the origin. Then, we have the following:

$$(11.3.2) \quad n_j = \ell_j \quad \text{and} \quad \Delta_j(\beta_{j,k})_{k=1}^j = \omega_j(\delta_{j,k})_{k=1}^j \quad \text{for each } j = 1, 2, \dots, r = \rho$$

\iff

$$(11.3.3) \quad V(g_r) \equiv V(\phi_\rho) \quad (\text{Multiseq})$$

\implies

$$(11.3.4) \quad C(g_r : t) \equiv C(\phi_\rho : t) \quad (\text{Multiseq})$$

where $V(g_r) \equiv C(g_r : t)$ and $V(\phi_\rho) \equiv C(\phi_\rho : t)$ (Multiseq)

\implies

$$(11.3.5) \quad C(g_r : t) \text{ and } C(\phi_\rho : t) \text{ have the same standard Puiseux expansion}$$

Moreover, if g_r and ϕ_ρ is irreducible in $\mathbb{C}\{y, z\}$, it can be easily proved by Theorem 7.3 and Theorem 10.2 that $V(g_r) \equiv V(\phi_\rho)$ (Multiseq) if and only if $g_r \stackrel{\text{divisor}}{\sim} \phi_\rho$ under the standard resolutions. \square

§11.3. The proof of Theorem 11.2

Proof of Theorem 11.2. First of all, note by Sublemma 5.2 of Theorem 5.0 that the multiplicity of g_r at $(y, z) = (0, 0)$ is $n = \prod_{k=1}^r n_k$ because $1 \leq n_1 < \beta_{1,1}$.

For the proof of theorem, it suffices to prove that **Fact(A)** is true because of the following:

(i) The proof of **Fact(A)** with Theorem 8.10 and Definition 11.1 implies clearly that **Fact(B)** is true.

(ii) It is trivial by Theorem 7.3, Theorem 7.7, and Theorem 10.2 that **Fact(C)** is true.

Fact(A) For the proof, **Fact(A)** is divided into two steps:

Fact(A-1) Then, the curve $C(g_r : t)$ of (11.2.1) is the Puiseux expansion at $(y, z) = (0, 0)$.

Fact(A-2) Then, $V(g_r) \equiv C(g_r : t)$ (Multiseq). For notation, $D = D_{g_r}$ is defined by the curve which is parametrized by $D(t) = D_{g_r}(t) = C(g_r : t)$ if necessary.

Fact(A-1) There is nothing to prove for **Fact(A-1)**.

Fact(A-2) Then, the proof will be by induction on the multiplicity $n = \prod_{k=1}^r n_k$ of g_r at $(y, z) = (0, 0)$ that $V(g_r) \equiv C(g_r : t)$ (Multiseq) where $1 \leq n_1 < \beta_{1,1}$ and $n_i \geq 2$ for $2 \leq i \leq r$. For notation, we write $D(t) = D_{g_r}(t) = C(g_r : t)$ if necessary.

For the induction proof of **Fact(A-2)**, it suffices to consider the following two cases:

Case[I] $n = 2$ and Case[II] $n \geq 3$.

Note by (11.2.1) that $\gcd(n, \alpha_1) = n_2 n_3 \cdots n_r \leq n = n_1 n_2 \cdots n_r$ with $\gcd(n_1, \beta_{1,1}) = 1$. So, for each of Case[I] and Case[II], it suffices to consider two subcases, respectively:

$$(11.2.4) \quad \text{(a) } \gcd(n, \alpha_1) < n \quad \text{and} \quad \text{(b) } \gcd(n, \alpha_1) = n.$$

Case[I] of Fact(A-2) Let $n = 2$. By (11.2.4), there are two subcases:

Case[Ia] $1 = \gcd(n, \alpha_1) < n = 2$ and Case[Ib] $\gcd(n, \alpha_1) = n = 2$.

Case[Ia] of Fact(A-2) Let $n = 2$ with $d_1 = \gcd(n, \alpha_1) = 1$. Then, it is trivial that the local defining equation g_1 for an analytic variety $V(g_1)$ at the origin is analytically defined by $g_1(y, z) = z^2 + y^{\beta_{1,1}}$ where $2 < \beta_{1,1} = \alpha_1$ and $\gcd(2, \alpha_1) = 1$.

On the other hand, since $\gcd(2, \alpha_1) = 1$ then the irreducible parametrization of $C(g_1 : t)$ for the above g_1 can be written by $y = t^2$ and $z = t^{\alpha_1}$ with $\alpha_1 = \beta_{1,1}$.

Then, it is clear that $V(g_1) \equiv C(g_1 : t)$ (Multiseq). Thus, the proof of Case[Ia] is done.

Case[Ib] of Fact(A-2) Let $n = 2$ with $d_1 = \gcd(n, \alpha_1) = n = 2$. That, is, $d_1 = \gcd(2, \alpha_1) = 2$.

Then, it is trivial that $g_2(y, z)$ of Sequences[I] with five conditions is written in the form

$$(11.2.5) \quad g_2 = (z^{n_1} + y^{\beta_{1,1}})^{n_2} + \varepsilon y^{\beta_{2,1}} z^{\beta_{2,2}} = (z + y^{\beta_{1,1}})^2 + \varepsilon y^{\beta_{2,1}} z^{\beta_{2,2}} \quad \text{with} \\ \Delta_2(\beta_{2,1}, \beta_{2,2}) = n_1 \beta_{2,1} + \beta_{1,1} \beta_{2,2} = \beta_{2,1} + \beta_{1,1} \beta_{2,2} > n_2 n_1 \beta_{1,1} = 2\beta_{1,1}, \quad \text{and} \\ \gcd(n_2, \Delta_2(\beta_{2,1}, \beta_{2,2})) = \gcd(2, \Delta_2(\beta_{2,1}, \beta_{2,2})) = 1,$$

where $\Delta_2(t_1, t_2)$ is defined by $n_1 t_1 + \beta_{1,1} t_2 = t_1 + \beta_{1,1} t_2$ and ε is a unit in $\mathbb{C}\{y, z\}$.

By Theorem 10.3 or by a nonsingular change of coordinates at the origin, it can be easily proved that $V(g_2) \equiv V(h)$ (Multiseq) where $h(y, z) = z^2 + y^{\Delta_2(\beta_{2,1}, \beta_{2,2})}$.

On the other hand, since $\alpha_1 = 2\beta_{1,1}$ by (11.2.5) and $\alpha_2 - \alpha_1 = \hat{\Delta}_2(\beta_{2,1}, \beta_{2,2}) = \Delta_2(\beta_{2,1}, \beta_{2,2}) - n_2 n_1 \beta_{1,1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) - 2\beta_{1,1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) - \alpha_1$ by (11.2.1), then $C(g_2 : t)$ of (11.2.1) can be represented by $y = t^2$ and $z = t^{2\beta_{1,1}} + t^{2\beta_{1,1} + \hat{\Delta}_2(\beta_{2,1}, \beta_{2,2})}$. Noting that $2\beta_{1,1} + \hat{\Delta}_2(\beta_{2,1}, \beta_{2,2}) = \Delta_2(\beta_{2,1}, \beta_{2,2})$, let C^* be defined by $y = t^2$ and $z = t^{\Delta_2(\beta_{2,1}, \beta_{2,2})}$.

Then, it is clear that $C^* \equiv C(g_2 : t)$ (Multiseq), $C^* \equiv V(h)$ (Multiseq) and $V(g_2) \equiv V(h)$ (Multiseq). So, $V(g_2) \equiv C(g_2 : t)$ (Multiseq). Thus, we proved that Case[Ib] is true, and therefore the proof of Case[I] is done.

Case[II] of Fact(A-2) Let $n \geq 3$. For the induction proof, suppose we have shown by Case[I] that Fact(A-2) is true when g_r has a multiplicity at the origin which is either less than n or equal to two. Now, assuming that g_r has a multiplicity n at the origin, then it is enough to consider two such subcases, respectively:

Case[IIa] $\gcd(n, \alpha_1) < n$ ($n_1 > 1$). Case[IIb] $\gcd(n, \alpha_1) = n$ ($n_1 = 1$).

Case[IIa] of Fact(A-2) Let $\gcd(n, \alpha_1) < n$. To avoid the complexity of notations for the proof, we may assume that $V(g_r)$ and $D(t) = C(g_r : t)$ of Fact(A-1) satisfy the corresponding properties, respectively in the assumption of this theorem up to the same notations.

Since $\gcd(n, \alpha_1) < n$ with $2 \leq n < \alpha_1$, let q be the positive integer such that

$$(11.2.6) \quad qn_1 < \beta_{1,1} < (q+1)n_1, \quad \text{that is,} \quad qn < \alpha_1 < (q+1)n,$$

where $q \geq 1$, $d_1 = \gcd(n, \alpha_1) = n_2 n_3 \cdots n_r$, $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$.

In preparation for the proof of Case[IIa], it suffices to prove the following two sublemmas, Sublemma 11.2.1 for Case[IIa] and Sublemma 11.2.2 for Case[IIa] respectively, because of the following:

(i) Firstly, we show by Sublemma 11.2.1 for Case[IIa] that we find the representation of $V^{(q)}(g_r)$ and $D^{(q)}(t)$ in Sublemma 11.2.1 for Case[IIa], using the same method as we have done in Lemma 4.2 where the q -th proper transforms of $V(g_r)$ and $D(t) = C(g_r : t)$ are denoted by $V^{(q)}(g_r)$ and $D^{(q)}(t)$, respectively. In addition, we compute by Sublemma 11.2.1 for Case[IIa] that if we write $\text{Multiseq}(V(g_r)) = \{a_1, a_2, \dots, a_m\}$ and $\text{Multiseq}(D(t)) = \{b_1, b_2, \dots, b_s\}$ then $a_i = b_i = n$ for $i = 1, 2, \dots, q$.

(ii) Secondly, using Sublemma 11.2.1 for Case[IIa], it is enough to show by Sublemma 11.2.2 for Case[IIa] that $V^{(q)}(g_r) \equiv D^{(q)}(t)$ (Multiseq).

(iii) After then, Sublemma 11.2.1 and Sublemma 11.2.2 for Case[IIa] imply by (i) and Corollary 3.8 that $\{[\text{Mult}(V(g_r))]\} = \text{Join}(\{[n : qn]\}, \{[\text{Mult}(V^{(q)}(g_r))]\})$ and $\{[\text{Mult}(D(t))]\} = \text{Join}(\{[n : qn]\}, \{[\text{Mult}(D^{(q)}(t))]\})$ are equal.

Sublemma 11.2.1 for Case[IIa] of Fact(A-2) Claim the following:

(a) Whenever we use the composition of q iterations of blow-ups, denoted by $\tau_q = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_q : M^{(q)} \rightarrow \mathbb{C}^2$ where q is a positive integer defined by (11.2.6), then it can be easily proved by Lemma 4.2 that just one of the local coordinates for each blow-up π_i with $1 \leq i \leq q$ is needed for the study of the i -th proper transforms of both $V(g_r)$ and $D(t)$

simultaneously under $\tau_i = \pi_1 \circ \pi_2 \circ \dots \circ \pi_i$ because $V(g_r)$ and $D(t)$ are irreducible in $\mathbb{C}\{y, z\}$. Note that the i -th proper transforms of $V(g_r)$ and $D(t) = C(g_r : t)$ are denoted by $V^{(i)}(g_r)$ and $D^{(i)}(t)$ respectively. Then by Lemma 4.2, τ_i can be represented, as a composition of analytic mappings, as follows:

$$(11.2.7) \quad \tau_i(v_i, u_i) = (y, z) = (v_i, v_i^i u_i),$$

where for brevity of notation, (v_i, u_i) is called one coordinate patch of the local coordinates for the i -th blow-up $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ at some point of $M^{(i-1)}$ with $M^0 = \mathbb{C}^2$. If we write $\text{Multiseq}(V(g_r)) = \{a_1, a_2, \dots, a_m\}$ and $\text{Multiseq}(D(t)) = \{b_1, b_2, \dots, b_s\}$, then $a_i = b_i = n$ for $i = 1, 2, \dots, q$.

(b) For the proof, first construct the local defining equation for the proper transform $D^{(q)}(t)$ of $D(t)$ under τ_q , and next construct the local defining equation for the proper transform $V^{(q)}(g_r)$ of $V(g_r)$ under τ_q , denoted by $(g_r \circ \tau_q)_{proper}$, as follows:

(b1) The local defining equation for the curve $D^{(q)}$ is as follows:

$$(11.2.8) \quad D^{(q)}(t) := \begin{cases} v = t^n, \\ u = t^{\alpha_1 - qn} + t^{\alpha_2 - qn} + \dots + t^{\alpha_r - qn}. \end{cases}$$

Since $0 < \alpha_1 - qn < n$, the parametrization in (11.2.8) is not the Puiseux expansion.

Let $\Omega = \Omega(s)$ be another irreducible curve with isolated singularity at $0 \in \mathbb{C}$, which is defined by the Puiseux expansion with a parameter s , as follows:

$$(11.2.9) \quad \Omega(s) := \begin{cases} v = s^n + s^{n+\alpha_2-\alpha_1} + \dots + s^{n+\alpha_r-\alpha_1}, \\ u = s^{\alpha_1 - qn}, \end{cases}$$

because $0 < \alpha_1 - qn < n$ and $0 < \alpha_1 < \alpha_2 < \dots < \alpha_r$.

Then, it is clear by Theorem 8.8 and Theorem 8.10 that $D^{(q)}(t) \equiv \Omega(t)$ (Multiseq).

(b2) For each $t = 1, 2, \dots, q$, along $E_t = \{v_t = 0\}$ the local defining equation $(g_r \circ \tau_t)_{total}$ for the t -th total transform of $V(g_r)$ is as follows:

$$(11.2.10) \quad \begin{aligned} (g_j \circ \tau_t)_{total} &= v_t^{tn_1 n_2 \dots n_j} (g_j \circ \tau_t)_{proper} \quad \text{for } 1 \leq j \leq r, \\ (g_1 \circ \tau_t)_{proper} &= u_t^{n_1} + \varepsilon_1 v_t^{\beta_{1,1} - tn_1}, \dots, \\ (g_j \circ \tau_t)_{proper} &= (g_{j-1} \circ \tau_t)_{proper}^{n_j} + \varepsilon_j u_t^{\beta_{j,2}} v_t^{\omega_{t,j}} \prod_{k=3}^j (g_{k-2} \circ \tau_t)_{proper}^{\beta_{j,k}} \\ &\text{with } \omega_{t,j} = \beta_{j,1} + t\beta_{j,2} + tn_1\beta_{j,3} + \dots + tn_1 n_2 \dots n_{j-2} \beta_{j,j} - tn_1 n_2 \dots n_j > 0, \end{aligned}$$

such that $\omega_{t,j} > 0$ for all $t = 1, 2, \dots, q$ and all $j = 2, 3, \dots, r$ because $\beta_{1,1} - n_1 q > 0$. Note that each $\varepsilon_i = \varepsilon_i(u_t, v_t)$ is a unit in $\mathbb{C}\{u_t, v_t\}$ for $1 \leq i \leq r$.

(b3) Moreover, the multiplicity of $(g_r \circ \tau_q)_{proper}$ at $(v_q, u_q) = (0, 0)$ is $\alpha_1 - qn$, which is less than the multiplicity n of g_r at $(y, z) = (0, 0)$. \square

Proof of Sublemma 11.2.1 for Case[IIa] As some applications of Theorem 3.6, Sublemma 5.1, Sublemma 5.2, Sublemma 5.3, Sublemma 5.4, Theorem 8.8 and Theorem 8.10, the proof of this sublemma can be easily done. \square

Sublemma 11.2.2 for Case[IIa] of Fact(A-2) Claim the following:

For the proof of $V^{(q)}(g_r) \equiv \Omega(s)$ (Multiseq), since the multiplicity of $(g_r \circ \tau_q)_{proper}$ at $(v_q, u_q) = (0, 0)$ is $\alpha_1 - qn < n$ by Sublemma 11.2.1 for Case[IIa], it suffices to show by the induction method that the following two facts are true:

Fact(i) of Sublemma 11.2.2 $V^{(q)}(g_r)$ satisfies the same kind of properties as $V(g_r)$ does in the assumption of Fact(A-2).

Fact(ii) of Sublemma 11.2.2 $\Omega(s)$ of (11.2.9) satisfies the same kind of properties relative to $V^{(q)}(g_r)$ as $D(t)$ of (11.2.1) does relative to $V(g_r)$ in the assumption of Fact(A-2).

In order to prove that Fact(i) and Fact(ii) are true, we will represent Fact(i) and Fact(ii) in more detail.

For notation, rewrite $(g_1 \circ \tau_q)_{proper}, (g_2 \circ \tau_q)_{proper}, \dots, (g_r \circ \tau_q)_{proper}$ of (11.2.10) by h_1, h_2, \dots, h_r , respectively in $\mathbb{C}\{v, u\}$ as follows: Note that $(v, u) = (v_q, u_q)$.

$$(11.2.11) \quad h_1 = \varepsilon_1 v^{m_1} + u^{\gamma_{1,1}} \quad \text{where} \quad 1 \leq m_1 = \beta_{1,1} - n_1 q < \gamma_{1,1} = n_1, \\ h_j = h_{j-1}^{m_j} + \varepsilon_j u^{\gamma_{j,1}} v^{\gamma_{j,2}} h_1^{\gamma_{j,3}} \cdots h_{j-2}^{\gamma_{j,j}} \quad \text{for } j = 2, 3, \dots, r, \\ \text{where } 2 \leq m_j = n_j, \gamma_{j,1} = \beta_{j,2}, \\ \gamma_{j,2} = \omega_{q,j} = \beta_{j,1} + q\beta_{j,2} + qn_1\beta_{j,3} + \cdots + qn_1n_2 \cdots n_{j-2}\beta_{j,j} - qn_1n_2 \cdots n_j > 0, \\ \gamma_{j,3} = \beta_{j,3}, \dots, \gamma_{j,j} = \beta_{j,j}.$$

Note that $0 < m_1 = \beta_{1,1} - n_1 q < n_1$ and each $\varepsilon_i = \varepsilon_i(u, v)$ is a unit in $\mathbb{C}\{u, v\}$ for $1 \leq i \leq r$.

Then, apply the notations in (11.2.11) to Fact(i) and Fact(ii), as follows:

Fact(i) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) Applying the same kind of notations as we have used for $\{X_k : k = 1, 2, \dots, r\}$ in the assumption of this theorem to the proof of Fact(i) of this sublemma, we claim that Fact(i) can be rewritten as follows:

Sequences[I]⁽¹⁾ of the r-th type: Let $\{Y_k : k = 1, 2, \dots, r\}$ with $Y_k \subset N_0$,
 $\{h_k : k = 1, 2, \dots, r\}$ with $h_k = (g_k \circ \tau_q)_{proper}$ in $\mathbb{C}\{u, v\}$,
 $\{\Xi_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$
be three sequences, satisfying the following five conditions for each k.

Five conditions are denoted by **The 1st Cond⁽¹⁾, ..., The 5-th Cond⁽¹⁾** of **Sequences[I]⁽¹⁾**.

[I]-(1) The 1st Cond⁽¹⁾ of Sequences[I]⁽¹⁾:

(1a) $Y_1 = \{m_1, \gamma_{1,1}\}$ with $1 \leq m_1 = \beta_{1,1} - n_1 q < \gamma_{1,1} = n_1$.

(1b) $Y_j = \{m_j, \gamma_{j,1}, \gamma_{j,2}, \dots, \gamma_{j,j}\}$ with $m_j \geq 2$, where $j = 2, \dots, r$.

[I]-(2) The 2nd Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

(2a) $h_1 = v^{m_1} + \varepsilon_1 u^{\gamma_{1,1}}$.

(2b) $h_j = h_{j-1}^{m_j} + \varepsilon_j u^{\gamma_{j,1}} v^{\gamma_{j,2}} h_1^{\gamma_{j,3}} \cdots h_{j-2}^{\gamma_{j,j}}$ for $j = 2, \dots, r$.

Note that each $\varepsilon_j = \varepsilon_j(u, v)$ is a unit in $\mathbb{C}\{u, v\}$ for $1 \leq j \leq r$.

[I]-(3) The 3rd Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

(3a) $\Xi_1(t) = t$ for each $t \in N_0$.

(3b) $\Xi_j(t_k)_{k=1}^j = t_j \Xi_{j-1}(\gamma_{j-1,k})_{k=1}^j + m_{j-1} \Xi_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$.

[I]-(4) The 4-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

(4a) $\Xi_1(\gamma_{1,1}) = \gamma_{1,1} > m_1 \geq 1$.

(4b) $\Xi_j(\gamma_{j,k})_{k=1}^j > m_j m_{j-1} \Xi_{j-1}(\gamma_{j-1,k})_{k=1}^{j-1}$ for $j = 2, \dots, r$.

[I]-(5) The 5-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ :

(5) $\gcd(m_j, \Xi_j(\gamma_{j,k})_{k=1}^j) = 1$ for $j = 1, \dots, r$.

Fact(ii) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) For the given $V(h_r) = \{(u, v) : h_r(u, v) = 0\}$ in Fact(i), the Puiseux expansion $\Omega(s)$ at $s = 0$ for the curve Ω is given as follows:

$$(11.2.12) \quad \Omega(s) := \begin{cases} v = s^n + s^{n+\alpha_2-\alpha_1} + \cdots + s^{n+\alpha_r-\alpha_1}, \\ u = s^{\alpha_1-qn}, \end{cases}$$

where $0 < \alpha_1 - qn < n$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r$.

As we have seen in (11.2.1), $\Omega(s)$ of (11.2.12) satisfies the following property:

$$(11.2.13) \quad m_1 m_2 \cdots m_r = \alpha_1 - qn, \quad \gamma_{1,1} m_2 \cdots m_r = n, \quad \text{and} \quad \text{for } j = 1, \dots, r,$$

$$(n + \alpha_j - \alpha_1) - (n + \alpha_{j-1} - \alpha_1) = \widehat{\Xi}_j(\gamma_{j,k})_{k=1}^j m_{j+1} m_{j+2} \cdots m_r.$$

Note that $\alpha_j - \alpha_{j-1} = (n + \alpha_j - \alpha_1) - (n + \alpha_{j-1} - \alpha_1)$ for $j = 1, \dots, r$. \square

Proof of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) For the proof of Sublemma 11.2.2 for Case[IIa] of Fact(A-2), It suffices to prove Fact(i) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) and Fact(ii) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2), respectively.

Proof of Fact(i) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) For the proof of Fact(i), it suffices to show that the properties in The 4-th Cond⁽¹⁾ and The 5-th Cond⁽¹⁾ of Sequences[I]⁽¹⁾ are true.

For the proofs of the properties in The 4-th Cond⁽¹⁾ and The 5-th Cond⁽¹⁾, it is trivial by (11.2.11) that $\Xi_1(\gamma_{1,1}) = \gamma_{1,1} > m_1 \geq 1$ and $1 = \gcd(n_1, \beta_{1,1}) = \gcd(n_1, \beta_{1,1} - qn_1) = \gcd(m_1, \gamma_{1,1})$, and so it suffices to show that the following assertion is true.

Assertion Let w and ℓ be arbitrary positive integers such that $2 \leq w \leq \ell \leq r$. Then, we have the following inequalities:

$$\begin{aligned}
(11.2.14) \quad (i) \quad & \Xi_w(\gamma_{\ell,k})_{k=1}^w = \Delta_w(\beta_{\ell,k})_{k=1}^w + qn_1^2 \cdots n_{w-1}^2 (\beta_{\ell,w+1} + n_w \beta_{\ell,w+2} + \cdots \\
& + n_w n_{w+1} \cdots n_{\ell-2} \beta_{\ell,\ell} - n_w n_{w+1} \cdots n_{\ell}). \\
(ii) \quad & \Xi_w(\gamma_{w,k})_{k=1}^w = \Delta_w(\beta_{w,k})_{k=1}^w - qn_1^2 \cdots n_{w-1}^2 n_w. \\
(iii) \quad & \widehat{\Xi}_w(\gamma_{w,k})_{k=1}^w = \Xi_w(\gamma_{w,k})_{k=1}^w - m_w m_{w-1} \Xi_{w-1}(\gamma_{w-1,k})_{k=1}^{w-1} \\
& = \Delta_w(\beta_{w,k})_{k=1}^w - n_w n_{w-1} \Delta_{w-1}(\beta_{w-1,k})_{k=1}^{w-1} = \widehat{\Delta}_w(\beta_{w,k})_{k=1}^w > 0. \\
(iv) \quad & \gcd(\Xi_w(\gamma_{w,k})_{k=1}^w, m_w) = 1.
\end{aligned}$$

So, it is clear that the inequalities in (iii) and (iv) of (11.2.14) imply the properties in The 4-th Cond⁽¹⁾ and The 5-th Cond⁽¹⁾.

Proof of the Assertion: The proof will be by induction on the integer $w \geq 1$. So, for the proof of this assertion, it suffices to consider two subcases, respectively:

Subcase(1) $w = 1$ and Subcase(2) $w \geq 2$.

Subcase(1) for the Assertion If $w = 1$, then, there is nothing to prove.

Subcase(2) for the Assertion Let $w \geq 2$. Next, by using the induction assumption on the integer $w \geq 1$, suppose we have shown that the above assertion is true on the integer w with $r - 1 \geq \ell \geq w \geq 1$. Assuming that $r \geq \ell \geq w + 1 > 2$, then it suffices to show that the proof of four equalities in (11.2.14) of the assertion just follows from the proofs of (a), (b), (c) and (d):

(a) Compute $\Xi_{w+1}(\gamma_{\ell,k})_{k=1}^{w+1}$ by (11.2.11) and The 3-th Cond⁽¹⁾, as follows:

$$\begin{aligned}
(11.2.15) \quad & \Xi_{w+1}(\gamma_{\ell,k})_{k=1}^{w+1} = \gamma_{\ell,w+1} \Xi_w(\gamma_{w,k})_{k=1}^w + m_w \Xi_w(\gamma_{\ell,k})_{k=1}^w \\
& = \beta_{\ell,w+1} (\Delta_w(\beta_{w,k})_{k=1}^w - qn_1^2 \cdots n_{w-1}^2 n_w) + n_w \{ \Delta_w(\beta_{\ell,k})_{k=1}^w + qn_1^2 \cdots n_{w-1}^2 \\
& \quad (\beta_{\ell,w+1} + n_w \beta_{\ell,w+2} + n_w n_{w+1} \beta_{\ell,w+3} + \cdots + n_w n_{w+1} \cdots n_{\ell-2} \beta_{\ell,\ell} - n_w n_{w+1} \cdots n_{\ell}) \} \\
& = \Delta_{w+1}(\beta_{\ell,k})_{k=1}^{w+1} + qn_1^2 \cdots n_{w-1}^2 n_w^2 (\beta_{\ell,w+2} + n_{w+1} \beta_{\ell,w+3} + \cdots \\
& \quad + n_{w+1} n_{w+2} \cdots n_{\ell-2} \beta_{\ell,\ell} - n_{w+1} n_{w+2} \cdots n_{\ell}),
\end{aligned}$$

which gives the proof of (i) of (11.2.14).

(b) From (11.2.15) in (a), if $\ell = w + 1$, then

$$(11.2.16) \quad \Xi_{w+1}(\gamma_{w+1,k})_{k=1}^{w+1} = \Delta_{w+1}(\beta_{w+1,k})_{k=1}^{w+1} - qn_1^2 \cdots n_{w-1}^2 n_w^2 n_{w+1},$$

and so the proof of (ii) of (11.2.14) is done.

(c) By (b) and (11.2.11), we have

$$\begin{aligned}
(11.2.17) \quad & \widehat{\Xi}_{w+1}(\gamma_{w+1,k})_{k=1}^{w+1} = \Xi_{w+1}(\gamma_{w+1,k})_{k=1}^{w+1} - m_{w+1} m_w \Xi_w(\gamma_{w,k})_{k=1}^w \\
& = \Delta_{w+1}(\beta_{w+1,k})_{k=1}^{w+1} - qn_1^2 \cdots n_w^2 n_{w+1} - n_{w+1} n_w \{ \Delta_w(\beta_{w,k})_{k=1}^w - qn_1^2 \cdots n_{w-1}^2 n_w \} \\
& = \Delta_{w+1}(\beta_{w+1,k})_{k=1}^{w+1} - n_{w+1} n_w \Delta_w(\beta_{w,k})_{k=1}^w = \widehat{\Delta}_{w+1}(\beta_{w+1,k})_{k=1}^{w+1} > 0.
\end{aligned}$$

Thus, the proof of (iii) of (11.2.14) is done.

(d) By (c) and (11.2.11) again, we have

$$(11.2.18) \quad \begin{aligned} \gcd(\Xi_{w+1}(\gamma_{w+1,k})_{k=1}^{w+1}, m_{w+1}) &= \gcd(\widehat{\Xi}_{w+1}(\gamma_{w+1,k})_{k=1}^{w+1}, m_{w+1}) \\ &= \gcd(\widehat{\Delta}_{w+1}(\beta_{w+1,k})_{k=1}^{w+1}, n_{w+1}) = \gcd(\Delta_{w+1}(\beta_{w+1,k})_{k=1}^{w+1}, n_{w+1}) = 1. \end{aligned}$$

Thus, the proof of (iv) of (11.2.14) is done, and so we proved that Subcase(2) is true. Therefore, the proof of the assertion is done, and so we finished the proof of Fact(i) of Sublemma 11.2.2.

Proof of Fact(ii) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) It suffices to show that $\Omega(s)$ of (11.2.12) satisfies the equalities of (11.2.13). By use of (11.2.11) and (11.2.14), and (11.2.1) of the assumption, the proof is as follows:

$$(11.2.19) \quad \begin{aligned} (i) \quad m_1 m_2 \cdots m_r &= (\beta_{1,1} - qn_1)n_2 n_3 \cdots n_r \\ &= \beta_{1,1} n_2 n_3 \cdots n_r - qn_1 n_2 \cdots n_r = \alpha_1 - qn, \\ (ii) \quad \gamma_{1,1} m_2 \cdots m_r &= n_1 n_2 n_3 \cdots n_r = n, \\ (iii) \quad \alpha_{j+1} - \alpha_j &= \widehat{\Delta}_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} n_{j+2} n_{j+3} \cdots n_r \quad \text{for } 1 \leq j \leq r-1 \\ &= \widehat{\Xi}_{j+1}(\gamma_{j+1,k})_{k=1}^{j+1} m_{j+2} m_{j+3} \cdots m_r. \end{aligned}$$

Thus, the proof of Fact(ii) of Sublemma 11.2.2 for Case[IIa] of Fact(A-2) is done, and so we finished the proof of Sublemma 11.2.2 for Case[IIa] of Fact(A-2). \square

Therefore, we can show by the proofs of Sublemma 11.2.1 and Sublemma 11.2.2 that Case[IIa] of Fact(A-2) is true. \square

Case[IIb] of Fact(A-2) Let $\gcd(n, \alpha_1) = n$. Then, $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$ with $d_1 = \gcd(n, \alpha_1) = n > 1$, and so $d_1 = n_2 n_3 \cdots n_r$ by (11.2.2), noting that $n_1 = 1 < \beta_{1,1}$.

Since $1 = n_1 < \beta_{1,1}$ in Case[IIb], then the representation of g_r in the assumption of Case[IIb] and the representation of g_r in the assumptions of Sequences[I] of Theorem 10.3 are the same. Since g_r is irreducible in $\mathbb{C}\{y, z\}$, then by the assumption of Theorem 10.3 we can use the same results and notations as in the conclusions of Theorem 10.3:

$$(11.2.20) \quad \omega_j(\sigma_{j,k})_{k=1}^j = \Delta_{j+1}(\beta_{j+1,k})_{k=1}^{j+1} \quad \text{for each } j = 1, 2, \dots, r-1,$$

$$(11.2.21) \quad \begin{aligned} \widehat{\omega}_q(\sigma_{q,k})_{k=1}^q &= \omega_q(\sigma_{q,k})_{k=1}^q - c_q c_{q-1} \omega_{q-1}(\sigma_{q-1,k})_{k=1}^{q-1} \\ &= \Delta_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,k})_{k=1}^q \quad \text{for each } q = 2, 3, \dots, r-1, \\ &= \widehat{\Delta}_{q+1}(\beta_{q+1,k})_{k=1}^{q+1} > 0, \end{aligned}$$

$$(11.2.22) \quad g_r \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ if and only if } h_{r-1} \text{ is irreducible in } \mathbb{C}\{y, z\},$$

$$(11.2.23) \quad g_r \text{ and } h_{r-1} \text{ have the same multiplicity sequence.}$$

Noting by Theorem 10.3 that $c_j = n_{j+1}$ for $1 \leq j \leq r-1$, we showed that $\gcd(n, \xi_1) < n$, $\gcd(n, \xi_1) = 1$ and $2 \leq c_1 < \sigma_{1,1}$ or $2 \leq n < \xi_1$ in Sequences[II] where $n = c_1 c_2 \cdots c_{r-1}$, $\xi_1 = \sigma_{1,1} c_2 \cdots c_{r-1}$ and n is the multiplicity of h_{r-1} at $(y, z) = (0, 0)$, and so the results of Case[IIa] can be applied to Case[IIb].

Applying Case[IIa] to this case, then for a given $\{(y, z) : h_{r-1}(y, z) = 0\}$ in [II]-(2) of Sequences[II] of Theorem 10.3, we can compute the Puiseux expansion $C(h_{r-1} : t)$ for the curve such that $V(h_{r-1}) \equiv C(h_{r-1} : t)$ (Multiseq) where $V(g_r) = \{(y, z) : g_r(y, z) = 0\}$ in the assumption, as follows:

$$(11.2.24) \quad C(h_{r-1} : t) := \begin{cases} y = t^c, \\ z = t^{\xi_1} + t^{\xi_2} + \cdots + t^{\xi_{r-1}}, \end{cases}$$

$$\text{such that} \quad c = c_1 c_2 \cdots c_{r-1} \quad \text{and} \quad \xi_1 = \sigma_{1,1} c_2 c_3 \cdots c_{r-1},$$

$$\xi_j = \xi_{j-1} + \widehat{\omega}_j(\sigma_{j,k})_{k=1}^j c_{j+1} c_{j+2} \cdots c_{r-1} \quad \text{for } 2 \leq j \leq r-1.$$

Now, in order to finish the proof of the subcase Case[IIb] of Case[II], since n is not a divisor of α_2 and $c = n$ is not a divisor of ξ_1 , then it remains to show by Case[IIa] that the following equalities are true:

$$(11.2.25) \quad n = c \quad \text{and} \quad \xi_j = \alpha_{j+1} \quad \text{for } j = 1, 2, \dots, r-1.$$

For the proof, first compute $\{\xi_k : k = 1, 2, \dots, r-1\}$ as follows:

(i) By (11.2.20) and (11.2.24), and by (11.2.1), we have

$$(11.2.26) \quad \begin{aligned} \xi_1 &= \sigma_{1,1} c_2 c_3 \cdots c_{r-1} = \Delta_2(\beta_{2,1}, \beta_{2,2}) n_3 n_4 \cdots n_r, \\ \alpha_2 &= \alpha_1 + \widehat{\Delta}_2(\beta_{2,1}, \beta_{2,2}) n_3 n_4 \cdots n_r \\ &= \beta_{1,1} n_2 n_3 \cdots n_r + (\Delta_2(\beta_{2,1}, \beta_{2,2}) - n_2 n_1 \beta_{1,1}) n_3 n_4 \cdots n_r \\ &= \Delta_2(\beta_{2,1}, \beta_{2,2}) n_3 n_4 \cdots n_r, \end{aligned}$$

because $n_1 = 1$. So, $\xi_1 = \alpha_2$.

(ii) By (11.2.20), (11.2.21) and (11.2.24), we have for each $j = 2, 3, \dots, r$,

$$(11.2.27) \quad \begin{aligned} \xi_{j-1} - \xi_{j-2} &= \widehat{\omega}_{j-1}(\sigma_{j-1,k})_{k=1}^{j-1} c_j c_{j+1} \cdots c_{r-1} \\ &= \widehat{\Delta}_j(\beta_{j,k})_{k=1}^j n_{j+1} n_{j+2} \cdots n_r = \alpha_j - \alpha_{j-1}. \end{aligned}$$

So, the proof of (11.2.25) follows very easily from (11.2.26) and (11.2.27). Thus, the proof of Case[IIb] is done, which implies the completion for the proof of Case[II].

Therefore, we showed by Case[I] and Case[II] that Fact(A) is true.

The proofs of Fact(B) and Fact(C) It is trivial by Theorem 7.3, Theorem 7.7, and Theorem 10.2 that Fact(B) and Fact(C) are true.

Thus, we finished the proof of the theorem. \square

The proof of Corollary 11.3 just follows from Theorem 11.2 with Theorem 10.2 and Theorem 8.10.

§11.4. The second half of The 1st Algorithm(Theorem 11.4)

Now, in order to solve Problem[1-C] mentioned in §7.0 of §7, it suffices to show how to use the Euclidean algorithm in (3) of Corollary 7.6 only.

Theorem 11.4(Theorem 1.6:Algorithm for finding the unique element of Family(1) corresponding to any given standard Puiseux expansion of Family(2)). Theorem 11.4 has the same statement as Theorem 1.6 does in §1. \square

Proof of Theorem 11.4. The proof of Theorem 11.4 follows from Theorem 7.7 of §7. \square

Example 11.4.1 for the second half of The 1st algorithm in Theorem 11.4: Example 11.4.1 and Example 1.6.3 of §1 are the same.

§11.5. An application of The 1st Algorithm

Theorem 11.5. As an application of The 1st Algorithm, *instead of using the 2nd, the 3rd and the 4-th algorithms, we can show directly that there exists a one-to-one correspondence between four families, Family(1), Family(2), Family(3) and Family(4) as we have seen in Definition 1.2 and Definition 2.4, which was already proved.* \square

Part[C](Chapter VIII, . . . , Chapter XI)

A complete and explicit irreducibility algorithms for the W-polys of two complex variables with proofs and related topics in the Puiseux expansions

Chapter VIII: The new terminology and notations in preparation for finding irreducibility criterion of W-polys in $\mathbb{C}\{y, z\}$

§12. The generalized standard irreducible W-polys of the recursive r-type with Theorem 12.0

In order to succeed in the computation of Explicit algorithm, it is very interesting and important to prove that we can define the new terminology, irreducible Weierstrass polynomials of two complex variables of the recursive type, called “the generalized standard Puiseux polynomial in $\mathbb{C}[y, z]$ of the recursive type” throughout this paper, which will be well-defined. In preparation, we need Definition 12.0.0 and Theorem 12.0.

Definition 12.0.0. Let N_0 be the set of nonnegative integers and N_0^k be its k -dimensional copy, and N be the set of positive integers. Let r be an arbitrary positive integer.

[A] $g_r \in \mathbb{C}\{y, z\}$ is called either a generalized semi-quasi-Puiseux germ of the recursive r-type or a generalized semi-quasi Puiseux convergent power series of the recursive r-type if there are sequences $\{X_k : k = 1, 2, \dots, r\}$ with $X_k \subset N_0$, $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ and $\{\Delta_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r\}$ satisfying the following four conditions:

Four conditions are denoted by The 1-th Cond⁽⁰⁾, . . . , The 4-th Cond⁽⁰⁾ for brevity.

The 1st Cond⁽⁰⁾ The family $\{X_\ell : \ell = 1, 2, \dots, r\}$ with $X_\ell \subset N_0$ is as follows:

- (1)(1a) $X_1 = \{n_1\} \cup \{\beta_{1,i,1} : 0 \leq i < n_1\}$ with $n_1 \geq 2$ and $\beta_{1,0,1} \geq 1$ where $X_1 \subset N$.
 (1b) $X_j = \{n_j\} \cup \{\beta_{j,i,1} : 0 \leq i < n_j\} \cup \{\beta_{j,i,2} : 0 \leq i < n_j\} \cup \dots \cup \{\beta_{j,i,j} : 0 \leq i < n_j\}$ with $n_j \geq 2$ where $j = 2, 3, \dots, r$.

For each $j = 2, 3, \dots, r$, assume that at least one of $\beta_{j,0,1}, \beta_{j,0,2}, \dots, \beta_{j,0,j}$ is nonzero.

The 2nd Cond⁽⁰⁾ For each $j = 1, 2, \dots, r$, let $g_j = g_j(y, z, c_j)$ be in $\mathbb{C}\{y, z\}$ where all the c_j are complex numbers, each of which is defined by the following way:

- (2)(2a) $g_1 = z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} + c_1 \sum_{i=1}^{n_1-1} \varepsilon_{1,i} y^{\beta_{1,i,1}} z^i$ with $\varepsilon_{1,0} = 1$.
 (2b) $g_j = g_{j-1}^{n_j} + \varepsilon_{j,0} y^{\beta_{j,0,1}} z^{\beta_{j,0,2}} g_1^{\beta_{j,0,3}} g_2^{\beta_{j,0,4}} \dots g_{j-2}^{\beta_{j,0,j}}$
 $+ c_j \sum_{i=1}^{n_j-1} \varepsilon_{j,i} y^{\beta_{j,i,1}} z^{\beta_{j,i,2}} g_1^{\beta_{j,i,3}} \dots g_{j-2}^{\beta_{j,i,j}} g_{j-1}^i$, where $j = 2, 3, \dots, r$.

Note that each $\varepsilon_{j,i} = \varepsilon_{j,i}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq j \leq r$ and $0 \leq i < n_j$, if exists. As far as analytic equivalence of isolated plane curve singularities defined by all g_j , $1 \leq j \leq r$, is concerned, then we may assume that $\varepsilon_{1,0}$ is equal to one by a suitable nonsingular change of coordinates at the origin in \mathbb{C}^2 .

The 3rd Cond⁽⁰⁾ Let $\{\Delta_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r\}$ be a sequence such that each Δ_k is an integer-valued function defined by the following way:

- (3) (3a) $\Delta_1(t) = t$ for each $t \in N_0$.
 (3b) $\Delta_j(t_k)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$.

The 4-th Cond⁽⁰⁾ Then, the following inequalities hold: Note that $r \geq 2$.

- (4)(4a) $\Delta_1(\beta_{1,i,1}) = \beta_{1,i,1} > 0$ for $0 \leq i < n_1$.
 (4b) $\Delta_j(\beta_{j,i,k})_{k=1}^j > (n_j - i) n_{j-1} \Delta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1}$ for $0 \leq i < n_j$ where $j = 2, \dots, r$.

[B] Let $g_r \in \mathbb{C}\{y, z\}$ be a generalized semi-quasi-Puiseux germ of the recursive r-type as in [A]. There are two additional conditions, denoted by The 5-th Cond⁽⁰⁾ and The 6-th Cond⁽⁰⁾.

The 5-th Cond⁽⁰⁾ For each $q = 1, 2, \dots, r$, the following inequalities hold:

$$(5)(5a) \quad \gcd(n_q, \Delta_q(\beta_{q,0,k})_{k=1}^q) = 1 \quad \text{for } 1 \leq q \leq r.$$

$$(5b) \quad \frac{\Delta_q(\beta_{q,i,k})_{k=1}^q}{n_q - i} > \frac{\Delta_q(\beta_{q,0,k})_{k=1}^q}{n_q} \quad \text{for } 0 < i < n_q.$$

The 6-th Cond⁽⁰⁾ The following inequalities hold: Note that $2 \leq j \leq r$.

$$(6)(6a) \quad 2 \leq n_1 < \beta_{1,0,1}.$$

$$(6b) \quad n_j \geq 2, \beta_{j,i,1} > 0, \text{ and } 0 \leq \beta_{j,i,k} < n_{k-1} \text{ for } 2 \leq j \leq r, 0 \leq i < n_j \text{ and } 2 \leq k \leq j.$$

Now, we define the new terminology in [B1], [B2] and [B3] of [B].

[B1] It is said that $g_r \in \mathbb{C}\{y, z\}$ is a generalized quasi-Puiseux germ of the recursive r-type if g_r in [A] satisfies an additional condition, denoted by The 5-th Cond⁽⁰⁾, equivalently, if $g_r \in \mathbb{C}\{y, z\}$ satisfies the above five conditions, i.e., The 1-th Cond⁽⁰⁾, ..., The 4-th Cond⁽⁰⁾ and The 5-th Cond⁽⁰⁾.

[B2] $g_r \in \mathbb{C}\{y, z\}$ is called a generalized Puiseux germ of the recursive r-type in $\mathbb{C}\{y, z\}$ if g_r in [A] satisfies The 5-th Cond⁽⁰⁾ and an inequality in (6a) of The 6-th Cond⁽⁰⁾, without mentioning any other equalities in (6b).

[B3] $g_r \in \mathbb{C}\{y, z\}$ is called a generalized standard Puiseux germ of the recursive r-type in $\mathbb{C}\{y, z\}$ if g_r in [A] satisfies The 5-th Cond⁽⁰⁾ and The 6-th Cond⁽⁰⁾.

[C] Let $g_r \in \mathbb{C}\{y, z\}$ be a generalized standard Puiseux germ of the recursive r-type in [B3]. If each unit $\varepsilon_{j,i} = \varepsilon_{j,i}(y, z)$ is equal to $\varepsilon_{j,i}(y, 0)$ for $1 \leq j \leq r$ and $0 \leq i < n_j$ if exists in The 2-th Cond⁽⁰⁾ of [A], g_r is called a generalized standard Puiseux Weierstrass polynomial of the recursive r-type. In particular, if each unit $\varepsilon_{j,i} = \varepsilon_{j,i}(y, z)$ is equal to an integer one for $1 \leq j \leq r$ and $0 \leq i < n_j$ if exists, as in The 2-th Cond⁽⁰⁾ of [A], then g_r is called the generalized standard Puiseux polynomial of the recursive r-type.

Theorem 12.0(To find the necessary and sufficient condition for a generalized semi-quasi-Puiseux convergent power series in $\mathbb{C}\{y, z\}$ of the recursive type to be irreducible in $\mathbb{C}\{y, z\}$).

Assumptions Let g_r be a generalized semi-quasi-Puiseux germ of the recursive r-type, satisfying the same properties and notations as in [A] of Definition 12.0.0.

In addition, assume that we have the following: $\gcd(n_1, \beta_{1,0,1}) = 1$.

Conclusions For each $j = 1, 2, \dots, r$, let $(0, 0)$ be the singularity of an analytic variety $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ except for $V(g_1)$ with $\beta_{1,0,1} = 1$.

For brevity of representation, we may assume that the above g_r satisfies all the equalities in (5a) of The 5-th Cond⁽⁰⁾ of [B] of Definition 12.0.0, without mentioning all the inequalities in (5b) of The 5-th Cond⁽⁰⁾.

Then, we get two statements [A] and [B] as follows:

[A] g_r is irreducible in $\mathbb{C}\{y, z\}$

$$\iff g_1, \dots, g_{r-1} \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \frac{\Delta_r(\beta_{r,i,k})_{k=1}^r}{n_r - i} > \frac{\Delta_r(\beta_{r,0,k})_{k=1}^r}{n_r} \text{ for } 0 < i < n_j$$

$$\iff \frac{\Delta_j(\beta_{j,i,k})_{k=1}^j}{n_j - i} > \frac{\Delta_j(\beta_{j,0,k})_{k=1}^j}{n_j} \text{ for each } j = 1, 2, \dots, r \text{ and for } 0 < i < n_j.$$

[B] Let g_r be irreducible in $\mathbb{C}\{y, z\}$.

[B1] Let $V(y^\gamma g_r) = \{(y, z) : y^\gamma g_r(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by

$$(12.0.0) \quad y^\gamma g_r(y, z) \quad \text{such that} \quad \begin{cases} \gamma = 1, & \text{if } \beta_{1,0,1} = 1, \\ \gamma = 0, & \text{if } \beta_{1,0,1} > 1. \end{cases}$$

Then, $y^\gamma g_r \in$ the type $[r]$ under the standard resolution, denoted by τ , in the sense of Definition 2.5. Also, if $\beta_{1,0,1} = 1$ then $g_r \in$ the type $[r-1]$ under the standard resolution.

[B2] In particular, $z^\delta y g_r \in$ the type $[r]$ under the same standard resolution τ , whether δ is either one or zero.

[C] Let g_r be irreducible in $\mathbb{C}\{y, z\}$.

[C1] For each $j = 1, 2, \dots, r$, let $V(\psi_j) = \{(y, z) : \psi_j(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 defined by

$$(**) \quad \psi_j(y, z) = g_j(y, z, 0).$$

Then, the singularity of both $V(\psi_j)$ and $V(g_j(y, z, c_j))$ have the same divisor under the standard resolutions.

[C2] In particular, for each $j = 1, 2, \dots, r$, let $V(H_j) = \{(y, z) : H_j(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is defined as follows:

- (i) $H_1 = z^{n_1} + y^{\beta_{1,0,1}}.$
- (ii) $H_j = H_{j-1}^{n_j} + y^{\beta_{j,0,1}} z^{\beta_{j,0,2}} H_1^{\beta_{j,0,3}} \dots H_{j-2}^{\beta_{j,0,j}}$ for $j = 2, 3, \dots, r$.

Then, the singularity of both $V(g_j(y, z, c_j))$ and $V(H_j)$ have the same divisor under the standard resolutions. \square

§12.1. In preparation for the proof of Theorem 12.0

In this section, for the proof of Theorem 12.0 we will prepare the statements of five sublemmas without proofs, consisting of Sublemma 12.1, Sublemma 12.2, ..., Sublemma 12.5. In §13, we will finish the proof of Theorem 12.0 with Corollary 12.6, using the proofs of these five sublemmas.

Sublemma 12.1. Assumptions Suppose that the same properties and notations as in **Assumptions** of Theorem 12.0 hold.

For any integer $r \geq 2$, let $\Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2})$ and $\Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j$ with $3 \leq j \leq r$ be the notations defined as follows : Note that $\Delta_2(t_1, t_2) = n_1 t_1 + \beta_{1,0,1} t_2$ for each $(t_1, t_2) \in N_0^2$.

$$(12.1.1) \quad \Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) = \Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) \quad \text{for } 0 \leq i < n_2.$$

$$\Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j = \Delta_2(\beta_{j,i,1}, \beta_{j,i,2}) + n_1 \beta_{1,0,1} \beta_{j,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{j,i,4} \\ + n_1 \beta_{1,0,1} n_2 n_3 \beta_{j,i,5} + \dots + n_1 \beta_{1,0,1} n_2 \dots n_{j-2} \beta_{j,i,j} \quad \text{for } 0 \leq i < n_j.$$

Conclusions Then, we have the following:

$$(12.1.2) \quad \Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) > n_1 \beta_{1,0,1} (n_2 - i) \quad \text{on } g_2.$$

$$\Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j > n_1 \beta_{1,0,1} n_2 n_3 \dots n_{j-1} (n_j - i) \quad \text{on } g_j. \quad \square$$

Sublemma 12.2. Assumptions Suppose that the same properties and notations as in **Assumptions** of Theorem 12.0 hold. Let r be arbitrary integer with $r \geq 1$.

In addition, we need the following assumption: Note that $\gcd(n_1, \beta_{1,0,1}) = 1$.

$$(12.2.0) \quad g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}, \quad \text{equivalently,} \quad \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \quad \text{for } 0 < i < n_1.$$

But, note that g_r may not be irreducible in $\mathbb{C}\{y, z\}$ for some $r \geq 2$.

Conclusions Then, we get the following:

(a) $g_r = g_r(y, z)$ can be written in the form

$$(12.2.1) \quad g_r = (z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}})^{d_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta \quad \text{with } \varepsilon_{1,0} = 1 \text{ and} \\ \text{with } n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} d_r,$$

where if $r \geq 2$ then $d_r = \prod_{k=2}^r n_k$ and $d_1 = 1$, and a unit $\varepsilon_{1,0} = \varepsilon_{1,0}(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}d_r$.

(b) For each $r \geq 1$, we have the following:

(b1) The multiplicity of $g_r(0, z)$ at $z = 0$ is $n_1 \prod_{k=2}^r n_k$ when $g_r = g_r(y, z)$.

(b2) The multiplicity of $g_r(y, 0)$ at $y = 0$ is $\beta_{1,0,1} \prod_{k=2}^r n_k$ when $g_r = g_r(y, z)$.

(c) For each $r \geq 1$, we have the following:

(c1) If $n_1 < \beta_{1,0,1}$ then $\alpha + \beta > n_1 \prod_{k=2}^r n_k$, and so the multiplicity of g_r at $(y, z) = (0, 0)$ is $n_1 \prod_{k=2}^r n_k$.

(c2) If $n_1 > \beta_{1,0,1}$ then $\alpha + \beta > \beta_{1,0,1} \prod_{k=2}^r n_k$, and so the multiplicity of g_r at $(y, z) = (0, 0)$ is $\beta_{1,0,1} \prod_{k=2}^r n_k$. \square

Sublemma 12.3. Assumptions Suppose that the same properties and notations as in **Assumptions** of Theorem 12.0 hold. Let $r \geq 2$ be an arbitrary positive integer.

In addition, assume that we have the following: Note that $\gcd(n_1, \beta_{1,0,1}) = 1$ by the assumption of Theorem 12.0, but that the condition in (12.3.0) does not belong to the assumption of Theorem 12.0.

$$(12.3.0) \quad g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}, \quad \text{equivalently,} \quad \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \quad \text{for } 0 < i < n_1.$$

Since $\gcd(n_1, \beta_{1,0,1}) = 1$ with $n_1 \geq 2$ and $\beta_{1,0,1} \geq 1$, then there are two nonnegative integers $a > 0$ and $b \geq 0$ such that $a\beta_{1,0,1} - bn_1 = 1$.

For given two integers $a > 0$ and $b \geq 0$, let $\Omega_2 : N_0^2 \rightarrow N_0$ be a function defined by

$$(12.3.1) \quad \Omega_2(t_1, t_2) = at_1 + bt_2.$$

Let $\Omega_2^\#(\beta_{2,i,1}, \beta_{2,i,2})$ and $\Omega_j^\#(\beta_{j,i,k})_{k=1}^j$ with $3 \leq j \leq r$ be the notations defined as follows:

$$(12.3.2) \quad \begin{aligned} \Omega_2^\#(\beta_{2,i,1}, \beta_{2,i,2}) &= \Omega_2(\beta_{2,i,1}, \beta_{2,i,2}) \quad \text{for } 0 \leq i < n_2. \\ \Omega_j^\#(\beta_{j,i,k})_{k=1}^j &= \Omega_2(\beta_{j,i,1}, \beta_{j,i,2}) + bn_1\beta_{j,i,3} + bn_1n_2\beta_{j,i,4} \\ &\quad + \cdots + bn_1n_2 \cdots n_{j-2}\beta_{j,i,j} \quad \text{for } 0 \leq i < n_j. \end{aligned}$$

Conclusions Then, we get the following:

$$(12.3.3) \quad \begin{aligned} \Omega_2^\#(\beta_{2i1}, \beta_{2i2}) &\geq bn_1n_2 \quad \text{for } 0 \leq i < n_2. \\ \Omega_j^\#(\beta_{jik})_{k=1}^j &\geq bn_1n_2n_3 \cdots n_{j-1}(n_j - i) \quad \text{for } 0 \leq i < n_j. \square \end{aligned}$$

Sublemma 12.4. Assumptions Suppose that the same properties and notations as in **Assumptions** of Theorem 12.0 hold. As in Sublemma 12.3, additionally assume that we have the following:

$$(*1) \quad g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}, \quad \text{equivalently,} \quad \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \quad \text{for } 0 < i < n_1.$$

But, note that g_r may not be irreducible in $\mathbb{C}\{y, z\}$ for some $r \geq 2$.

Conclusions For each $j = 1, 2, \dots, r$, let $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ be an analytic variety at the origin in \mathbb{C}^2 . For the construction of the statement of the conclusion, let $V(G) = \{(y, z) : G(y, z) = 0\}$ be another analytic variety with an isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(12.4.0) \quad \begin{aligned} g_0 &= z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}} \quad \text{with a unit } \varepsilon_{1,0} \in \mathbb{C}\{y, z\}, \\ G &= y^\gamma g_0, \end{aligned}$$

satisfying the properties (i) and (ii):

- (i) If $\beta_{1,0,1} = 1$, then $\gamma = 1$.
- (ii) If $\beta_{1,0,1} \geq 2$, then $\gamma = 0$.

Let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(G)$. If $V(g_1)$ in the assumption of Theorem 12.0 has the singular point at the origin, then as compared with the above τ_m , exactly the same τ_m can be also used for the standard resolution of the singular point of $V(yg_1)$ as far as the blow-ups process is concerned.

(a)(a1) We can use just one coordinate patch of the local coordinates for each blow-up π_i of τ_m with $1 \leq i \leq m$ in the sense of Lemma 2.12.

(a2) Just as above, we can use the same τ_m for the composition of the first finite number m of successive blow-ups in process of the standard resolution of the singular point $(0,0)$ of $V(g_j)$ for all $j = 2, 3, \dots, r$.

(a3) Also, we can use just the common one coordinate patch of the given local coordinates for each blow-up π_i of the above τ_m in (a1), in order to study any of $V^{(i)}(g_j)$ for all $j = 2, 3, \dots, r$ and all $i = 1, 2, \dots, m$ in the sense of Lemma 2.14.

(b) For brevity of notation, by (a3) let (v, u) be the common one of the local coordinates for the m -th blow-up $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ at $(0,0)$ which is the quasisingular point of $V^{(m-1)}(G)$. Being viewed as an analytic mapping, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ can be written in the form

$$(12.4.1) \quad \tau_m(v, u) = (y, z) = (v^{n_1} u^a, v^{\beta_{1,0,1}} u^b),$$

where (i) $a > 0$ and $b \geq 0$ are some nonnegative integers such that $a\beta_{1,0,1} - bn_1 = 1$,
(ii) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.

(c) As in Sublemma 12.1 and Sublemma 12.3, use the same notations for a sequence $\{\Delta_i^\# : N_0^i \rightarrow N_0, \text{ functions for } i=2, \dots, r\}$ and $\{\Omega_i^\# : N_0^i \rightarrow N_0, \text{ functions for } i=2, \dots, r\}$ where $\Omega_2 : N_0^2 \rightarrow N_0$ is a function defined by $\Omega_2(t_1, t_2) = at_1 + bt_2$ for given two nonnegative integers a and b in (b_1) of (12.4.1), and we may start with assuming that $\varepsilon_{1,0} = 1$ in $V(y^\gamma g_0) = \{y^\gamma(z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}}) = 0\}$, in order to study $V^{(i)}(g_j)$ for all $i = 1, 2, \dots, m$, and all $j = 1, 2, \dots, r$. Whether $\beta_{1,0,1} \geq 2$ or $\beta_{1,0,1} = 1$, we may write that $((y^\gamma g_0) \circ \tau_m)_{\text{proper}} = u^{a,\gamma}(1 + \varepsilon_{1,0}u)$ with $\varepsilon_{1,0} = 1$, without complexity of the notation if necessary, noting that if $\beta_{1,0,1} = 1$, $V(g_0)$ and $V(g_1)$ have no singularity at the origin.

Now, along $v = 0$, $(g_j \circ \tau_m)_{\text{total}}$ with $(g_j \circ \tau_m)_{\text{proper}}$ can be written as follows:

$$(12.4.2) \quad \begin{aligned} ((y^\gamma g_0) \circ \tau_m)_{\text{total}} &= v^{(\gamma+\beta_{1,0,1})n_1} u^{bn_1+a\gamma} ((y^\gamma g_0) \circ \tau_m)_{\text{proper}} \quad \text{with} \\ ((y^\gamma g_0) \circ \tau_m)_{\text{proper}} &= (1 + \varepsilon_{1,0}u) \quad \text{with } \varepsilon_{1,0} = 1, \\ (g_1 \circ \tau_m)_{\text{total}} &= v^{n_1\beta_{1,0,1}} u^{bn_1} (g_1 \circ \tau_m)_{\text{proper}} \quad \text{with} \\ (g_1 \circ \tau_m)_{\text{proper}} &= (1 + \varepsilon_{1,0}u) + c_1 \sum_{i=1}^{n_1-1} \varepsilon'_{1,i} v^{\Delta_2^\#(\beta_{1,i,1,i})-n_1\beta_{1,0,1}} u^{\Omega_2^\#(\beta_{1,i,1,i})-bn_1} \\ &= 1 + \varepsilon_{1,0}\bar{u} = 1 + \bar{u} \quad \text{for brevity of notation,} \\ (g_j \circ \tau_m)_{\text{total}} &= v^{n_1\beta_{1,0,1}d_j} u^{bn_1d_j} (g_j \circ \tau_m)_{\text{proper}} \quad \text{with} \\ (g_j \circ \tau_m)_{\text{proper}} &= (g_{j-1} \circ \tau_m)_{\text{proper}}^{n_j} + \{\varepsilon'_{j,0} v^{\Delta_j^\#(\beta_{j,0,k})_{k=1}^j - n_1\beta_{1,0,1}d_j} u^{\Omega_j^\#(\beta_{j,0,k})_{k=1}^j - bn_1d_j} \\ &\quad \times (g_1 \circ \tau_m)_{\text{proper}}^{\beta_{j,0,3}} (g_2 \circ \tau_m)_{\text{proper}}^{\beta_{j,0,4}} \cdots (g_{j-2} \circ \tau_m)_{\text{proper}}^{\beta_{j,0,j}}\} \\ &\quad + c_j \cdot \left\{ \sum_{i=1}^{n_j-1} \varepsilon'_{j,i} v^{\Delta_j^\#(\beta_{j,i,k})_{k=1}^j - n_1\beta_{1,0,1}d_{j-1}(n_j-i)} u^{\Omega_j^\#(\beta_{j,i,k})_{k=1}^j - bn_1d_{j-1}(n_j-i)} \right. \\ &\quad \left. \times (g_1 \circ \tau_m)_{\text{proper}}^{\beta_{j,i,3}} (g_2 \circ \tau_m)_{\text{proper}}^{\beta_{j,i,4}} \cdots (g_{j-2} \circ \tau_m)_{\text{proper}}^{\beta_{j,i,j}} (g_{j-1} \circ \tau_m)_{\text{proper}}^i \right\}, \end{aligned}$$

where if $j \geq 2$ then $d_j = \prod_{k=2}^j n_k$ and $d_1 = 1$, and each $\varepsilon'_{j,i} = \varepsilon_{j,i} \circ \tau_m(v, u)$ is a unit in $\mathbb{C}\{v, 1+u\}$ for $2 \leq j \leq r$ and $0 \leq i < n_j$.

Note by Sublemma 12.1 and Sublemma 12.3 that for $j = 2, 3, \dots, r$ and $i = 0, 1, \dots, n_j-1$,

$$(12.4.3) \quad \begin{aligned} \Delta_j^\#(\beta_{j,i,k})_{k=1}^j &> n_1\beta_{1,0,1}n_2 \cdots n_{j-1}(n_j-i) = n_1\beta_{1,0,1}d_{j-1}(n_j-i) \quad \text{and} \\ \Omega_j^\#(\beta_{j,i,k})_{k=1}^j &\geq bn_1n_2 \cdots n_{j-1}(n_j-i) = bn_1d_{j-1}(n_j-i). \end{aligned}$$

Moreover, $(y^{\beta_{j,i,1}} z^{\beta_{j,i,2}} g_1^{\beta_{j,i,3}} g_2^{\beta_{j,i,4}} \cdots g_{j-2}^{\beta_{j,i,j}}) \circ \tau_m(v, u)$ can be viewed as

$$(12.4.4) \quad u^{\Delta_j^\#(\beta_{j,i,k})_{k=1}^j} v^{\Delta_j^\#(\beta_{j,i,k})_{k=1}^j} (g_1 \circ \tau_m)_{proper}^{\beta_{j,i,3}} (g_2 \circ \tau_m)_{proper}^{\beta_{j,i,4}} \cdots (g_{j-2} \circ \tau_m)_{proper}^{\beta_{j,i,j}},$$

where (c_1) for $j = 1, 2, \dots, r$ and for $1 \leq i < n_j$, $(\beta_{j,i,k})_{k=1}^j \in N_0^j$ as in [A] of Definition 12.0.0,

(c_2) $\Delta_j^\#(\beta_{j,i,k})_{k=1}^j = \Delta_2(\beta_{j,i,1}, \beta_{j,i,2}) + n_1 \beta_{1,0,1} \beta_{j,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{j,i,4} + \cdots + n_1 \beta_{1,0,1} n_2 \cdots n_{j-2} \beta_{j,i,j}$ by the definition of $\Delta_j^\#(\beta_{j,i,k})_{k=1}^j$ in Sublemma 12.1,

(c_3) $\Omega_j^\#(\beta_{j,i,k})_{k=1}^j = \Omega_2(\beta_{j,i,1}, \beta_{j,i,2}) + b n_1 \beta_{j,i,3} + b n_1 n_2 \beta_{j,i,4} + \cdots + b n_1 n_2 \cdots n_{j-2} \beta_{j,i,j}$ by the definition of $\Omega_j^\#(\beta_{j,i,k})_{k=1}^j$ in Sublemma 12.3.

(d) Let $\tau_m^{-1}(0, 0) = \cup_{i=1}^m E_i$ where E_i is an exceptional curve of the first kind. For $j = 1, 2, \dots, r$, let

$$(12.4.5) \quad (g_j \circ \tau_m)_{divisor} = V^{(m)}(g_j) + \sum_{i=1}^m e_{j,i} E_i,$$

where $V^{(m)}(g_j)$ is the proper transform of $V(g_j)$ under τ_m .

Then we have the following: Note again that τ_m is the composition of a finite number m of successive blow-ups, which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ or $V(g_1)$.

$(d1)$ If $\beta_{1,0,1} \geq 2$, then $e_{j+1,i} = n_{j+1} e_{j,i}$ for any $j \geq 1$ and for $i = 1, 2, \dots, m$. If $\beta_{1,0,1} = 1$, then $e_{j+1,i} = n_{j+1} e_{j,i}$ for any $j \geq 2$ and for $i = 1, \dots, m$.

In particular, $e_{j,m} = n_1 \beta_{1,0,1} n_2 \cdots n_j$ for $j = 2, \dots, r$, and if $j = 1$ with $\beta_{1,0,1} > 1$, then $e_{j,m} = n_1 \beta_{1,0,1}$.

$(d2)$ $V^{(m)}(g_j) \cap (\cup_{i=1}^m E_i) = V^{(m)}(g_j) \cap E_m = \{(v, 1 + \varepsilon_{1,0} u) = (0, 0)\}$ for any $j = 2, \dots, r$ where $1 + \varepsilon_{1,0} u = (g_1 \circ \tau_m)_{proper}$.

$(d3)$ If $\beta_{1,0,1} \geq 2$, then for any $j = 1, 2, \dots, r$, $g_j \in$ the type [1] under τ_m . If $\beta_{1,0,1} = 1$, then for any $j = 1, 2, \dots, r$, $g_j \in$ the type [0] under τ_m .

If $\beta_{1,0,1} \geq 1$, note that for all $j = 1, 2, \dots, r$, $z^\delta y g_j \in$ the type [1] under τ_m whether $\delta = 1$ or $\delta = 0$, by Theorem 3.6. \square

Sublemma 12.5. Assumptions Suppose that the same properties and notations as in **Assumptions** of Theorem 12.0 hold. As in either Sublemma 12.3 or Sublemma 12.4, additionally assume that we have the following:

$$(12.5.0) \quad g_1 \text{ is irreducible in } \mathbb{C}\{y, z\}, \quad \text{equivalently,} \quad \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \quad \text{for } 0 < i < n_1.$$

Let r be an arbitrary positive integer with $r \geq 2$. Throughout this sublemma, we will use the same notations and consequences as in Sublemma 12.4, in order to get the representation for the conclusion of this sublemma.

Conclusions As $\{g_k : k = 1, 2, \dots, r\}$ with $g_k \in \mathbb{C}\{y, z\}$ satisfies four conditions in the assumptions of Theorem 12.0, denoted by **The 1-th Cond⁽⁰⁾**, \dots , **The 4-th Cond⁽⁰⁾** in Definition 12.0.0, then $(g_r \circ \tau_m)_{proper} \in \mathbb{C}\{v, 1 + \bar{u}\}$ is a generalized semi-quasi-Puiseux germ of the recursive $(r - 1)$ type in the sense of Definition 12.0.0 where $\{(g_k \circ \tau_m)_{proper} : k = 2, 3, \dots, r\}$ with $(g_k \circ \tau_m)_{proper} \in \mathbb{C}\{v, 1 + \bar{u}\}$ satisfies the same kind of four conditions, which will be denoted by **The 1-th Cond⁽¹⁾**, \dots , **The 4-th Cond⁽¹⁾**. Note that $\{(g_k \circ \tau_m)_{proper} : k = 2, 3, \dots, r\}$ has been already well-defined by Sublemma 12.4.

In more detail, in order to construct four conditions recursively, which will be denoted by **The 1-th Cond⁽¹⁾**, \dots , **The 4-th Cond⁽¹⁾**, we prefer to add one more condition to the above four conditions, denoted by **The 5-th Cond⁽¹⁾**, for convenience of representation.

By using the same kind of properties and notations as in Definition 12.0.0, the desired construction is as follows:

Let $\{Y_k : k = 1, 2, \dots, r-1\}$ with $Y_k \subset N_0$,
 $\{h_k : k = 1, 2, \dots, r-1\}$ with $h_k = (g_{k+1} \circ \tau_m)_{proper}$ in $\mathbb{C}\{v, 1 + \bar{u}\}$,
 $\{\Xi_k : N_0^k \rightarrow N_0$ is an integer-valued function for $k = 1, 2, \dots, r-1\}$
be three sequences, satisfying the following five conditions for each k :

Such conditions are denoted by **The 1-th Cond⁽¹⁾**, ..., **The 5-th Cond⁽¹⁾**.

The 1-th Cond⁽¹⁾: Let $\{Y_j : j = 1, 2, \dots, r-1\}$ with $Y_j \subset N_0$ be defined by

$$(12.5.1) \quad Y_1 = \{s_1\} \cup \{\gamma_{1,i,1} : 0 \leq i < s_1\} \text{ with } s_1 \geq 2 \text{ and } \gamma_{1,0,1} \geq 1, \\ Y_j = \{s_j\} \cup \{\gamma_{j,i,1} : 0 \leq i < s_j\} \cup \{\gamma_{j,i,2} : 0 \leq i < s_j\} \cup \{\gamma_{j,i,j} : 0 \leq i < s_j\} \\ \text{with } s_j \geq 2,$$

such that for each $j = 1, 2, \dots, r-1$, $e_{1,m} = n_1 \Delta_1(\beta_{1,0,1}) = n_1 \beta_{1,0,1}$, and

$$(12.5.1.1) \quad s_1 = n_2 \geq 2, \gamma_{1,i,1} = \Delta_2^\sharp(\beta_{2,i,k})_{k=1}^2 - n_1 \beta_{1,0,1}(n_2 - i) > 0 \text{ for } 0 \leq i < s_1, \\ s_j = n_{j+1} \geq 2, \gamma_{j,i,1} = \Delta_{j+1}^\sharp(\beta_{j+1,i,k})_{k=1}^{j+1} - n_1 \beta_{1,0,1} n_2 n_3 \cdots n_j (n_{j+1} - i) > 0 \text{ and} \\ \gamma_{j,i,2} = \beta_{j+1,i,3}, \gamma_{j,i,3} = \beta_{j+1,i,4}, \gamma_{j,i,4} = \beta_{j+1,i,5}, \dots, \gamma_{j,i,j} = \beta_{j+1,i,j+1} \text{ for } 0 \leq i < s_j,$$

noting that $\gamma_{1,i,1}, \gamma_{2,i,1}, \dots, \gamma_{r-1,i,1}$ are positive by Sublemma 12.1.

The 2-th Cond⁽¹⁾: Let $(g_2 \circ \tau_m)_{proper}, (g_3 \circ \tau_m)_{proper}, \dots, (g_r \circ \tau_m)_{proper}$ be denoted by h_1, h_2, \dots, h_{r-1} , respectively in $\mathbb{C}\{v, u+1\}$ as follows: For brevity of notation, we can define a local biholomorphic mapping ϕ from $(u, v) = (-1, 0)$ to $(\bar{u}, v) = (-1, 0)$ such that $1 + \bar{u} = (g_1 \circ \tau_m)_{proper} = (1 + \varepsilon_{1,0}u) + c_1 \sum_{i=1}^{n_1-1} \varepsilon'_{1,i} v^{\Delta_2^\sharp(\beta_{1,i,1,i}) - n_1 \beta_{1,0,1}} u^{\Omega_2^\sharp(\beta_{1,i,1,i}) - b n_1}$ with $\varepsilon_{1,0} = 1$ and $\phi(u, v) = (\bar{u}, v)$.

$$(12.5.2) \quad (g_2 \circ \tau_m)_{total} = v^{n_2 e_{1,m}} (g_2 \circ \tau_m)_{proper} = v^{n_2 e_{1,m}} h_1 \quad \text{with}$$

$$h_1 = (1 + \bar{u})^{s_1} + \eta_{1,0} v^{\gamma_{1,0,1}} + c_2 \sum_{i=1}^{s_1-1} \eta_{1,i} v^{\gamma_{1,i,1}} (1 + \bar{u})^i \text{ with } \eta_{1,0} = 1,$$

$$(g_j \circ \tau_m)_{total} = v^{n_j n_{j-1} \cdots n_2 e_{1,m}} (g_j \circ \tau_m)_{proper} = v^{n_j n_{j-1} \cdots n_2 e_{1,m}} h_{j-1} \quad \text{with}$$

$$h_{j-1} = h_{j-2}^{s_{j-1}-1} + \eta_{j-1,0} v^{\gamma_{j-1,0,1}} (1 + \bar{u})^{\gamma_{j-1,0,2}} h_1^{\gamma_{j-1,0,3}} \cdots h_{j-3}^{\gamma_{j-1,0,j-1}} \\ + c_j \sum_{i=1}^{s_{j-1}-1} \eta_{j-1,i} v^{\gamma_{j-1,i,1}} (1 + \bar{u})^{\gamma_{j-1,i,2}} \alpha h_1^{\gamma_{j-1,i,3}} \cdots h_{j-3}^{\gamma_{j-1,i,j-1}} h_{j-2}^i,$$

where $\eta_{j,i} = \eta_{j,i}(v, 1 + \bar{u})$ is a unit in $\mathbb{C}\{v, 1 + \bar{u}\}$ for $1 \leq j \leq r-1$ and $1 \leq i \leq s_j-1$, noting that $\eta_{ji} = \varepsilon'_{j+1,i} u^{\Omega_{j+1}^\sharp(\beta_{j+1,i,k})_{k=1}^{j+1} - b n_1 n_2 \cdots n_j (n_{j+1} - i)}$. Here, we may assume by a nonsingular change of coordinates that η_1 can be equal to an integer one for the standard resolution of the quasisingular point $(v, 1 + \bar{u}) = (0, 0)$ of $V(h_k)$ for $1 \leq k \leq r-1$.

The 3-th Cond⁽¹⁾: Let $\{\Xi_j : N_0^j \rightarrow N_0$ is an integer-valued function for $j=1, 2, \dots, r-1\}$ be a sequence defined by the following:

$$(12.5.3) \quad \Xi_1(t) = t \text{ for each } t \in N_0.$$

$$\Xi_{j-1}(t_k)_{k=1}^{j-1} = t_{j-1} \Xi_{j-2}(\gamma_{j-2,0,k})_{k=1}^{j-2} + s_{j-2} \Xi_{j-2}(t_k)_{k=1}^{j-2} \text{ for each } (t_k)_{k=1}^{j-1} \in N_0^{j-1}.$$

The (4 α)-th Cond⁽¹⁾: For each $q = 1, 2, \dots, r-1$, we have the following: Note that $r \geq 2$.

$$(12.5.4\alpha) \quad \Xi_1(\gamma_{1,i,1}) = \Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - n_1 \beta_{1,0,1}(n_2 - i) > 0, \\ \Xi_q(\gamma_{q,i,k})_{k=1}^q - (s_q - i) s_{q-1} \Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1} \quad \text{for } 2 \leq q \leq r-1 \\ = \Delta_{q+1}(\beta_{q+1,i,k})_{k=1}^{q+1} - (n_{q+1} - i) n_q \Delta_q(\beta_{q,0,k})_{k=1}^q > 0 \quad \text{for } 0 \leq i < s_q.$$

The 4-th Cond⁽¹⁾: By The (4 α)-th Cond⁽¹⁾, for $q = 1, 2, \dots, r-1$, it is clear:

$$(12.5.4) \quad \begin{aligned} \Xi_1(\gamma_{1,i,1}) &= \gamma_{1,i,1} > 0 \quad \text{for } 0 \leq i < s_1. \\ \Xi_q(\gamma_{q,i,k})_{k=1}^q &> (s_q - i)s_{q-1}\Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1} \quad \text{for } 0 \leq i < s_q. \end{aligned}$$

The (5 α)-th Cond⁽¹⁾: By The (4 α)-th Cond⁽¹⁾, we have the following for $q = 1, 2, \dots, r-1$:

$$(12.5.5\alpha)(12.5.5\alpha.1) \quad \gcd(s_q, \Xi_q(\gamma_{q,0,k})_{k=1}^q) = \gcd(n_{q+1}, \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1}).$$

$$(12.5.5\alpha)(12.5.5\alpha.2) \quad \begin{aligned} &\frac{\Delta_{q+1}(\beta_{q+1,i,k})_{k=1}^{q+1}}{n_{q+1} - i} > \frac{\Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1}}{n_{q+1}} \quad \text{for } 0 < i < n_{q+1} \\ &\iff \\ &\frac{\Xi_q(\gamma_{q,i,k})_{k=1}^q}{s_q - i} > \frac{\Xi_q(\gamma_{q,0,k})_{k=1}^q}{s_q} \quad \text{for } 0 < i < s_q. \quad \square \end{aligned}$$

§13. The proofs of Theorem 12.0 with five sublemmas and corollaries in §12

§13.1. For the proofs of five sublemmas

Using the same method as we have used in the proofs of Sublemma 5.1, Sublemma 5.2, ..., Sublemma 5.5 and following the same kind of properties and notations as we have seen in the proof of Sublemma 5.1, Sublemma 5.2, ..., Sublemma 5.5, the proofs of Sublemma 12.1, Sublemma 12.2, ..., Sublemma 12.5 can be generalized.

Proof of Sublemma 12.1. If $r = 2$, then it is trivial to prove that $\Delta_2^\#(\beta_{2,i,1}, \beta_{2,i,2}) > n_1\beta_{1,0,1}(n_2 - i)$ on g_2 , because $\Delta_2^\#(\beta_{2,i,1}, \beta_{2,i,2}) = \Delta_2(\beta_{2,i,1}, \beta_{2,i,2})$ by (12.1.1) and $\Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) > n_1\beta_{1,0,1}(n_2 - i)$ by The 4-th Cond⁽⁰⁾ in the assumptions of Theorem 12.0.

Let $r \geq 3$. For any $\ell = 3, 4, \dots, r$, it is trivial to note by definition of $\Delta_\ell^\#(\beta_{\ell,i,k})_{k=1}^\ell$ in (12.1.1) that the following three equalities are the same, and so we can write $c = \Delta_\ell^\#(\beta_{\ell,i,k})_{k=1}^\ell - n_1\beta_{1,0,1}n_2n_3 \cdots n_{\ell-1}(n_\ell - i)$ for convenience of notation:

$$(12.1.3) \quad c = \Delta_\ell^\#(\beta_{\ell,i,k})_{k=1}^\ell - n_1\beta_{1,0,1}n_2n_3 \cdots n_{\ell-1}(n_\ell - i) \quad \text{by (c) of Sublemma 12.4}$$

$$\iff$$

$$(12.1.4) \quad c = \Delta_2(\beta_{\ell,i,1}, \beta_{\ell,i,2}) + n_1\beta_{1,0,1}\beta_{\ell,i,3} + n_1\beta_{1,0,1}n_2\beta_{\ell,i,4} + n_1\beta_{1,0,1}n_2n_3\beta_{\ell,i,5} \\ + \cdots + n_1\beta_{1,0,1}n_2 \cdots n_{\ell-2}\beta_{\ell,i,\ell} - n_1\beta_{1,0,1}n_2n_3 \cdots n_{\ell-1}(n_\ell - i)$$

$$\iff$$

$$(12.1.5) \quad c = \Delta_2(\beta_{\ell,i,1}, \beta_{\ell,i,2}) - \{(n_\ell - i)n_{\ell-1} \cdots n_2 - \beta_{\ell,i,\ell}n_{\ell-2}n_{\ell-3} \cdots n_2 \\ - \beta_{\ell,i,\ell-1}n_{\ell-3}n_{\ell-4} \cdots n_2 - \cdots - \beta_{\ell,i,4}n_2 - \beta_{\ell,i,3}\}n_1\beta_{1,0,1}.$$

So, for any integer $\ell \geq 3$, it suffices to show that the above integer c of (12.1.5) can be equal to an integer $\xi_{\ell-2} > 0$, where $\xi_{\ell-2}$ is the $(\ell - 2)$ -th element of a positive sequence $\{\xi_j : j = 1, 2, \dots, \ell - 2\}$ such that each ξ_j satisfies the following properties:

$$(12.1.6) \quad \xi_0 = \Delta_\ell(\beta_{\ell,i,k})_{k=1}^\ell - (n_\ell - i)n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1} > 0, \\ \xi_1 = \Delta_{\ell-1}(\beta_{\ell,i,k})_{k=1}^{\ell-1} - \{(n_\ell - i)n_{\ell-1} - \beta_{\ell,i,\ell}\}n_{\ell-2}\Delta_{\ell-2}(\beta_{\ell-2,0,k})_{k=1}^{\ell-2} > 0, \\ \xi_2 = \Delta_{\ell-2}(\beta_{\ell,i,k})_{k=1}^{\ell-2} - \{(n_\ell - i)n_{\ell-1}n_{\ell-2} - \beta_{\ell,i,\ell}n_{\ell-2} - \beta_{\ell,i,\ell-1}\}n_{\ell-3}\Delta_{\ell-3}(\beta_{\ell-3,0,k})_{k=1}^{\ell-3} > 0, \\ \dots\dots\dots \\ \xi_j = \Delta_{\ell-j}(\beta_{\ell,i,k})_{k=1}^{\ell-j} - \{(n_\ell - i)n_{\ell-1} \cdots n_{\ell-j} - \beta_{\ell,i,\ell}n_{\ell-2}n_{\ell-3} \cdots n_{\ell-j} \\ - \beta_{\ell,i,\ell-1}n_{\ell-3}n_{\ell-4} \cdots n_{\ell-j} - \cdots - \beta_{\ell,i,\ell-j+2}n_{\ell-j} \\ - \beta_{\ell,i,\ell-j+1}\} \times n_{\ell-j-1}\Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} > 0 \quad \text{for } 3 \leq j \leq \ell - 2.$$

Let $\ell \geq 3$ be chosen arbitrary. Now, we will show by the induction method on the nonnegative integer $j \leq \ell - 2$ that ξ_j is positive for all j .

It is trivial by The 4-th Cond⁽⁰⁾ in the assumption of Theorem 12.0 that $\xi_0 > 0$.

In order to prove that ξ_1 is positive, first of all, it is easy to observe the following by the definition of $\Delta_\ell(\beta_{\ell,i,k})_{k=1}^\ell$:

$$(12.1.7) \quad 0 < \xi_0 = \Delta_\ell(\beta_{\ell,i,k})_{k=1}^\ell - (n_\ell - i)n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1} \\ = \beta_{\ell,i,\ell}\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1} + n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell,i,k})_{k=1}^{\ell-1} - (n_\ell - i)n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1} \\ = n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell,i,k})_{k=1}^{\ell-1} - ((n_\ell - i)n_{\ell-1} - \beta_{\ell,i,\ell})\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1}.$$

Since $\Delta_{\ell-1}(\beta_{\ell-1,0,k})_{k=1}^{\ell-1} > n_{\ell-1}n_{\ell-2}\Delta_{\ell-2}(\beta_{\ell-2,0,k})_{k=1}^{\ell-2}$ by The 4-th Cond⁽⁰⁾ in the assumption of Theorem 12.0, then the third inequality of (12.1.7) implies that

$$(12.1.8) \quad n_{\ell-1}\Delta_{\ell-1}(\beta_{\ell,i,k})_{k=1}^{\ell-1} > ((n_\ell - i)n_{\ell-1} - \beta_{\ell,i,\ell})n_{\ell-1}n_{\ell-2}\Delta_{\ell-2}(\beta_{\ell-2,0,k})_{k=1}^{\ell-2}, \\ \text{whether or not } (n_\ell - i)n_{\ell-1} - \beta_{\ell,i,\ell} > 0.$$

Dividing both sides on (12.1.8) by $n_{\ell-1}$, then

$$(12.1.9) \quad \Delta_{\ell-1}(\beta_{\ell,i,k})_{k=1}^{\ell-1} > ((n_{\ell}-i)n_{\ell-1} - \beta_{\ell,i,\ell})n_{\ell-2}\Delta_{\ell-2}(\beta_{\ell-2,0,k})_{k=1}^{\ell-2},$$

which is equivalent to the fact that $\xi_1 > 0$.

By the induction assumption on the positive integer $j \leq \ell - 2$, suppose we have shown that ξ_j is positive with $1 \leq j \leq \ell - 3$. To prove that ξ_{j+1} is positive, for convenience of notations, let ξ_j of (12.1.6) be written again in the form

$$(12.1.10) \quad \begin{aligned} \xi_j &= \Delta_{\ell-j}(\beta_{\ell,i,k})_{k=1}^{\ell-j} - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} > 0 \quad \text{with} \\ \omega_j &= (n_{\ell}-i)n_{\ell-1} \cdots n_{\ell-j} - \beta_{\ell,i,\ell} n_{\ell-2} n_{\ell-3} \cdots n_{\ell-j} \\ &\quad - \beta_{\ell,i,\ell-1} n_{\ell-3} n_{\ell-4} \cdots n_{\ell-j} - \cdots - \beta_{\ell,i,\ell-j+2} n_{\ell-j} - \beta_{\ell,i,\ell-j+1}. \end{aligned}$$

Now, by (12.1.10) and the definition of $\Delta_{\ell-j}(\beta_{\ell,i,k})_{k=1}^{\ell-j}$ only, it is easy to prove that

$$(12.1.11) \quad \begin{aligned} 0 < \xi_j &= \Delta_{\ell-j}(\beta_{\ell,i,k})_{k=1}^{\ell-j} - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} \\ &= \beta_{\ell,i,\ell-j} \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} + n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} \\ &\quad - \omega_j n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} \\ &= n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} - (\omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j}) \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1}. \end{aligned}$$

Since $\Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1} > n_{\ell-j-1} n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,0,k})_{k=1}^{\ell-j-2}$ by The 4-th Cond⁽⁰⁾ in the assumption of Theorem 12.0, we get the following from the last equality in (12.1.11):

$$(12.1.12) \quad \begin{aligned} n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} &> (\omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j}) \Delta_{\ell-j-1}(\beta_{\ell-j-1,0,k})_{k=1}^{\ell-j-1}, \quad \text{and so} \\ n_{\ell-j-1} \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} &> (\omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j}) n_{\ell-j-1} n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,0,k})_{k=1}^{\ell-j-2}, \\ \text{whether or not } \omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j} &> 0. \end{aligned}$$

Dividing both sides of (12.1.12) by $n_{\ell-j-1}$, then we get

$$(12.1.13) \quad \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} > (\omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j}) n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,0,k})_{k=1}^{\ell-j-2}.$$

Before we prove that $\xi_{j+1} > 0$, then it is trivial to observe by (12.1.10) that ξ_{j+1} of (12.1.6) can be rewritten as follows:

$$(12.1.14) \quad \xi_{j+1} = \Delta_{\ell-j-1}(\beta_{\ell,i,k})_{k=1}^{\ell-j-1} - (\omega_j n_{\ell-j-1} - \beta_{\ell,i,\ell-j}) n_{\ell-j-2} \Delta_{\ell-j-2}(\beta_{\ell-j-2,0,k})_{k=1}^{\ell-j-2}.$$

Then, it is clear by (12.1.13) that ξ_{j+1} is positive for all $j = 1, 2, \dots, \ell - 3$. Since $c = \xi_{\ell-2}$ for some integer $\ell \geq 3$ by (12.1.6), the proof is done. \square

Proof of Sublemma 12.2. We prove (a), (b), (c) and (d), respectively.

(a) In preparation for the proof of an equality in (12.2.1), it is clear by (2a) of The 2-th Cond⁽⁰⁾ in the assumptions of this theorem and by an inequality in (12.2.0) of this sublemma that the following are true:

If $r = 1$ then $g_1 = g_1(y, z)$ can be written in the form

$$(12.2.2) \quad \begin{aligned} g_1 &= \Sigma_{1,0} + \Sigma_{1,1}, \\ \text{where } \Sigma_{1,0} &= z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} \quad \text{with } \varepsilon_{1,0} = 1, \\ \Sigma_{1,1} &= \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(1)} y^{\alpha} z^{\beta} \quad \text{with } n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1}, \end{aligned}$$

where a unit $\varepsilon_{1,0} = \varepsilon_{1,0}(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$ if necessary, and the $c_{\alpha,\beta}^{(1)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}$, if exist.

Also, note by (12.2.2) that for any positive integer ℓ , g_1^ℓ can be written in the form

$$(12.2.3) \quad \begin{aligned} g_1^\ell &= (\Sigma_{1,0} + \Sigma_{1,1})^\ell = (\Sigma_{1,0})^\ell + \sum_{k=1}^{\ell-1} \binom{\ell}{k} (\Sigma_{1,0})^k (\Sigma_{1,1})^{\ell-k} + (\Sigma_{1,1})^\ell \\ &= \Sigma_{1,0}^{(\ell)} + \Sigma_{1,1}^{(\ell)} \quad \text{for any integer } \ell \geq 2, \end{aligned}$$

where write $\Sigma_{1,0}^{(\ell)} = (\Sigma_{1,0})^\ell = (z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}})^\ell$ with $\varepsilon_{1,0} = 1$,

$$\text{and } \Sigma_{1,1}^{(\ell)} = \sum_{k=1}^{\ell-1} \binom{\ell}{k} (\Sigma_{1,0})^k (\Sigma_{1,1})^{\ell-k} + (\Sigma_{1,1})^\ell.$$

It is clear that for any monomial $y^\alpha z^\beta \in (\Sigma_{1,0})^\ell$, $n_1\alpha + \beta_{1,0,1}\beta = n_1\beta_{1,0,1}\ell$. So, to prove that $n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}\ell$ for any monomial $y^\alpha z^\beta \in g_1^\ell - (\Sigma_{1,0})^\ell$, is equivalent to prove by (12.2.3) that for any nonzero monomial $y^\alpha z^\beta \in \Sigma_{1,1}^{(\ell)}$

$$(12.2.4) \quad n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}\ell.$$

To prove (12.2.4), it suffices to show that the following three claims hold by using an equation of $\Sigma_{1,1}^{(\ell)}$ in (12.2.3): Let $\ell \geq 1$ be an arbitrary integer.

$$(12.2.5) \quad \text{Claim(i)} \quad \text{For any monomial } y^\alpha z^\beta \in (\Sigma_{1,0})^\ell, \quad n_1\alpha + \beta_{1,0,1}\beta = n_1\beta_{1,0,1}\ell.$$

$$\text{Claim(ii)} \quad \text{For any monomial } y^\alpha z^\beta \in (\Sigma_{1,1})^\ell, \quad n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}\ell.$$

$$\begin{aligned} \text{Claim(iii)} \quad &\text{For any monomial } y^\gamma z^\delta \in (\Sigma_{1,0})^k (\Sigma_{1,1})^{\ell-k}, \\ &n_1\gamma + \beta_{1,0,1}\delta > n_1\beta_{1,0,1}k + n_1\beta_{1,0,1}(\ell - k) = n_1\beta_{1,0,1}\ell. \end{aligned}$$

Note by (12.2.2) that the proof of three claims in (12.2.5) is trivial, and so the proof of (12.2.4) is done.

Now, for the proof of the sublemma, using equations in (12.2.2), (12.2.3) and (12.2.4), it suffices to show that for any integer $r \geq 2$, g_r of (12.2.1) can be generally represented in the following form: Let $\ell \geq 2$ be an arbitrary integer.

$$(12.2.6)(12.2.6.1) \quad \begin{aligned} g_r &= \Sigma_{r,0} + \Sigma_{r,1}, \\ \Sigma_{r,0} &= (\Sigma_{1,0})^{d_r} \quad \text{and } \Sigma_{1,0} = z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}} \quad \text{with } \varepsilon_{1,0} = 1 \text{ and } d_r = n_2 \cdots n_r, \\ \Sigma_{r,1} &= \sum_{\gamma, \delta \geq 0} c_{\gamma,\delta}^{(1)} y^\gamma z^\delta \quad \text{with } n_1\gamma + \beta_{1,0,1}\delta > n_1\beta_{1,0,1}d_r, \quad \text{and then} \end{aligned}$$

$$(12.2.6.2) \quad \begin{aligned} g_r^\ell &= \Sigma_{r,0}^{(\ell)} + \Sigma_{r,1}^{(\ell)} \quad \text{for any integer } \ell \geq 2, \\ \Sigma_{r,0}^{(\ell)} &= (\Sigma_{1,0})^{d_r\ell} = (\Sigma_{r,0})^\ell, \\ \Sigma_{r,1}^{(\ell)} &= \sum_{\gamma, \delta \geq 0} c_{\gamma,\delta}^{(\ell)} y^\gamma z^\delta \quad \text{with } n_1\gamma + \beta_{1,1}\delta > n_1\beta_{1,0,1}d_r\ell, \end{aligned}$$

where $\varepsilon_{1,0}$ is assumed to be one in $\mathbb{C}\{y, z\}$ if necessary, and the $c_{\gamma,\delta}^{(\ell)}$ are nonzero complex numbers for some nonnegative integers γ and δ such that $n_1\gamma + \beta_{1,0,1}\delta > n_1\beta_{1,0,1}d_r\ell$.

For the induction proof of an equality in (12.2.6), it suffices to consider the following two cases:

Case(I) $r = 2$ and Case(II) $2 < r$.

Case(I) Let $r = 2$. Recall by (2b) of The 2-th Cond⁽⁰⁾ and (4b) of The 4-th Cond⁽⁰⁾ in the assumptions of this theorem and by (12.1.2) of Sublemma 12.1 that g_2 can be written in the form

$$(12.2.7)(12.2.7.1) \quad g_2 = g_1^{n_2} + \varepsilon_{2,0} y^{\beta_{2,0,1}} z^{\beta_{2,0,2}} + c_2 \sum_{i=1}^{n_2-1} \varepsilon_{2,i} y^{\beta_{2,i,1}} z^{\beta_{2,i,2}} g_1^i \quad \text{and}$$

$$(12.2.7.2) \quad \Delta_2^\#(\beta_{2,i,1}, \beta_{2,i,2}) = \Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) > n_1 \beta_{1,0,1} (n_2 - i) \quad \text{on } g_2.$$

In order to prove an equality in (12.2.6) for $r = 2$, since $n_1 \beta_{2,0,1} + \beta_{1,0,1} \beta_{2,0,2} > n_2 n_1 \beta_{1,0,1}$ by (4b) of The 4-th Cond⁽⁰⁾ in the assumption of Theorem 12.0, then it suffices to show that the following two claims hold by using equations in (12.2.7) with (12.2.4):

$$(12.2.8) \quad \begin{aligned} \text{Claim(i)} \quad & \text{For any monomial } y^\alpha z^\beta \in (g_1)^{n_2} - (\Sigma_{1,0})^{n_2}, \quad n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} n_2. \\ \text{Claim(ii)} \quad & \text{For any monomial } y^\gamma z^\delta \in y^{\beta_{2,i,1}} z^{\beta_{2,i,2}} g_1^i, \quad n_1 \gamma + \beta_{1,0,1} \delta \geq \\ & \Delta_2^\#(\beta_{2,i,1}, \beta_{2,i,2}) + n_1 \beta_{1,0,1} i > n_1 \beta_{1,0,1} (n_2 - i) + n_1 \beta_{1,0,1} i = n_1 \beta_{1,0,1} n_2. \end{aligned}$$

Thus, the proof of two claims in (12.2.8) is trivial by (12.2.4) and (12.2.7.2), and so the proof of (12.2.6.1) is done for $r = 2$. Also, if $r = 2$, the proof of (12.2.6.2) is trivial by the same method as we have used in the proof for (12.2.4) with (12.2.3). Thus, the proof of Case(I) is done.

Case(II) Let $2 < r$. Now, suppose we have proved by induction assumption on the positive integer $j < r$ that the representation of g_j in (12.2.6) is true for $2 \leq j < r$. Then, recall by (2b) of The 2-th Cond⁽⁰⁾ and (4b) of The 4-th Cond⁽⁰⁾ in the assumptions of this theorem and by (12.1.2) of Sublemma 12.1 that g_{j+1} can be rewritten as follows:

$$(12.2.9) \quad \begin{aligned} g_{j+1} &= g_j^{n_{j+1}} + \varepsilon_{j+1,0} y^{\beta_{j+1,0,1}} z^{\beta_{j+1,0,2}} g_1^{\beta_{j+1,0,3}} \cdots g_{j-1}^{\beta_{j+1,0,j+1}} \\ &\quad + c_{j+1} \sum_{i=1}^{n_{j+1}-1} \varepsilon_{j+1,i} y^{\beta_{j+1,i,1}} z^{\beta_{j+1,i,2}} g_1^{\beta_{j+1,i,3}} \cdots g_{j-1}^{\beta_{j+1,i,j+1}} g_j^i, \\ \Delta_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} &> n_1 \beta_{1,0,1} n_2 n_3 \cdots n_j (n_{j+1} - i) \quad \text{on } g_{j+1}, \end{aligned}$$

where $\varepsilon_{j+1,0}$ and $\varepsilon_{j+1,i}$ are units in $\mathbb{C}\{y, z\}$, and the c_{j+1} are nonzero complex numbers.

First of all, applying the induction assumption on g_j , then for any $k = 1, 2, \dots, j$ and any integer $\ell > 0$, suppose we have shown that the following are true:

$$(12.2.10)(12.2.10.1) \quad \begin{aligned} g_k &= \Sigma_{k,0} + \Sigma_{k,1}, \\ \Sigma_{k,0} &= (\Sigma_{1,0})^{d_k} \quad \text{and } \Sigma_{1,0} = z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} \quad \text{with } \varepsilon_{1,0} = 1 \text{ and } d_k = n_2 \cdots n_k, \\ \Sigma_{k,1} &= \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(1)} y^\gamma z^\delta \quad \text{with } n_1 \gamma + \beta_{1,0,1} \delta > n_1 \beta_{1,0,1} d_k, \quad \text{and then} \\ (12.2.10.2) \quad g_k^\ell &= \Sigma_{k,0}^{(\ell)} + \Sigma_{k,1}^{(\ell)} \quad \text{for any integer } \ell \geq 2, \\ \Sigma_{k,0}^{(\ell)} &= (\Sigma_{1,0})^{d_k \ell} = (\Sigma_{k,0})^\ell, \\ \Sigma_{k,1}^{(\ell)} &= \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(\ell)} y^\gamma z^\delta \quad \text{with } n_1 \gamma + \beta_{1,1} \delta > n_1 \beta_{1,0,1} d_k \ell, \end{aligned}$$

where $\varepsilon_{1,0}$ is assumed to be one in $\mathbb{C}\{y, z\}$ if necessary, and the $c_{\gamma, \delta}^{(\ell)}$ are nonzero complex numbers for some nonnegative integers γ and δ such that $n_1 \gamma + \beta_{1,0,1} \delta > n_1 \beta_{1,0,1} d_k \ell$.

First, it is clear by (12.2.10) that for any nonzero monomial $y^{\alpha_k} z^{\beta_k} \in g_k^\ell = \Sigma_{k,0}^{(\ell)} + \Sigma_{k,1}^{(\ell)}$,

$$(12.2.11) \quad n_1 \alpha_k + \beta_{1,0,1} \beta_k \geq n_1 \beta_{1,0,1} d_k \ell \quad \text{on } g_k.$$

In order to prove that (12.2.6) is true on g_{j+1} , it is clear by (12.2.9) and (12.2.11) that any nonzero monomial $y^\gamma z^\delta \in g_{j+1} - g_j^{n_{j+1}}$ can be represented as follows:

$$(12.2.12) \quad y^\gamma z^\delta = y^{\beta_{j+1,i,1}} z^{\beta_{j+1,i,2}} \prod_{k=1}^j y^{\alpha_k} z^{\beta_k} \quad \text{and} \\ y^{\alpha_k} z^{\beta_k} \in g_k^{\beta_{j+1,i,k+2}} \quad \text{with } n_1 \alpha_k + \beta_{1,0,1} \beta_k \geq n_1 \beta_{1,0,1} d_k \beta_{j+1,i,k+2}.$$

So, by (12.2.11), (12.2.12) and Sublemma 12.1, for any nonzero monomial $y^\gamma z^\delta \in g_{j+1} - g_j^{n_{j+1}}$,

$$(12.2.13) \quad n_1 \gamma + \beta_{1,0,1} \delta \\ \geq n_1 \beta_{j+1,i,1} + \beta_{1,0,1} \beta_{j+1,i,2} + \sum_{k=1}^{j-1} (n_1 \alpha_k + \beta_{1,0,1} \beta_k) + n_1 \beta_{1,0,1} d_j i \\ \geq n_1 \beta_{j+1,i,1} + \beta_{1,0,1} \beta_{j+1,i,2} + \sum_{k=1}^{j-1} (n_1 \beta_{1,0,1} d_k \beta_{j+1,i,k+2}) + n_1 \beta_{1,0,1} d_j i \\ = \Delta_{j+1}^\# (\beta_{j+1,i,k})_{k=1}^{j+1} + n_1 \beta_{1,0,1} d_j i \\ > n_1 \beta_{1,0,1} d_j (n_{j+1} - i) + n_1 \beta_{1,0,1} d_j i = n_1 \beta_{1,0,1} d_j n_{j+1} = n_1 \beta_{1,0,1} d_{j+1}.$$

Since any nonzero monomial $y^\gamma z^\delta \in g_j^{n_{j+1} - \sum_{j,0}^{(n_{j+1})}}$ implies that $n_1 \gamma + \beta_{1,0,1} \delta > n_1 \beta_{1,0,1} d_{j+1}$, then if $r = j + 1$, the proof of (12.2.6.1) is done. Also, if $r = j + 1$, then the proof of (12.2.6.2) is trivial by the same method as we have used in the proof for (12.2.4) with (12.2.3). Then, the proof of Case(II) is done. So, we finished the proof of (a).

(b) To prove (b1), it is enough to consider $g_r(0, z)$ from $g_r(y, z)$ of (12.2.1). Then

$$(12.2.14) \quad g_r(0, z) = z^{n_1 n_2 \cdots n_r} + \sum c_{0,\beta}^{(r)} z^\beta \quad \text{with} \\ \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_r.$$

Thus, $\beta > n_1 n_2 \cdots n_r$, and so it is done. Also, the proof of (b2) can be done similarly.

(c) To prove (c1), suppose that $n_1 < \beta_{1,0,1}$. By (a), $\beta_{1,0,1}(\alpha + \beta) > n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_r$ implies that $\alpha + \beta > n_1 \prod_{k=2}^r n_k$. Thus, the proof of (c1) is done. Similarly, (c2) can be proved.

Thus, the proof of this sublemma is finished. \square

Proof of Sublemma 12.3. To prove (12.3.3) for any $r \geq 2$, it is enough to show by (12.3.1) and (12.3.2) that the following equation in (12.3.4) is nonnegative:

$$(12.3.4) \quad \Omega_r^\#(\beta_{r,i,k})_{k=1}^r - b n_1 n_2 \cdots n_{r-1} (n_r - i) = a \beta_{r,i,1} + b D \geq 0 \quad \text{with} \\ D = \beta_{r,i,2} + n_1 \beta_{r,i,3} + n_1 n_2 \beta_{r,i,4} + \cdots + n_1 n_2 \cdots n_{r-2} \beta_{r,i,r} - n_1 n_2 \cdots n_{r-1} (n_r - i),$$

by definition of $\Omega_r^\#(\beta_{r,i,k})_{k=1}^r$ in (12.3.2) where $a > 0$ and $b \geq 0$.

Then, it remains to prove by (12.3.4) that $a \beta_{r,i,1} + b D \geq 0$ depending on D , and so it suffices to consider the following two cases:

Case(i) Let $D \geq 0$. It is clear that $a \beta_{r,i,1} + b D \geq 0$ because $a > 0$, and also $b \geq 0$ with $\beta_{r,i,1}$ nonnegative. Thus, the proof of Case(i) is done.

Case(ii) Let $D < 0$. In order to prove that $a \beta_{r,i,1} + b D \geq 0$, first note that the inequality $\Delta_r^\#(\beta_{r,i,k})_{k=1}^r - n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{r-1} (n_r - i) > 0$ of Sublemma 12.1 can be equivalently rewritten as follows:

$$(12.3.5) \quad 0 < \Delta_r^\#(\beta_{r,i,k})_{k=1}^r - n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{r-1} (n_r - i) \\ = n_1 \beta_{r,i,1} + \beta_{1,0,1} \beta_{r,i,2} + n_1 \beta_{1,0,1} \beta_{r,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{r,i,4} \\ + \cdots + n_1 \beta_{1,0,1} n_2 \cdots n_{r-2} \beta_{r,i,r} - n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{r-1} (n_r - i) \quad \text{by (12.1.1)} \\ = n_1 \beta_{r,i,1} + \beta_{1,0,1} D, \quad \text{where} \\ D = \beta_{r,i,2} + n_1 \beta_{r,i,3} + \cdots + n_1 n_2 \cdots n_{r-2} \beta_{r,i,r} - n_1 n_2 \cdots n_{r-1} (n_r - i) \quad \text{by (12.3.4).}$$

Since $-D > 0$ and $n_1 \geq 2 > 0$, then the inequality $n_1\beta_{r,i,1} + \beta_{1,0,1}D > 0$ in (12.3.5) can be equivalently represented as follows:

$$(12.3.6) \quad \frac{\beta_{r,i,1}}{-D} > \frac{\beta_{1,0,1}}{n_1}.$$

Also, $a\beta_{1,0,1} - bn_1 = 1$ with $a > 0$ implies that $\frac{\beta_{1,0,1}}{n_1} > \frac{b}{a}$. Therefore, we proved by (12.3.6) that $\frac{\beta_{r,i,1}}{-D} > \frac{b}{a}$, that is, $a\beta_{r,i,1} + bD > 0$. Thus, the proof of Case(ii) is done.

Therefore, we showed by Case(i) and Case(ii) that the equation in (12.3.4) is nonnegative, and so the proof of this sublemma is finished. \square

Proof of Sublemma 12.4. Following the same assumptions and notations as in Sublemma 12.1, Sublemma 12.2 and Sublemma 12.3, then (a) of Sublemma 12.2 is true, and so for each $j = 2, 3, \dots, r$, $g_j = g_j(y, z)$ of (12.2.1) can be easily rewritten as follows:

$$(12.4.6) \quad g_j = (z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}})^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} y^\alpha z^\beta \quad \text{with } \varepsilon_{1,0} = 1 \text{ and} \\ \text{with } n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}d_j, \quad \text{where } d_j = n_2n_3 \cdots n_j,$$

where $\varepsilon_{1,0} = \varepsilon_{1,0}(y, z)$ is assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}d_j$.

For the proof of this sublemma, it suffices to follow Step(I) and Step(II) in order:

Step(I) First, we will show how to apply Theorem 3.6 to the proof of (a), (b), and (d_2) and (d_3) of (d) in this sublemma.

Step(II) Next, the remaining part of this sublemma will be proved computationally.

Step(I) In preparation for the proof of (a), (b), (d_2) and (d_3) , it is clear that the equation of g_j of (12.4.6) satisfies the same kind of properties as f does in the assumption of Theorem 3.6, which can be represented as follows:

g_j of (12.4.6) satisfies the same kind of assumption as in Theorem 3.6. Let $V(g_0) = \{(y, z) : g_0(y, z) = 0\}$, $V(f) = \{(y, z) : f(y, z) = 0\}$ and $V(G) = \{(y, z) : G(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively as follows: For brevity of notation, substitute g_j of (12.4.6) by f , for an application of Theorem 3.6.

$$(12.4.7) \quad g_0 = z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}} \quad \text{with } \varepsilon_{1,0} = 1, \\ f = g_0^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} y^\alpha z^\beta \quad \text{with } n_1\alpha + \beta_{1,0,1}\beta > n_1\beta_{1,0,1}d_j, \\ F = y^{\delta_1} z^{\delta_2} f, \\ G = y^\gamma g_0,$$

satisfying the properties (i), (ii), (iii), (iv), (v) and (vi):

- (i) $\gcd(n_1, \beta_{1,0,1}) = 1$ with $n_1 \geq 2$ and $\beta_{1,0,1} \geq 1$.
- (ii) $d_j = n_2n_3 \cdots n_j$ is a positive integer with $d_j \geq 2$, and $d_1 = 1$ if necessary.
- (iii) $\varepsilon_{1,0}$ is assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1\alpha + k_1\beta > n_1\beta_{1,0,1}d_j$, if exist.
- (iv) Assume that $V(f)$ has an isolated singular point at the origin as a reduced variety.
- (v) If $\beta_{1,0,1} = 1$, then $\gamma = 1$, and if $\beta_{1,0,1} \geq 2$, then $\gamma = 0$.
- (vi) In addition, assume that each δ_i is either a positive integer or 0 for $i = 1, 2$, as far as $V(F)$ has an isolated singular point at the origin as a reduced variety, even if $d_j \geq 1$.

So, g_j of (12.4.6) has the same kind of conclusion as in Theorem 3.6, up to change of notations:

The same kind of conclusion as in Theorem 3.6 As we have seen in Theorem 3.6, let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(G) = V(y^\gamma g_0)$. Therefore, by the conclusion of Theorem 3.6, there is nothing to prove for (a), (b), (d2) and (d3) in this sublemma.

Step(II) To finish the proof of this sublemma, it remains to prove (c) and the remaining part (d₁) of (d). Now, we will use the same kind of notations and properties as in (a), (b), (d2) and (d3) as follows:

For (a), (b) and (d2), along $v = 0$ $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ as a composition of analytic mappings and $(f \circ \tau_m)_{total}$ can be rewritten in the following form: Note that $2 \leq j \leq r$.

$$(12.4.8) \quad \begin{aligned} \tau_m(v, u) &= (y, z) = (v^{n_1} u^a, v^{\beta_{1,0,1}} u^b), \\ (f \circ \tau_m)_{total} &= (f \circ \tau_m)(v, u) = v^{e_{j,m}} u^{\rho_{j,m}} (f \circ \tau_m)_{proper} \quad \text{with } g_j = f, \\ (f \circ \tau_m)_{proper} &= (1 + \varepsilon_{1,0} u)^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} v^{n_1 \alpha + \beta_{1,0,1} \beta - n_1 \beta_{1,0,1} d_j} u^{a\alpha + b\beta - bn_1 d_j}, \end{aligned}$$

where

- (i) a and b are some nonnegative integers such that $a\beta_{1,0,1} - bn_1 = 1$ and $\varepsilon_{1,0}$ is assumed to be one in $\mathbb{C}\{y, z\}$,
- (ii) $e_{j,m} = n_1 \beta_{1,0,1} d_j$ and $\rho_{j,m} = bn_1 d_j$ and $\rho_{\alpha, \beta} = a\alpha + b\beta - bn_1 d_j \geq 0$,
- (iii) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.
- (iv) $V^{(m)}(g_j) \cap (\cup_{i=1}^m E_i) = V^{(m)}(g_j) \cap E_m = \{(v, 1 + \varepsilon_{1,0} u) = (0, 0)\}$ for any $j = 2, \dots, r$.

For (d3), after m iterations of blow-ups, denoted by τ_m , we have the following consequences:

- $$(12.4.9) \quad \begin{aligned} \text{(i)} \quad & \text{If } \beta_{1,0,1} = 1, \text{ then } f \in \text{the type}[0] \text{ under } \tau_m, \text{ and} \\ & \text{if } \beta_{1,0,1} \geq 2, \text{ then } f \in \text{the type}[1] \text{ under } \tau_m. \\ \text{(ii)} \quad & \text{Whether } \beta_{1,0,1} = 1 \text{ or } \beta_{1,0,1} \geq 2, \text{ then } F \in \text{the type}[1] \text{ under } \tau_m. \end{aligned}$$

Remark 12.4.1

(a) In the assumption of Theorem 3.6, the construction for $G(y, z) = z^\gamma g_0$ with $g_0 = z^{n_1} + y^{k_1}$ was defined as follows: Note that $\gcd(n_1, k_1) = 1$.

Let $1 \leq n_1 < k_1$, and if $n_1 = 1$, then $\gamma = 1$, and if $n_1 \geq 2$, then $\gamma = 0$.

(b) In the conclusion of Theorem 3.6, whether $n_1 = 1$ or $2 \leq n_1 < k_1$, or $2 \leq k_1 < n_1$, there are some nonnegative integers a and b such that $bn_1 - ak_1 = 1$ because of [I] and [II] in Theorem 3.6.

In preparation for the proof of (c), apply the above conclusion with (12.4.8), to $g_{j+1} = g_0^{d_{j+1}} + \sum_{\alpha, \beta} c_{\alpha, \beta}^{(j+1)} y^\alpha z^\beta$ in (12.4.6) where $g_0 = z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}}$ with $\varepsilon_{1,0} = 1$. Then for any $j = 1, 2, \dots, r-1$, we have the following:

$$(12.4.10) \quad \begin{aligned} (g_{j+1} \circ \tau_m)_{total} &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_0 \circ \tau_m)_{proper}^{d_{j+1}} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j+1)} (v^{n_1} u^a)^\alpha (v^{\beta_{1,0,1}} u^b)^\beta \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} \{ (g_0 \circ \tau_m)_{proper}^{d_{j+1}} \\ &\quad + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j+1)} v^{n_1 \alpha + \beta_{1,0,1} \beta - n_1 \beta_{1,0,1} d_{j+1}} u^{a\alpha + b\beta - bn_1 d_{j+1}} \} \\ &= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_{j+1} \circ \tau_m)_{proper}, \end{aligned}$$

where $d_{j+1} = n_2 n_3 \cdots n_{j+1}$, $e_{j+1,m} = n_1 \beta_{1,0,1} d_{j+1}$ and $\rho_{j+1,m} = bn_1 d_{j+1}$, noting by (12.4.7) and (ii) of (12.4.8) that $n_1 \alpha + \beta_{1,0,1} \beta - n_1 \beta_{1,0,1} d_{j+1} > 0$ and $a\alpha + b\beta - bn_1 d_{j+1} \geq 0$.

On the other hand, recall by (2b) of The 2-th Cond⁽⁰⁾ in the assumption of the theorem and by (12.4.6) that

$$\begin{aligned}
(12.4.11) \quad g_{j+1} &= g_j^{n_{j+1}} + \varepsilon_{j+1,0} y^{\beta_{j+1,0,1}} z^{\beta_{j+1,0,2}} g_1^{\beta_{j+1,0,3}} \cdots g_{j-1}^{\beta_{j+1,0,j+1}} \\
&\quad + c_{j+1} \sum_{i=1}^{n_{j+1}-1} \varepsilon_{j+1,i} y^{\beta_{j+1,i,1}} z^{\beta_{j+1,i,2}} g_1^{\beta_{j+1,i,3}} \cdots g_{j-1}^{\beta_{j+1,i,j+1}} g_j^i, \\
g_j &= g_0^{n_2 \cdots n_j} + \sum_{\gamma, \delta \geq 0} c_{\gamma, \delta}^{(j)} y^\gamma z^\delta, \\
g_0 &= z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} \quad \text{with } \varepsilon_{1,0} = 1, \\
\text{where} \quad (i) \quad \Delta_2(\gamma, \delta) &= n_1 \gamma + \beta_{1,0,1} \delta > n_1 \beta_{1,0,1} n_2 \cdots n_j \quad \text{by (12.4.6),} \\
(ii) \quad \Delta_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} &> n_1 \beta_{1,0,1} n_2 n_3 \cdots n_j (n_{j+1} - i) \quad \text{by (12.1.2),} \\
(iii) \quad \Omega_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} &\geq b n_1 n_2 n_3 \cdots n_j (n_{j+1} - i) \quad \text{by (12.3.3),} \\
(iv) \quad \varepsilon_{j+1,i} &\text{ is a unit in } \mathbb{C}\{y, z\} \quad \text{for } 0 \leq i \leq n_{j+1} - 1.
\end{aligned}$$

Now, apply (12.4.8) or (12.4.10), to g_{j+1} in (12.4.11).

Then, we have the following:

$$\begin{aligned}
(12.4.12) \quad (g_{j+1} \circ \tau_m)_{total} &= ((g_j \circ \tau_m)(v, u))^{n_{j+1}} + (\varepsilon_{j+1,0} \circ \tau_m)(v, u)((y \circ \tau_m)(v, u))^{\beta_{j+1,0,1}} \\
&\quad \times ((z \circ \tau_m)(v, u))^{\beta_{j+1,0,2}} ((g_1 \circ \tau_m)(v, u))^{\beta_{j+1,0,3}} \cdots ((g_{j-1} \circ \tau_m)(v, u))^{\beta_{j+1,0,j+1}} \\
&\quad + c_{j+1} \sum_{i=1}^{n_{j+1}-1} \{(\varepsilon_{j+1,i} \circ \tau_m)(v, u)((y \circ \tau_m)(v, u))^{\beta_{j+1,i,1}} ((z \circ \tau_m)(v, u))^{\beta_{j+1,i,2}} \\
&\quad \times ((g_1 \circ \tau_m)(v, u))^{\beta_{j+1,i,3}} \cdots ((g_{j-1} \circ \tau_m)(v, u))^{\beta_{j+1,i,j+1}} ((g_j \circ \tau_m)(v, u))^i\}, \\
&= \{v^{e_{j,m}} u^{\rho_{j,m}} (g_j \circ \tau_m)_{proper}\}^{n_{j+1}} + \varepsilon'_{j+1,0} (v^{n_1} u^a)^{\beta_{j+1,0,1}} (v^{\beta_{1,0,1}} u^b)^{\beta_{j+1,0,2}} \\
&\quad \times \{v^{n_1 \beta_{1,0,1}} u^{b n_1} (g_1 \circ \tau_m)_{proper}\}^{\beta_{j+1,0,3}} \{v^{e_{2,m}} u^{\rho_{2,m}} (g_2 \circ \tau_m)_{proper}\}^{\beta_{j+1,0,4}} \cdots \\
&\quad \times \{v^{e_{j-1,m}} u^{\rho_{j-1,m}} (g_{j-1} \circ \tau_m)_{proper}\}^{\beta_{j+1,0,j+1}} \\
&\quad + c_{j+1} \sum_{i=1}^{n_{j+1}-1} \{\varepsilon'_{j+1,i} (v^{n_1} u^a)^{\beta_{j+1,i,1}} (v^{\beta_{1,0,1}} u^b)^{\beta_{j+1,i,2}} \\
&\quad \times [v^{n_1 \beta_{1,0,1}} u^{b n_1} (g_1 \circ \tau_m)_{proper}]^{\beta_{j+1,i,3}} [v^{e_{2,m}} u^{\rho_{2,m}} (g_2 \circ \tau_m)_{proper}]^{\beta_{j+1,i,4}} \cdots \\
&\quad \times [v^{e_{j-1,m}} u^{\rho_{j-1,m}} (g_{j-1} \circ \tau_m)_{proper}]^{\beta_{j+1,i,j+1}} [v^{e_{j,m}} u^{\rho_{j,m}} (g_j \circ \tau_m)_{proper}]^i\} \\
&= v^{e_{j+1,m}} u^{\rho_{j+1,m}} \{ (g_j \circ \tau_m)_{proper}^{n_{j+1}} + \varepsilon'_{j+1,0} v^{\Delta_2^\#(\beta_{j+1,0,1}, \beta_{j+1,0,2}) - e_{j+1,m}} \\
&\quad \times u^{\Omega_2^\#(\beta_{j+1,0,1}, \beta_{j+1,0,2}) - b n_1 n_2 n_3 \cdots n_{j+1}} (g_1 \circ \tau_m)_{proper}^{\beta_{j+1,0,3}} \cdots (g_{j-1} \circ \tau_m)_{proper}^{\beta_{j+1,0,j+1}} \\
&\quad + c_{j+1} \sum_{i=1}^{n_{j+1}-1} \varepsilon'_{j+1,i} v^{\Delta_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} - e_{j+1,m} + e_{j,m} i} u^{\Omega_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} - b n_1 n_2 \cdots n_j (n_{j+1} - i)} \\
&\quad \times (g_1 \circ \tau_m)_{proper}^{\beta_{j+1,i,3}} \cdots (g_{j-1} \circ \tau_m)_{proper}^{\beta_{j+1,i,j+1}} (g_j \circ \tau_m)_{proper}^i \} \\
&= v^{e_{j+1,m}} u^{\rho_{j+1,m}} (g_{j+1} \circ \tau_m)_{proper} \quad \text{because of the following two equations in (12.4.13),}
\end{aligned}$$

where

- (a) $\varepsilon'_{j+1,i} = (\varepsilon_{j+1,i} \circ \tau_m)(v, u)$ is a unit in $\mathbb{C}\{v, 1+u\}$ for $0 \leq i < n_{j+1}$ and
- (b) noting by (12.4.7) and (12.4.8) that $d_j = n_2 n_3 \cdots n_j$, $e_{j,m} = n_1 \beta_{1,0,1} d_j$ and $\rho_{j,m} = b n_1 d_j$ for $j = 2, 3, \dots, r$, then $d_j n_{j+1} = d_{j+1}$, $e_{j,m} n_{j+1} = e_{j+1,m}$ and $\rho_{j,m} n_{j+1} = \rho_{j+1,m}$.

Thus, the proof for the representation in (12.4.12) just follows from Equation(1) and Equation(2) of (12.4.13):

Equation(1) of (5.4.13).

$$\begin{aligned}
& (n_1\beta_{j+1,i,1} + \beta_{1,0,1}\beta_{j+1,i,2}) + n_1\beta_{1,0,1}\beta_{j+1,i,3} + e_{2,m}\beta_{j+1,i,4} + \cdots + e_{j-1,m}\beta_{j+1,i,j+1} \\
&= \Delta_2(\beta_{j+1,i,1}, \beta_{j+1,i,2}) + n_1\beta_{1,0,1}\beta_{j+1,i,3} + n_1\beta_{1,0,1}d_2\beta_{j+1,i,4} + \cdots + n_1\beta_{1,0,1}d_{j-1}\beta_{j+1,i,j+1} \\
&= \Delta_2(\beta_{j+1,i,1}, \beta_{j+1,i,2}) + n_1\beta_{1,0,1}\beta_{j+1,i,3} + n_1\beta_{1,0,1}n_2\beta_{j+1,i,4} + \cdots + n_1\beta_{1,0,1}n_2 \cdots n_{j-1}\beta_{j+1,i,j+1} \\
&= \Delta_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} > e_{j+1,m} - ie_{j,m} = n_1\beta_{1,0,1}d_j(n_{j+1} - i) \text{ by (12.1.2) and (b) of (12.4.12).}
\end{aligned}$$

Equation(2) of (12.4.13).

$$\begin{aligned}
& a\beta_{j+1,i,1} + b\beta_{j+1,i,2} + bn_1\beta_{j+1,i,3} + \rho_{2,m}\beta_{j+1,i,4} + \cdots + \rho_{j-1,m}\beta_{j+1,i,j+1} \\
&= \Omega_2(\beta_{j+1,i,1}, \beta_{j+1,i,2}) + bn_1\beta_{j+1,i,3} + bn_1d_2\beta_{j+1,i,4} + \cdots + bn_1d_{j-1}\beta_{j+1,i,j+1} \\
&= \Omega_2(\beta_{j+1,i,1}, \beta_{j+1,i,2}) + bn_1\beta_{j+1,i,3} + bn_1n_2\beta_{j+1,i,4} + \cdots + bn_1n_2n_3 \cdots n_{j-1}\beta_{j+1,i,j+1} \\
&= \Omega_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} \geq \rho_{j+1,m} - i\rho_{j,m} = bn_1d_j(n_{j+1} - i) \text{ by (12.3.2), (12.3.3) and (b) of (12.4.12).}
\end{aligned}$$

Thus, the proof of (c) is done. Now, it suffices to show that the remaining part (d1) of (d) in this sublemma is true, which just follows from Corollary 3.8. So, the proof of the sublemma is finished. \square

Proof of Sublemma 12.5. First of all, let $\{Y_k : k = 1, 2, \dots, r-1\}$ with $Y_k \subset N_0$, $\{h_k : k = 1, 2, \dots, r-1\}$ with $h_k = (g_{k+1} \circ \tau_m)_{proper}$ in $\mathbb{C}\{v, 1+\bar{u}\}$ and $\{\Xi_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r-1\}$ be three different sequences where each Ξ_k is an integer-valued function, satisfying the given three conditions, denoted by **The 1-th Cond**⁽¹⁾, ..., **The 3-th Cond**⁽¹⁾, in the conclusion of this sublemma. After the proof of Sublemma 12.4 was done, it is easy to observe without any need of the proof that this three sequences with three conditions are well-constructed.

For the proof of this sublemma, it suffices to show that this three sequences satisfy the remaining two conditions **The 4 α -th Cond**⁽¹⁾ and **The 5 α -th Cond**⁽¹⁾ in Conclusions of this sublemma, because there is nothing to prove that the truth of **The 4 α -th Cond**⁽¹⁾ implies that of **The 4-th Cond**⁽¹⁾. So, for the proof, firstly we will prove by **[I]** that **The 4 α -th Cond**⁽¹⁾ is true, and secondly, by **[II]** that **The 5-th Cond**⁽¹⁾ is true.

[I] For the proof of the truth of **The 4 α -th Cond**⁽¹⁾, it remains to prove the second inequality in (12.5.4 α) by using the following three steps: Let ℓ and q be an arbitrary positive integer such that $r-1 \geq \ell \geq q \geq 2$.

$$\begin{aligned}
\text{Step(i)} \quad \Xi_q(\gamma_{\ell,i,k})_{k=1}^q &= \Delta_{q+1}(\beta_{\ell+1,i,k})_{k=1}^{q+1} + n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,0,1} \\
&\quad \times \{\beta_{\ell+1,i,q+2} + n_{q+1}\beta_{\ell+1,i,q+3} + n_{q+1}n_{q+2}\beta_{\ell+1,i,q+4} \\
&\quad + \cdots + n_{q+1}n_{q+2} \cdots n_{\ell-1}\beta_{\ell+1,i,\ell+1} - n_{q+1}n_{q+2} \cdots n_\ell(n_{\ell+1} - i)\}.
\end{aligned}$$

Step(ii) In particular, if $\ell = q$ then

$$\Xi_q(\gamma_{q,i,k})_{k=1}^q = \Delta_{q+1}(\beta_{q+1,i,k})_{k=1}^{q+1} - (n_{q+1} - i)n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,0,1} \quad \text{from Step(i).}$$

$$\begin{aligned}
\text{Step(iii)} \quad \Xi_q(\gamma_{q,i,k})_{k=1}^q &- (s_q - i)s_{q-1}\Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1} \\
&= \Delta_{q+1}(\beta_{q+1,i,k})_{k=1}^{q+1} - (n_{q+1} - i)n_q\Delta_q(\beta_{q,0,k})_{k=1}^q > 0 \quad \text{from Step(ii).}
\end{aligned}$$

We will prove Step(i), Step(ii) and Step(iii) in order, by induction on the integer $q \geq 2$.

So, it is enough to consider two cases, respectively:

Case(I) $q = 2$, and Case(II) $q \geq 2$.

Case(I): Let $q = 2$. Note by **The 3-th Cond**⁽¹⁾ that $\Xi_2(t_1, t_2) = t_2\Xi_1(\gamma_{1,0,1}) + s_1\Xi_1(t_1) = t_2\gamma_{1,0,1} + s_1t_1$ for each $(t_1, t_2) \in N_0^2$.

$$\begin{aligned}
\text{Step(i)} \quad & \Xi_2(\gamma_{\ell,i,1}, \gamma_{\ell,i,2}) \\
&= s_1\gamma_{\ell,i,1} + \gamma_{1,0,1}\gamma_{\ell,i,2} \\
&= n_2\{\Delta_2^\sharp(\beta_{\ell+1,i,1}, \beta_{\ell+1,i,2}) - n_1\beta_{1,0,1}n_2 \cdots n_\ell(n_{\ell+1} - i)\} \\
&\quad + \{\Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2\}\beta_{\ell+1,i,3} \quad \text{by (12.5.1)} \\
&= n_2\{\Delta_2(\beta_{\ell+1,i,1}, \beta_{\ell+1,i,2}) + n_1\beta_{1,0,1}\beta_{\ell+1,i,3} + n_1\beta_{1,0,1}n_2\beta_{\ell+1,i,4} + \cdots \\
&\quad + n_1\beta_{1,0,1}n_2 \cdots n_{\ell-1}\beta_{\ell+1,i,\ell+1} - n_1\beta_{1,0,1}n_2 \cdots n_\ell(n_{\ell+1} - i)\} \\
&\quad + \{\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2\}\beta_{\ell+1,i,3} \quad \text{by (12.1.1)} \\
&= \Delta_3(\beta_{\ell+1,i,1}, \beta_{\ell+1,i,2}, \beta_{\ell+1,i,3}) + n_2^2n_1\beta_{1,0,1}\{\beta_{\ell+1,i,4} + n_3\beta_{\ell+1,i,5} + n_3n_4\beta_{\ell+1,i,6} \\
&\quad + \cdots + n_3n_4 \cdots n_{\ell-1}\beta_{\ell+1,i,\ell+1} - n_3n_4 \cdots n_\ell(n_{\ell+1} - i)\},
\end{aligned}$$

by the definition of $\Delta_3(\beta_{\ell+1,i,1}, \beta_{\ell+1,i,2}, \beta_{\ell+1,i,3})$ only, which implies the proof of Step(i).

Step(ii) In particular, if $\ell = 2$ then an equation of Step(i) gives

$$\Xi_2(\gamma_{2,i,1}, \gamma_{2,i,2}) = \Delta_3(\beta_{3,i,1}, \beta_{3,i,2}, \beta_{3,i,3}) - n_2^2n_1\beta_{1,0,1}(n_3 - i).$$

Thus, the proof of Step(ii) is done.

Step(iii) To prove that $\Xi_2(\gamma_{2,i,1}, \gamma_{2,i,2}) - (s_2 - i)s_1\Xi_1(\gamma_{1,0,1}) = \Delta_3(\beta_{3,i,1}, \beta_{3,i,2}, \beta_{3,i,3}) - (n_3 - i)n_2\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) > 0$, first note by (12.5.1.1) that

$$(s_2 - i)s_1\Xi_1(\gamma_{1,0,1}) = (s_2 - i)s_1\gamma_{1,0,1} = (n_3 - i)n_2\{\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2\}.$$

$$\begin{aligned}
\text{Then,} \quad & \Xi_2(\gamma_{2,i,1}, \gamma_{2,i,2}) - (s_2 - i)s_1\gamma_{1,0,1} \\
&= \Delta_3(\beta_{3,i,1}, \beta_{3,i,2}, \beta_{3,i,3}) - (n_3 - i)n_2^2n_1\beta_{1,0,1} - (n_3 - i)n_2\{\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2\} \\
&= \Delta_3(\beta_{3,i,1}, \beta_{3,i,2}, \beta_{3,i,3}) - (n_3 - i)n_2\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) > 0,
\end{aligned}$$

by Step(ii) and by **The 4-th Cond**⁽⁰⁾ in the assumption of Theorem 12.0, which implies the proof of Step(iii).

Thus, if $q = 2$, then we proved that the second inequality in (12.5.4 α) holds.

Case(II): Let $q \geq 2$. By the induction proof, suppose we have shown that all the equalities of Step(i), Step(ii) and Step(iii) are true on the integer $q \leq r - 1$ with $r - 1 \geq \ell \geq q$.

Then, it is enough to prove Step(i), Step(ii) and Step(iii) in order, on the integer $(q+1) \leq \ell$ as follows:

$$\begin{aligned}
\text{Step(i)} \quad & \Xi_{q+1}(\gamma_{\ell,i,k})_{k=1}^{q+1} \\
&= \gamma_{\ell,i,q+1}\Xi_q(\gamma_{q,0,k})_{k=1}^q + s_q\Xi_q(\gamma_{\ell,i,k})_{k=1}^q \quad \text{by definition of } \Xi_{q+1} \\
&= \beta_{\ell+1,i,q+2}\{\Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} - n_{q+1}n_q^2n_{q-1}^2 \cdots n_2^2n_1\beta_{1,0,1}\} \\
&\quad + n_{q+1}\{\Delta_{q+1}(\beta_{\ell+1,i,k})_{k=1}^{q+1} + n_q^2n_{q-1}^2 \cdots n_2^2n_1\beta_{1,0,1}[\beta_{\ell+1,i,q+2} \\
&\quad + n_{q+1}\beta_{\ell+1,i,q+3} + n_{q+1}n_{q+2}\beta_{\ell+1,i,q+4} + \cdots \\
&\quad + n_{q+1}n_{q+2} \cdots n_{\ell-1}\beta_{\ell+1,i,\ell+1} - n_{q+1}n_{q+2} \cdots n_\ell(n_{\ell+1} - i)]\} \\
&\quad \text{by the induction assumption on the integer } q \\
&= \Delta_{q+2}(\beta_{\ell+1,i,k})_{k=1}^{q+2} + n_{q+1}^2n_q^2n_{q-1}^2 \cdots n_2^2n_1\beta_{1,0,1} \\
&\quad \times \{\beta_{\ell+1,i,q+3} + n_{q+2}\beta_{\ell+1,i,q+4} + n_{q+2}n_{q+3}\beta_{\ell+1,i,q+5} + \cdots \\
&\quad + n_{q+2}n_{q+3} \cdots n_{\ell-1}\beta_{\ell+1,i,\ell+1} - n_{q+2}n_{q+3} \cdots n_\ell(n_{\ell+1} - i)\},
\end{aligned}$$

by the definition of $\Delta_{q+2}(\beta_{\ell+1,i,k})_{k=1}^{q+2}$ only, which implies the proof of Step(i).

Step(ii) In particular, if $\ell = q + 1$, then $\ell + 1 = q + 2 < q + 3$ and so

$$\Xi_{q+1}(\gamma_{q+1,i,k})_{k=1}^{q+1} = \Delta_{q+2}(\beta_{q+2,i,k})_{k=1}^{q+2} - (n_{q+2} - i)n_{q+1}^2 n_q^2 \cdots n_2^2 n_1 \beta_{1,0,1},$$

by Step(i) on the integer $q + 1$, which implies the proof of Step(ii) on the integer $q + 1$.

Step(iii) To prove that the equality in (12.5.4 α) is true, we have

$$\begin{aligned} & \Xi_{q+1}(\gamma_{q+1,i,k})_{k=1}^{q+1} - (s_{q+1} - i)s_q \Xi_q(\gamma_{q,0,k})_{k=1}^q \\ &= \{ \Delta_{q+2}(\beta_{q+2,i,k})_{k=1}^{q+2} - (n_{q+2} - i)n_{q+1}^2 n_q^2 \cdots n_2^2 n_1 \beta_{1,0,1} \} \\ & \quad - (n_{q+2} - i)n_{q+1} \{ \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} - n_{q+1} n_q^2 n_{q-1}^2 \cdots n_2^2 n_1 \beta_{1,0,1} \} \\ &= \Delta_{q+2}(\beta_{q+2,i,k})_{k=1}^{q+2} - (n_{q+2} - i)n_{q+1} \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} > 0, \end{aligned}$$

by Step(ii) on the integer q and $q + 1$, and by **The 4-th Cond**⁽⁰⁾ of Theorem 12.0, which implies the proof of Step(iii).

Thus, we proved that the second inequality in (12.5.3) is true, and so we can finish the proof of the truth of **The 4 α -th Cond**⁽¹⁾.

[II] By The (4 α)-th Cond⁽¹⁾ and The 5-th Cond⁽⁰⁾, the proof of the truth of an equality in (12.5.5 α .1) or (12.5.6.1) of **The 5-th Cond**⁽¹⁾ can be easily done: Note that $r \geq 2$.

$$\begin{aligned} (12.5.6.1) \quad & \gcd(\gamma_{1,0,1}, s_1) = \gcd(\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_2 n_1 \beta_{1,0,1}, n_2) = \gcd(\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}), n_2) = 1, \\ & \gcd(\Xi_q(\gamma_{q,0,k})_{k=1}^q, s_q) \quad \text{for } 2 \leq q \leq r - 1 \\ &= \gcd(\Xi_q(\gamma_{q,0,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1}, s_q) \quad \text{for } 2 \leq q \leq r - 1 \\ &= \gcd(\Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,0,k})_{k=1}^q, n_{q+1}) \quad \text{for } 0 \leq i < s_q. \\ &= \gcd(\Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1}, n_{q+1}) = 1 \quad \text{for } 0 \leq i < s_q. \end{aligned}$$

Moreover, by The (4 α)-th Cond⁽¹⁾ and The 4-th Cond⁽⁰⁾, the proof of the truth of an equality in (12.5.5 α .2) or (12.5.6.2) of **The 5-th Cond**⁽¹⁾ is easily done: For $q = 2, 3, \dots, r - 1$,

$$\begin{aligned} (12.5.6.2) \quad & \frac{\Xi_q(\gamma_{q,i,k})_{k=1}^q - (s_q - i)s_{q-1} \Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1}}{s_q - i} \\ & > \frac{\Xi_q(\gamma_{q,0,k})_{k=1}^q - s_q s_{q-1} \Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1}}{s_q} > 0 \quad \text{for } 0 < i < s_q, \\ \iff & \\ & \frac{\Delta_{q+1}(\beta_{q+1,i,k})_{k=1}^{q+1} - (n_{q+1} - i)n_q \Delta_q(\beta_{q,0,k})_{k=1}^q}{n_{q+1} - i} \\ & > \frac{\Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} - n_{q+1} n_q \Delta_q(\beta_{q,0,k})_{k=1}^q}{n_{q+1}} > 0 \quad \text{for } 0 < i < n_{q+1}. \end{aligned}$$

In particular, if $q = 1$, then note that an inequality in (12.5.6.2) of **The 5-th Cond**⁽¹⁾ holds if and only if the following inequality is true: Note that $s_1 = n_2$.

$$\begin{aligned} (12.5.6.2^*) \quad & \frac{\Xi_1(\gamma_{1,i,1})}{s_1 - i} > \frac{\Xi_1(\gamma_{1,0,1})}{s_1} > 0 \quad \text{for } 0 < i < s_1, \\ \iff & \\ & \frac{\Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) - n_1 \beta_{1,0,1} (n_2 - i)}{n_2 - i} > \frac{\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2}{n_2} > 0 \quad \text{for } 0 < i < n_2, \end{aligned}$$

which is equivalent to an equality in (5b) of **The 5-th Cond**⁽⁰⁾ of the assumption of this theorem.

§13.2. For the proof of Theorem 12.0

Now, we will prove Theorem 12.0 by using five sublemmas. For convenience of the proof of the theorem, we need another additional sublemma as follows.

Sublemma 12.7. Assumptions

(i) Suppose that the same properties and notations as in Assumptions of Theorem 12.0 hold.

(ii) Let r be arbitrary integer with $r \geq 1$. As we have seen in an additional condition of Conclusions of Theorem 12.0, we may assume that the above g_r satisfies all the equalities in (5a) of The 5-th Cond⁽⁰⁾ of [B] of Definition 12.0.0, without mentioning any inequality in (5b).

(iii) In addition, assume that g_s is irreducible in $\mathbb{C}\{y, z\}$ for some integer s where $1 \leq s \leq r$.

Conclusions Then, we have the following:

(a) g_1 is irreducible in $\mathbb{C}\{y, z\}$.

(b) For all $j = 1, 2, \dots, r$, $g_j = g_j(y, z)$ can be written in the form

$$(12.7.1) \quad g_j = (z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}})^{d_j} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(j)} y^\alpha z^\beta \quad \text{with } \varepsilon_{1,0} = 1 \text{ and} \\ \text{with } n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} d_j,$$

where if $j \geq 2$ then $d_j = \prod_{k=2}^j n_k$ with $d_1 = 1$, and a unit $\varepsilon_{1,0} = \varepsilon_{1,0}(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} d_j$.

Note that g_{s+1} may not be irreducible in $\mathbb{C}\{y, z\}$ for a given $s < r$.

Proof of Sublemma 12.7. If the proof of (a) is done, there is nothing prove for (b) because it was already proved by Sublemma 12.2 that for all $j = 1, 2, \dots, r$, $g_j = g_j(y, z)$ can be written in the form (12.7.1). So, the remaining proof of this sublemma is just to prove (a).

(a) Let g_s be irreducible in $\mathbb{C}\{y, z\}$ for some $s \geq 1$. Then, we have two cases:

Case(I) $s = 1$ and Case(II) $s \geq 2$.

Case(I) Let $s = 1$. There is nothing to prove by Sublemma 12.2.

Case(II) Let s be an integer with $2 \leq s \leq r$ such that g_s be irreducible in $\mathbb{C}\{y, z\}$. It suffices to prove that g_1 is irreducible in $\mathbb{C}\{y, z\}$. Assume the contrary, that is, g_1 is not irreducible in $\mathbb{C}\{y, z\}$. Then, it suffices to show by induction on the positive integer j that g_j is not irreducible in $\mathbb{C}\{y, z\}$ for all $j = 1, 2, \dots, r$, because it would be a contradiction to the assumption that g_s is irreducible in $\mathbb{C}\{y, z\}$.

In preparation for the above proof, we use the new notation in the following statements, denoted by Subdefinition 12.7.1 for Case(II) and Sublemma 12.7.2 for Case(II), depending on Case(II):

Subdefinition 12.7.1 for Case(II). Let $f(y, z)$ be analytic at the origin in \mathbb{C}^2 or $f \in \mathbb{C}\{y, z\}$. Then, we write $f(y, z) = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta$ for nonzero complex numbers $c_{\alpha, \beta}$ where α and β are some nonnegative integers. For notation, it is said that either $y^\gamma z^\delta$ belongs to $f(y, z)$ or $y^\gamma z^\delta \in \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta$ if $c_{\gamma, \delta} y^\gamma z^\delta$ is one of nonzero monomials which is in the representation of the convergent power series $\sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta$ of f at the origin in \mathbb{C}^2 . \square

Sublemma 12.7.2 for Case(II). Assumptions of Sublemma 12.7.2 for Case(II)

Whether or not $g_1 = g_1(y, z)$ in The 2-th Cond⁽⁰⁾ is irreducible in $\mathbb{C}\{y, z\}$, suppose by Assumptions of Theorem 12.0 that g_1 can be written in the form $g_1 = z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} + \sum_{i=1}^{n_1-1} \varepsilon_{1,i} y^{\beta_{1,i,1}} z^i$, where the $\beta_{1,i,1}$ are positive integers for $0 \leq i < n_1$, and $\varepsilon_{1,0} = \varepsilon_{1,0}(y, z)$ is a unit, and each $\varepsilon_{1,i} = \varepsilon_{1,i}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $0 < i < n_j$, if exists. Note that $\gcd(n_1, \beta_{1,0,1}) = 1$.

In addition, it was assumed in the process for the proof of Case(II) that g_1 is not irreducible in $\mathbb{C}\{y, z\}$.

Conclusions of Sublemma 12.7.2 for Case(II)

Then, g_1 satisfies Fact(i) and Fact(ii), as follows.

Fact(i): It is clear by $g_1 = g_1(y, z)$ in The 2-th Cond⁽⁰⁾ in the assumption of this theorem that $g_1 = g_1(y, z)$ can be written in the form

$$(12.7.2.1) \quad g_1 = \xi_{0,1}z^{n_1} + \xi_{1,0}y^{\beta_{1,0,1}} + \sum_{i=1}^{n_1-1} c_{1,i}y^{\alpha_i}z^i,$$

where $\xi_{0,1} = \xi_{0,1}(y, z)$ and $\xi_{1,0} = \xi_{1,0}(y, z)$ are units in $\mathbb{C}\{y, z\}$, and the $c_{1,i}$ are units in $\mathbb{C}\{y, z\}$ and the α_i are positive integers with $0 < \alpha_i < \beta_{1,0,1}$ for $1 \leq i < n_1$, and z^{n_1} and $y^{\beta_{1,0,1}}$ do not belong to $\sum_{i=1}^{n_1-1} c_i y^{\alpha_i} z^i$.

Fact(ii): By Theorem 3.5 and Sublemma 12.4, g_1 in Fact(i) may be rewritten in the form

$$(12.7.2.2) \quad g_1 = g_{1,1} + g_{1,2} \quad \text{with} \\ g_{1,1} = g_{1,1}(y, z) = (az^{n_1} + by^{\beta_{1,0,1}}) \quad \text{and} \quad g_{1,2} = g_{1,2}(y, z) = \sum_{\alpha, \beta \geq 0} c_{1,\alpha,\beta}^{(1)} y^\alpha z^\beta, \\ \text{such that} \quad z^{n_1} \notin g_{1,2} \quad \text{and} \quad y^{\beta_{1,0,1}} \notin g_{1,2} \quad \text{and} \\ \text{some} \quad y^{\alpha_0} z^{\beta_0} \in g_{2,2} \quad \text{with} \quad n_1 \alpha_0 + \beta_{1,0,1} \beta_0 < n_1 \beta_{1,0,1},$$

where a and b are nonzero complex numbers, and the $c_{1,\alpha,\beta}^{(1)}$ are nonzero complex numbers for some nonnegative integers α and β , if exists and $g_{1,2}(y, 0) = \eta_{1,0} \cdot y^k$ for some integer $k > \beta_{1,0,1}$ and $g_{1,2}(0, z) = \eta_{0,1} \cdot z^\ell$ for some integer $\ell > n_1$, and some units $\eta_{1,0}$ and $\eta_{0,1}$ in $\mathbb{C}\{y, z\}$, if exists, satisfying the following property:

(12.7.2.3) Let $m = \min\{n_1 \alpha + \beta_{1,0,1} \beta : y^\alpha z^\beta \text{ is arbitrary nonzero monomial in } g_1\}$. Then, there is a unique nonzero monomial $y^{\alpha_1} z^{\beta_1} \in g_{1,2}$ of g_1 such that $n_1 \alpha_0 + \beta_{1,0,1} \beta_0 = m < n_1 \beta_{1,0,1}$.

Moreover, for any integer $\ell > 0$, $g_1^\ell = (g_{1,1} + g_{1,2})^\ell$ can be rewritten as follows:

$$(12.7.2.4) \quad g_1^\ell = (g_{1,1})^\ell + \Sigma_{1,\ell} \quad \text{with} \quad \Sigma_{1,\ell} = \sum_{\gamma > 0, \delta > 0} c_{\ell,\gamma,\delta}^{(1)} y^\gamma z^\delta,$$

where the $c_{\ell,\gamma,\delta}^{(1)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \gamma + \beta_{1,0,1} \delta \neq \ell n_1 \beta_{1,0,1}$ with the following property:

- (b1) $(y^{\alpha_0} z^{\beta_0})^\ell$ is the unique nonzero monomial in $\Sigma_{1,\ell}$ of g_1^ℓ such that $\ell(n_1 \alpha_0 + \beta_{1,0,1} \beta_0) = \ell m < \ell(n_1 \beta_{1,0,1})$ where $\ell m = \min\{n_1 \alpha + \beta_{1,0,1} \beta : y^\alpha z^\beta \text{ is arbitrary nonzero monomials in } g_1^\ell\}$.
- (b2) $z^{\ell n_1} \in g_1^\ell$ and $y^{\ell \beta_{1,0,1}} \in g_1^\ell$, but $z^{\ell n_1} \notin \Sigma_{1,\ell}$ and $y^{\ell \beta_{1,0,1}} \notin \Sigma_{1,\ell}$. \square

Proof of Sublemma 12.7.2 for Case(II) It suffices to prove that the property in (12.7.2.3) is true. By Theorem 3.2 and Corollary 3.3, there is a nonzero monomial $y^\alpha z^\beta \in g_1$ in Sublemma 12.7.2 for Case(II) such that $n_1 \alpha + \beta_{1,0,1} \beta < n_1 \beta_{1,0,1}$ with $1 \leq \alpha < \beta_{1,0,1}$ and $1 \leq \beta < n_1$. Let $m = \min\{n_1 \alpha + \beta_{1,0,1} \beta : n_1 \alpha + \beta_{1,0,1} \beta < n_1 \beta_{1,0,1}\}$ such that $y^\alpha z^\beta \in g_1$. First, consider $S = \{(\alpha, \beta) : n_1 \alpha + \beta_{1,0,1} \beta = m\}$ such that $y^\alpha z^\beta \in g_1$. Next, define $\beta_0 = \min\{\beta : (\alpha, \beta) \in S \text{ is arbitrary}\}$ such that $n_1 \alpha_0 + \beta_{1,0,1} \beta_0 = m$ for some integer α_0 . Since it is clear that α_0 is unique for such β_0 , there is nothing to prove because $n_1 \alpha + \beta_{1,0,1} \beta < n_1 \beta_{1,0,1}$ with $1 \leq \alpha < \beta_{1,0,1}$ and $1 \leq \beta < n_1$ and $\gcd(n_1, \beta_{1,0,1}) = 1$. \square

Now, for the proof of the remaining for Case(II), using Subdefinition 12.7.1 and Sublemma 12.7.2 for Case(II), we show by induction on the poistive integer $j = 2, 3, \dots, r$ that g_j is not irreducible in $\mathbb{C}\{y, z\}$. To prove that g_j is not irreducible in $\mathbb{C}\{y, z\}$ for each $j = 2, 3, \dots, r$, it suffices to show by Theorem 3.2 and Theorem 3.5 that the following sublemma, called Sublemma 12.7.3 for Case(II), is true.

Sublemma 12.7.3 for Case(II). Assumptions of Sublemma 12.7.3 for Case(II) Let $j \geq 2$. To prove that g_j is not irreducible in $\mathbb{C}\{y, z\}$ for each $j = 2, 3, \dots, r$, we may use the same properties and notations as in the assumptions and conclusions of Sublemma 12.7.2 for Case(II) because g_1 is not irreducible in $\mathbb{C}\{y, z\}$.

For brevity of notation, g_j in The 2-th Cond⁽⁰⁾ in the assumption of this theorem can be rewritten as follows:

$$(12.7.3.0) \quad g_j = \sum_{j,0} + \sum_{j,1} + \sum_{j,2} \quad \text{with}$$

$$\sum_{j,0} = g_{j-1}^{n_j}, \quad \sum_{j,1} = \varepsilon_{j,0} \prod_{k=1}^j g_{k-2}^{\beta_{j,0,k}},$$

$$\sum_{j,2} = \sum_{i=1}^{n_j-1} \varepsilon_{j,i} \left(\prod_{k=1}^j g_{k-2}^{\beta_{j,i,k}} \right) g_{j-1}^i, \quad \text{where } g_{-1} = y \text{ and } g_0 = z.$$

Note that each $\varepsilon_{j,i} = \varepsilon_{j,i}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq j \leq r$ and $0 \leq i < n_j$, if exists.

Conclusions of Sublemma 12.7.3 for Case(II) For each $j = 2, 3, \dots, r$, it suffices to show by Theorem 3.2 and Theorem 3.5 that g_j satisfies Fact(i) and Fact(ii), as follows.

Fact(i): By the same properties and notations as in the assumption of Theorem 12.0 and Sublemma 12.7.2, $g_j = g_j(y, z)$ can be written in the form

$$(12.7.3.1) \quad g_j = \xi_{0,j} z^{d_j n_1} + \xi_{j,0} y^{d_j \beta_{1,0,1}} + \sum_{i=1}^{d_j n_1 - 1} c_{j,i} y^{\alpha_i} z^i,$$

where $\xi_{0,j} = \xi_{0,j}(y, z)$ and $\xi_{j,0} = \xi_{j,0}(y, z)$ are units in $\mathbb{C}\{y, z\}$, and the $c_{j,i}$ are units in $\mathbb{C}\{y, z\}$ and the α_i are positive integers with $0 < \alpha_i < d_j \beta_{1,0,1}$ for $1 \leq i < d_j n_1$, and $z^{d_j n_1}$ and $y^{d_j \beta_{1,0,1}}$ do not belong to $\sum_{i=1}^{d_j n_1 - 1} c_{j,i} y^{\alpha_i} z^i$.

Note that $g_j(y, 0) = \xi_{j,0}(y, 0) \cdot y^{d_j \beta_{1,0,1}}$ and $g_j(0, z) = \xi_{0,j}(0, z) \cdot z^{d_j n_1}$ are units in $\mathbb{C}\{y, z\}$ where $\xi_{0,j}(0, y)$ and $\xi_{j,0}(y, 0)$ are units in $\mathbb{C}\{y, z\}$.

Fact(ii): By Theorem 3.5 and Sublemma 12.7.2, g_j in Fact(i) may be rewritten in the form

$$(12.7.3.2) \quad g_j = g_{j,1} + g_{j,2}$$

$$g_{j,1} = g_{j,1}(y, z) = (az^{n_1} + by^{\beta_{1,0,1}})^{d_j} \text{ and } g_{j,2} = g_{j,2}(y, z) = \sum_{\alpha, \beta \geq 0} c_{1,\alpha,\beta}^{(j)} y^\alpha z^\beta$$

$$\text{with } z^{d_j n_1} \notin g_{j,2} \quad \text{and} \quad y^{d_j \beta_{1,0,1}} \notin g_{j,2},$$

$$\text{some } y^{d_j \alpha_0} z^{d_j \beta_0} \in g_{j,2} \quad \text{with} \quad n_1 d_j \alpha_0 + \beta_{1,0,1} d_j \beta_0 < n_1 \beta_{1,0,1} d_j$$

where a and b are nonzero complex numbers, and the $c_{1,\alpha,\beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β , if exists and $g_{j,2}(y, 0) = \eta_{1,0} \cdot y^k$ for some integer $k > d_j \beta_{1,0,1}$ and $g_{j,2}(0, z) = \eta_{0,1} \cdot z^\ell$ for some integer $\ell > d_j n_1$. Note that $\eta_{1,0}$ and $\eta_{0,1}$ are units in $\mathbb{C}\{y, z\}$. \square

Proof of Sublemma 12.7.3 for Case(II): The proof will be induction on a positive integer j where $2 \leq j \leq r$. Then, there are two subcases.

Case(II-a) $j = 2$ and Case(II-b) $j \geq 2$.

For convenience of notation, for the proof of Case(II-a) we will prove The statement for Case(II-a), and for the proof of Case(II-b) we will prove The Statement for Case(II-b), respectively.

The Statement for Case(II-a): Assumptions Suppose that g_1 is not irreducible in $\mathbb{C}\{y, z\}$.

Conclusions Then, g_2 satisfies Fact(i) and Fact(ii), as follows.

Fact(i): By the same properties and notations as in Sublemma 12.7.2, $g_2 = g_2(y, z)$ can be written in the form

$$(12.7.3.3) \quad g_2 = \xi_{0,2} z^{d_2 n_1} + \xi_{2,0} y^{d_2 \beta_{1,0,1}} + \sum_{i=1}^{d_2 n_1 - 1} c_{2,i} y^{\alpha_i} z^i \quad \text{with } d_2 = n_2,$$

where $\xi_{0,2} = \xi_{0,2}(y, z)$ and $\xi_{2,0} = \xi_{2,0}(y, z)$ are units in $\mathbb{C}\{y, z\}$, and the $c_{2,i}$ are units in $\mathbb{C}\{y, z\}$ and the α_i are positive integers with $0 < \alpha_i < d_2 \beta_{1,0,1}$ for $1 \leq i < d_2 n_1$, and $z^{d_2 n_1}$ and $y^{d_2 \beta_{1,0,1}}$ do not belong to $\sum_{i=1}^{d_2 n_1 - 1} c_{2,i} y^{\alpha_i} z^i$.

Note that $g_2(y, 0) = \xi_{2,0}(y, 0) \cdot y^{d_2 \beta_{1,0,1}}$ and $g_2(0, z) = \xi_{0,2}(0, z) \cdot z^{d_2 n_1}$ where $\xi_{2,0}(y, 0)$ and $\xi_{0,2}(0, z)$ are units in $\mathbb{C}\{y, z\}$.

Fact(ii): By Theorem 3.5 and Sublemma 12.4, g_j in Fact(i) may be rewritten in the form

$$(12.7.3.4) \quad g_2 = g_{2,1} + g_{2,2} \quad \text{with}$$

$$g_{2,1} = g_{2,1}(y, z) = (az^{n_1} + by^{\beta_{1,0,1}})^{d_2} \quad \text{and} \quad g_{2,2} = g_{2,2}(y, z) = \sum_{\alpha, \beta \geq 0} c_{1,\alpha,\beta}^{(2)} y^\alpha z^\beta,$$

such that $z^{d_2 n_1} \notin g_{2,2}$ and $y^{d_2 \beta_{1,0,1}} \notin g_{2,2}$ and

some $y^{d_2 \alpha_0} z^{d_2 \beta_0} \in g_{2,2}$ with $n_1 d_2 \alpha_0 + \beta_{1,0,1} d_2 \beta_0 < n_1 \beta_{1,0,1} d_2$,

where a and b are nonzero complex numbers, and the $c_{1,\alpha,\beta}^{(2)}$ are nonzero complex numbers for some nonnegative integers α and β , if exists and $g_{2,2}(y, 0) = \eta_{1,0} \cdot y^k$ for some integer $k > d_2 \beta_{1,0,1}$ and $g_{2,2}(0, z) = \eta_{0,1} \cdot z^\ell$ for some integer $\ell > d_2 n_1$. Note that $\eta_{1,0}$ and $\eta_{0,1}$ are units in $\mathbb{C}\{y, z\}$. \square .

Proof of The Statement for Case(II-a): For brevity of notation, g_2 in The 2-th Cond⁽⁰⁾ in the assumption of this theorem can be rewritten as follows:

$$(12.7.3.5) \quad g_2 = \sum_{2,0} + \sum_{2,1} + \sum_{2,2} \quad \text{with}$$

$$\sum_{2,0} = g_1^{n_2}, \quad \sum_{2,1} = \varepsilon_{2,0} y^{\beta_{2,0,1}} z^{\beta_{2,0,2}} \quad \text{and} \quad \sum_{2,2} = \sum_{i=1}^{n_2-1} \varepsilon_{2,i} y^{\beta_{2,i,1}} z^{\beta_{2,i,2}} g_1^i.$$

By The 4-th Cond⁽⁰⁾ in the assumption of this theorem and Sublemma 12.7.2, observe the following:

$$(12.7.3.6) \quad \Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) > n_1 \beta_{1,0,1} n_2 > mn_2, \text{ and}$$

$$\Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) + in_1 \beta_{1,0,1} > (n_2 - i) n_1 \beta_{1,0,1} + in_1 \beta_{1,0,1} = n_2 n_1 \beta_{1,0,1} > n_2 m.$$

To prove that g_2 satisfies Fact(i), it suffices to show by (12.7.3.5) that the following are true:

(a) It is clear by (12.7.2.4) of Sublemma 12.7.2 that $z^{n_2 n_1} \in \sum_{2,0} = g_1^{n_2}$ and $y^{n_2 \beta_{1,0,1}} \in \sum_{2,0}$.

(b) By (iii) of Sublemma 12.1, we are going to show that $z^{n_2 n_1} \notin \sum_{2,1}$ and $z^{n_2 n_1} \notin \sum_{2,2}$, and $y^{n_2 \beta_{1,0,1}} \notin \sum_{2,1}$ and $y^{n_2 \beta_{1,0,1}} \notin \sum_{2,2}$, respectively.

(b1) To show that $z^{n_2 n_1} \notin \sum_{2,1} = \varepsilon_{2,0} y^{\beta_{2,0,1}} z^{\beta_{2,0,2}}$, it suffices to prove that the following is true, which is trivial:

If $\beta_{2,0,1} = 0$ then $\beta_{2,0,2} > n_2 n_1$, because it is clear by (12.7.3.6) that $\beta_{1,0,1} \beta_{2,0,2} = n_1 \beta_{2,0,1} + \beta_{1,0,1} \beta_{2,0,2} = \Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) > n_1 \beta_{1,0,1} n_2$ implies $\beta_{2,0,2} > n_2 n_1$.

Similarly, it can be proved that $y^{n_2 \beta_{1,0,1}} \notin \sum_{2,1}$.

(b2) To show that $z^{n_2 n_1} \notin \sum_{2,2} = \sum_{i=1}^{n_2-1} \varepsilon_{2,i} y^{\beta_{2,i,1}} z^{\beta_{2,i,2}} g_1^i$, it suffices to prove that the following is true, which is trivial:

If $\beta_{2,i,1} = 0$ then $\beta_{2,i,2} > (n_2 - i) n_1$, because it is clear that $\beta_{1,0,1} \beta_{2,i,2} = n_1 \beta_{2,i,1} + \beta_{1,0,1} \beta_{2,i,2} = \Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) > (n_2 - i) n_1 \beta_{1,0,1}$ implies $\beta_{2,i,2} > (n_2 - i) n_1$. So, $\beta_{2,i,2} + in_1$ is an integer of $\min\{\delta: y^\gamma z^\delta \in \sum_{2,2} \text{ with } \gamma = 0\}$, which is greater than $n_2 n_1$.

Similarly, it can be proved that $y^{n_2\beta_{1,0,1}} \notin \sum_{2,2}$.

To prove that g_2 satisfies Fact(ii), it suffices to show that the following are true:

(a) It is clear by (12.7.2.4) of Sublemma 12.7.2 that $(y^{\alpha_0} z^{\beta_0})^{n_2}$ is the unique nonzero monomial in Σ_{1,n_2} of $g_1^{n_2}$ such that $n_2(n_1\alpha_0 + \beta_{1,0,1}\beta_0) = n_2m < n_2(n_1\beta_{1,0,1})$ such that $n_2m < n_1\alpha + \beta_{1,0,1}\beta$ for any nonzero monomial $y^\alpha z^\beta \in g_1^{n_2}$.

(b) It is clear by (12.7.3.6) that $n_2m < n_1\alpha + \beta_{1,0,1}\beta$ for any nonzero monomial $y^\alpha z^\beta \in \sum_{2,1}$, and also $n_2m < n_1\alpha + \beta_{1,0,1}\beta$ for any nonzero monomial $y^\alpha z^\beta \in \sum_{2,2}$. Thus, the proof of The Statement for Case(II-a) is done.

The Statement for Case(II-b): Assumptions Suppose that g_j satisfies Fact(i) and Fact(ii) for $j = 2, 3, \dots, w$ with $w < r$.

Conclusions For brevity of notation, g_{w+1} in The 2-th Cond⁽⁰⁾ in the assumption of this theorem can be rewritten as follows:

$$(12.7.3.7) \quad g_{w+1} = \sum_{w+1,0} + \sum_{w+1,1} + \sum_{w+1,2}$$

$$\sum_{w+1,0} = g_w^{n_{w+1}}, \quad \sum_{w+1,1} = \varepsilon_{w+1,0} \prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,0,k}},$$

$$\sum_{w+1,2} = \sum_{i=1}^{n_{w+1}-1} \varepsilon_{w+1,i} \left(\prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,i,k}} \right) g_w^i, \quad \text{where } g_{-1} = y \text{ and } g_0 = z.$$

To prove that g_{w+1} is not irreducible in $\mathbb{C}\{y, z\}$, it suffices to show by Theorem 3.3 and Theorem 3.5 that g_{w+1} satisfies Fact(i) and Fact(ii), as follows.

Fact(i): By the same properties and notations as in the assumption of Theorem 12.0 and Sublemma 12.7.2, $g_{w+1} = g_{w+1}(y, z)$ can be written in the form

$$(12.7.3.8) \quad g_{w+1} = \xi_{0,w+1} z^{d_{w+1}n_1} + \xi_{w+1,0} y^{d_{w+1}\beta_{1,0,1}} + \sum_{i=1}^{d_{w+1}n_1-1} c_{w+1,i} y^{\alpha_i} z^i,$$

where $\xi_{0,w+1} = \xi_{0,w+1}(y, z)$ and $\xi_{w+1,0} = \xi_{w+1,0}(y, z)$ are units in $\mathbb{C}\{y, z\}$, and the $c_{w+1,i}$ are units in $\mathbb{C}\{y, z\}$ and the α_i are positive integers with $0 < \alpha_i < d_{w+1}\beta_{1,0,1}$ for $1 \leq i < d_{w+1}n_1$, and $z^{d_{w+1}n_1}$ and $y^{d_{w+1}\beta_{1,0,1}}$ do not belong to $\sum_{i=1}^{d_{w+1}n_1-1} c_{w+1,i} y^{\alpha_i} z^i$.

Note that $g_{w+1}(y, 0) = \xi_{w+1,0}(y, 0) \cdot y^{d_{w+1}\beta_{1,0,1}}$ and $g_{w+1}(0, z) = \xi_{0,w+1}(0, z) \cdot z^{d_{w+1}n_1}$ where $\xi_{0,w+1}(0, y)$ and $\xi_{w+1,0}(y, 0)$ are units in $\mathbb{C}\{y, z\}$.

Fact(ii): By Theorem 3.5 and Sublemma 12.7.2, g_{w+1} in Fact(i) may be rewritten in the form

$$(12.7.3.9) \quad g_{w+1} = g_{w+1,1} + g_{w+1,2} \quad \text{with } g_{w+1,1} = g_{w+1,1}(y, z) \text{ and } g_{w+1,2} = g_{w+1,2}(y, z)$$

$$g_{w+1,1} = (az^{n_1} + by^{\beta_{1,0,1}})^{d_{w+1}} \text{ and } g_{w+1,2} = \sum_{\alpha, \beta \geq 0} c_{1,\alpha,\beta}^{(w+1)} y^\alpha z^\beta$$

$$\text{with } z^{d_{w+1}n_1} \notin g_{w+1,2} \quad \text{and} \quad y^{d_{w+1}\beta_{1,0,1}} \notin g_{w+1,2},$$

$$\text{some } y^{d_{w+1}\alpha_0} z^{d_{w+1}\beta_0} \in g_{w+1,2} \quad \text{with} \quad d_{w+1}(n_1\alpha_0 + \beta_{1,0,1}\beta_0) < n_1\beta_{1,0,1}d_{w+1},$$

where a and b are nonzero complex numbers, and the $c_{1,\alpha,\beta}^{(w+1)}$ are nonzero complex numbers for some nonnegative integers α and β , if exists and $g_{w+1,2}(y, 0) = \eta_{1,0} \cdot y^k$ for some integer $k > d_{w+1}\beta_{1,0,1}$ and $g_{w+1,2}(0, z) = \eta_{0,1} \cdot z^\ell$ for some integer $\ell > d_{w+1}n_1$. Note that $\eta_{1,0}$ and $\eta_{0,1}$ are units in $\mathbb{C}\{y, z\}$. \square

Proof of The statement for Case(II-b):

(a) Then, it is clear by (12.7.3.1) and (12.7.2.4) of Sublemma 12.7.2 that $\sum_{w+1,0} = g_w^{n_{w+1}}$ can be written as follows:

$\sum_{w+1,0} = (\xi_{0,w} z^{d_w n_1} + \xi_{w,0} y^{d_w \beta_{1,0,1}})^{n_{w+1}} + \sum_{\alpha > 0, \beta > 0} c_{w,\alpha,\beta}^{(w+1)} y^\alpha z^\beta$ where $z^{d_{w+1} n_1}$ and $y^{d_{w+1} \beta_{1,0,1}}$ do not belong to $\sum_{\alpha > 0, \beta > 0} c_{1,\alpha,\beta}^{(w+1)} y^\alpha z^\beta$. Note that $\xi_{0,w}$ and $\xi_{w,0}$ are units in $\mathbb{C}\{y, z\}$, and the $c_{w,\alpha,\beta}^{(w+1)}$ are units in $\mathbb{C}\{y, z\}$.

To prove that g_{w+1} satisfies Fact(i), it suffices to show by (12.7.3.7) that the following are true:

It is clear by (a) that $z^{d_{w+1} n_1} \in \sum_{w+1,0} = g_w^{n_{w+1}}$ and $y^{d_{w+1} \beta_{1,0,1}} \in \sum_{w+1,0}$.

(b) By (iii) of Sublemma 12.1, we show that $z^{d_{w+1} n_1} \notin \sum_{w+1,1}$ and $z^{d_{w+1} n_1} \notin \sum_{w+1,2}$, and $y^{d_{w+1} \beta_{1,0,1}} \notin \sum_{w+1,1}$ and $y^{d_{w+1} \beta_{1,0,1}} \notin \sum_{w+1,2}$ respectively, as follows.

(b1) To show that $z^{d_{w+1} n_1} \notin \sum_{w+1,1} = \varepsilon_{w+1,0} \prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,0,k}}$ where $g_{-1} = y$ and $g_0 = z$, it suffices to prove that the following is true:

Let $\sum_{w+1,1} = \sum_{w+1,1}(y, z)$, and then write $\sum_{w+1,1}(0, z) = \xi z^p$ is in $\mathbb{C}\{y, z\}$ where ξ is a unit in $\mathbb{C}\{z\}$. It suffices to prove by Sublemma 12.1 that $p > d_{w+1} n_1$. It is clear by construction of p that $p = \beta_{w+1,0,2} + n_1 \beta_{w+1,0,3} + n_1 n_2 \beta_{w+1,0,4} + \cdots + n_1 n_2 \cdots n_{w-1} \beta_{w+1,0,w+1}$.

Note by (12.1.1) of Sublemma 12.1 that $\Delta_j^\#(\beta_{j,0,k})_{k=1}^j = n_1 \beta_{j,0,1} + \beta_{1,0,1} \beta_{j,0,2} + n_1 \beta_{1,0,1} \beta_{j,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{j,i,4} + n_1 \beta_{1,0,1} n_2 n_3 \beta_{j,i,5} + \cdots + n_1 \beta_{1,0,1} n_2 \cdots n_{j-2} \beta_{j,i,j} > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} n_j$ for any $j \leq w+1$.

In particular, if $\beta_{j,0,1} = 0$ with $j = w+1$, then $\Delta_2^\#(\beta_{j,0,k})_{k=1}^j = p \beta_{1,0,1} > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_j$. Thus, it can be proved that $p > n_1 n_2 n_3 \cdots n_{j-1} n_j = n_1 d_{w+1}$.

Similarly, it can be proved that $y^{d_{w+1} \beta_{1,0,1}} \notin \sum_{w+1,1}$.

(b2) To show that $z^{d_{w+1} n_1} \notin (\prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,i,k}}) g_w^i$, one of nonzero monomials of $\sum_{w+1,2}$, it suffices to prove that the following is true:

Let $\sum_{w+1,2} = \sum_{w+1,2}(y, z)$, and then write $\sum_{w+1,2}(0, z) = \xi z^q$ is in $\mathbb{C}\{y, z\}$ where ξ is a unit in $\mathbb{C}\{z\}$. It suffices to prove by Sublemma 12.1 that $q > d_{w+1} n_1$.

It is clear by construction of q that $q = \beta_{w+1,i,2} + n_1 \beta_{w+1,i,3} + n_1 n_2 \beta_{w+1,i,4} + \cdots + n_1 n_2 \cdots n_{w-1} \beta_{w+1,i,w+1} + n_1 n_2 n_3 \cdots n_{j-1} i$.

Note by (12.1.1) of Sublemma 12.1 that $\Delta_j^\#(\beta_{j,i,k})_{k=1}^j + n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} i = n_1 \beta_{j,i,1} + \beta_{1,0,1} \beta_{j,i,2} + n_1 \beta_{1,0,1} \beta_{j,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{j,i,4} + \cdots + n_1 \beta_{1,0,1} n_2 \cdots n_{j-2} \beta_{j,i,j} + n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} i > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} n_j$ for any $j \leq w+1$.

In particular, if $\beta_{j,i,1} = 0$ with $j = w+1$, it is clear that $\Delta_j^\#(\beta_{j,i,k})_{k=1}^j + n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} i = q \beta_{1,0,1} > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} n_j$. Thus, it can be proved that $q > n_1 n_2 n_3 \cdots n_{j-1} n_j$ with $j = w+1$.

Similarly, it can be proved that $y^{d_{w+1} \beta_{1,0,1}} \notin \sum_{w+1,2}$.

To prove that g_{w+1} satisfies Fact(ii), it suffices to show by (12.7.3.7) that the following is true:

(12.7.3.10) There is a monomial $y^{d_{w+1} \alpha_0} z^{d_{w+1} \beta_0} \in g_w^{n_{w+1}}$

with $y^{d_{w+1} \alpha_0} z^{d_{w+1} \beta_0} \notin \sum_{w+1,1}$ and $y^{d_{w+1} \alpha_0} z^{d_{w+1} \beta_0} \notin \sum_{w+1,2}$

such that $n_1 d_{w+1} \alpha_0 + \beta_{1,0,1} d_{w+1} \beta_0 < n_1 \beta_{1,0,1} d_{w+1}$.

(a) It is clear by Sublemma 12.7.2, Sublemma 12.7.3.1 and Sublemma 12.7.3.2 that $y^{d_{w+1} \alpha_0} z^{d_{w+1} \beta_0}$ is the unique nonzero monomial in $g_w^{n_{w+1}} = \sum_{w+1,0}$ such that $d_{w+1}(n_1 \alpha_0 + \beta_{1,0,1} \beta_0) = d_{w+1} m < d_{w+1}(n_1 \beta_{1,0,1})$ such that $d_{w+1} m \leq d_{w+1}(n_1 \alpha + \beta_{1,0,1} \beta)$ for any nonzero monomial $y^\alpha z^\beta \in g_w^{n_{w+1}}$.

(b) It is clear by Sublemma 12.1 that $d_{w+1} m \leq d_{w+1}(n_1 \alpha + \beta_{1,0,1} \beta)$ for any nonzero monomial $y^\alpha z^\beta \in \sum_{w+1,1}$, and also $d_{w+1} m < d_{w+1}(n_1 \alpha + \beta_{1,0,1} \beta)$ for any nonzero monomial $y^\alpha z^\beta \in \sum_{w+1,2}$, because of the following:

Note by (12.1.1) of Sublemma 12.1 that $\Delta_j^\#(\beta_{j,i,k})_{k=1}^j + n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} i = n_1 \beta_{j,i,1} + \beta_{1,0,1} \beta_{j,i,2} + n_1 \beta_{1,0,1} \beta_{j,i,3} + n_1 \beta_{1,0,1} n_2 \beta_{j,i,4} + \cdots + n_1 \beta_{1,0,1} n_2 \cdots n_{j-2} \beta_{j,i,j} + n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} i > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_{j-1} n_j$ for any j .

(b1) It suffices to prove that for any nonzero monomial $y^\alpha z^\beta \in \sum_{w+1,1} = \varepsilon_{w+1,0} \prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,0,k}}$, $n_1 \alpha + \beta_{1,0,1} \beta > d_{w+1} m$. For brevity of notation, if $j = w+1$ then it remains to prove that

$n_1\beta_{j,0,1} + \beta_{1,0,1}\beta_{j,0,2} + m\beta_{j,0,3} + mn_2\beta_{j,0,4} + \cdots + mn_2 \cdots n_{j-2}\beta_{j,0,j} > mn_2n_3 \cdots n_{j-1}n_j$ for any j where $m = n_1\alpha_0 + \beta_{1,0,1}\beta_0$, that is, $\Delta_2(\beta_{j,0,1}, \beta_{j,0,2}) > mD$, where $D = \{n_2n_3 \cdots n_{j-1}n_j - (\beta_{j,i,3} + n_2\beta_{j,i,4} + n_2n_3\beta_{j,i,5} + \cdots + n_2 \cdots n_{j-2}\beta_{j,i,j})\}$.

Note by (12.1.1) of Sublemma 12.1 that $\Delta_j^\sharp(\beta_{j,0,k})_{k=1}^j > n_1\beta_{1,0,1}n_2n_3 \cdots n_{j-1}n_j$ can be rewritten by $\Delta_2(\beta_{j,0,1}, \beta_{j,0,2}) > n_1\beta_{1,0,1}D$. Then, it is clear that $\Delta_2(\beta_{j,0,1}, \beta_{j,0,2}) > mD$ whether or not D is positive.

(b2) It suffices to prove that for any nonzero monomial $y^\alpha z^\beta \in \varepsilon_{w+1,i} \prod_{k=1}^{w+1} g_{k-2}^{\beta_{w+1,i,k}} g_{w+1}^i$ of $\sum_{w+1,2}$, $n_1\alpha + \beta_{1,0,1}\beta > d_{w+1}m$. In more detail, it remains to prove that $n_1\beta_{j,i,1} + \beta_{1,0,1}\beta_{j,i,2} + m\beta_{j,i,3} + mn_2\beta_{j,i,4} + \cdots + mn_2 \cdots n_{j-2}\beta_{j,i,j} + mn_2n_3 \cdots n_{j-1}i > mn_2n_3 \cdots n_{j-1}n_j$ for any $j \leq w+1$, that is, $\Delta_2(\beta_{j,i,1}, \beta_{j,i,2}) > mD$ where $m = n_1\alpha_0 + \beta_{1,0,1}\beta_0$ and $D = \{n_2n_3 \cdots n_{j-1}(n_j - i) - (\beta_{j,i,3} + n_2\beta_{j,i,4} + n_2n_3\beta_{j,i,5} + \cdots + n_2 \cdots n_{j-2}\beta_{j,i,j})\}$ for $0 \leq i < n_j$.

Note by (12.1.1) of Sublemma 12.1 that $\Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j > n_1\beta_{1,0,1}n_2n_3 \cdots n_{j-1}(n_j - i)$ can be rewritten by $\Delta_2(\beta_{j,i,1}, \beta_{j,i,2}) > n_1\beta_{1,0,1}D$. Since $n_1\beta_{1,0,1} > m = n_1\alpha_0 + \beta_{1,0,1}\beta_0$, then, it is clear that $\Delta_2(\beta_{j,i,1}, \beta_{j,i,2}) > mD$ whether or not D is positive. Thus, the proof of The Statement for Case(II-b) is done. So, the proof of Sublemma 12.7.3 for Case(II) is finished.

Proof of Theorem 12.0. For example, let $r = 1$. By Theorem 3.2 and Corollary 3.3, there is nothing to prove for $[A]$. For the proof of $[B]$, if g_1 is irreducible in $\mathbb{C}\{y, z\}$ with $n_1 \geq 2$, then $y^\gamma g_1 \in$ the type[1] under the standard resolution by either Sublemma 12.4 or Theorem 3.6, where if $\beta_{101} = 1$ then $\gamma = 1$ and if $\beta_{101} > 1$ then $\gamma = 0$. In particular, $z^\delta y g_1 \in$ the type[1] under the standard resolution by Sublemma 12.4 or Theorem 3.6 whether $\delta = 1$ or $\delta = 0$. Thus, $[B]$ can be easily proved. So, if $r = 1$, the proofs of $[A]$ and $[B]$ are done.

For the main proof, let $r \geq 2$. If g_j is irreducible in $\mathbb{C}\{y, z\}$ for any $j \geq 1$, then it was already proved by Sublemma 12.7 that g_1 is irreducible in $\mathbb{C}\{y, z\}$. So, to prove the necessity and sufficiency of the condition in $[A]$ and the condition in $[B]$ at the same time, we may start with assuming that g_1 is irreducible in $\mathbb{C}\{y, z\}$ because it was already proved by Corollary 3.3 that g_1 is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\frac{\beta_{1,i,1}}{n_1-i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$, that is, the statement for $[A]$ with $r = 1$. Since $\frac{\beta_{1,i,1}}{n_1-i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$, we are going to follow the same notations and consequences as in Sublemma 12.4.

Therefore, as we have done in Sublemma 12.4, recall that $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ and $(g_j \circ \tau_m)_{total}$ with $(g_j \circ \tau_m)_{proper}$ satisfy the following four facts, where $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ is the composition of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ in the conclusion of Sublemma 12.4.

Fact[1] By (a) and (b) in the conclusion of Sublemma 12.4, let (v, u) be the common one of the local coordinates for the m -th blow-up $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ at $(0, 0)$ which is the quasisingular point of $V^{(m-1)}(y^\gamma g_1)$, in order to study any of $V^{(m)}(g_j)$ for all $j = 2, 3, \dots, r$ in the sense of Lemma 2.14. Being viewed as an analytic mapping, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ can be written in the form

$$(12.0.1) \quad \tau_m(v, u) = (y, z) = (v^{n_1}u^a, v^{\beta_{1,0,1}}u^b),$$

where (i) a and b are some nonnegative integers such that $a\beta_{1,0,1} - bn_1 = 1$,

(ii) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.

Fact[2] By (c) of the conclusion in Sublemma 12.4, along $v = 0$, $(g_j \circ \tau_m)_{total}$ with $(g_j \circ \tau_m)_{proper}$ can be written as follows: Use the same notations for $\Delta_2, \Delta_2^\sharp, \Omega_2^\sharp$ as in Sublemma 12.1, Sublemma 12.2, and Sublemma 12.3, and we may start with assuming that $\varepsilon_{1,0} = 1$ in $V(y^\gamma g_0) = \{y^\gamma(z^{n_1} + \varepsilon_{1,0}y^{\beta_{1,0,1}}) = 0\}$, in order to study $V^{(i)}(g_j)$ for all $i = 1, 2, \dots, m$, and all $j = 1, 2, \dots, r$. Whether $\beta_{1,0,1} \geq 2$ or $\beta_{1,0,1} = 1$, we may write that $(g_0 \circ \tau_m)_{proper} = (1 + \varepsilon_{1,0}u)$ and $(g_1 \circ \tau_m)_{proper} = (1 + \varepsilon_{1,0}\bar{u})$ with $\varepsilon_{1,0} = 1$, without complexity

of the notation if necessary, noting that if $\beta_{1,0,1} = 1$ then $V(g_0)$ has no singularity at the origin.

$$(12.0.2) \quad (g_1 \circ \tau_m)_{total} = v^{n_1 \beta_{1,0,1}} u^{bn_1} (g_1 \circ \tau_m)_{proper} \quad \text{with}$$

$$\begin{aligned} (g_1 \circ \tau_m)_{proper} &= (1 + \varepsilon_{1,0} u) + c_1 \sum_{i=1}^{n_1-1} \varepsilon'_{1,i} v^{\Delta_2^\sharp(\beta_{1,i,1},i) - n_1 \beta_{1,0,1}} u^{\Omega_2^\sharp(\beta_{1,i,1},i) - bn_1} \quad \text{with } \varepsilon_{1,0} = 1 \\ &= 1 + \varepsilon_{1,0} \bar{u} = 1 + \bar{u} \quad \text{for brevity of notation,} \\ (g_j \circ \tau_m)_{total} &= v^{n_1 \beta_{1,0,1} n_2 \cdots n_j} u^{bn_1 n_2 \cdots n_j} (g_j \circ \tau_m)_{proper} \quad \text{with} \\ (g_j \circ \tau_m)_{proper} &= (g_{j-1} \circ \tau_m)_{proper}^{n_j} + \varepsilon'_{j,0} v^{\Delta_j^\sharp(\beta_{j,0,k})_{k=1}^j - n_1 \beta_{1,0,1} n_2 \cdots n_j} u^{\Omega_j^\sharp(\beta_{j,0,k})_{k=1}^j - bn_1 n_2 \cdots n_j} \\ &\quad \times (g_1 \circ \tau_m)_{proper}^{\beta_{j,0,3}} (g_2 \circ \tau_m)_{proper}^{\beta_{j,0,4}} \cdots (g_{j-2} \circ \tau_m)_{proper}^{\beta_{j,0,j}} \\ &\quad + c_j \sum_{i=1}^{n_j-1} \varepsilon'_{j,i} v^{\Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j - n_1 \beta_{1,0,1} n_2 \cdots n_{j-1} (n_j - i)} u^{\Omega_j^\sharp(\beta_{j,i,k})_{k=1}^j - bn_1 n_2 \cdots n_{j-1} (n_j - i)} \\ &\quad \times (g_1 \circ \tau_m)_{proper}^{\beta_{j,i,3}} (g_2 \circ \tau_m)_{proper}^{\beta_{j,i,4}} \cdots (g_{j-2} \circ \tau_m)_{proper}^{\beta_{j,i,j}} (g_{j-1} \circ \tau_m)_{proper}^i, \end{aligned}$$

for $2 \leq j \leq r$ where each $\varepsilon'_{j,i} = \varepsilon_{j,i} \circ \tau_m(v, u)$ is a unit in $\mathbb{C}\{v, 1 + \bar{u}\}$ for $2 \leq j \leq r$ and $0 \leq i < n_j$, noting by Sublemma 12.1 and Sublemma 12.3 that for $0 \leq i < n_j$,

$$(12.0.3) \quad \Delta_j^\sharp(\beta_{j,i,k})_{k=1}^j > n_1 \beta_{1,0,1} n_2 \cdots n_{j-1} (n_j - i) \quad \text{and} \quad \Omega_j^\sharp(\beta_{j,i,k})_{k=1}^j \geq bn_1 n_2 \cdots n_{j-1} (n_j - i).$$

Here, we may write $\eta_{j-1,i} = \eta_{j-1,i}(v, 1 + \bar{u})$ is a unit in $\mathbb{C}\{v, 1 + \bar{u}\}$ for $2 \leq j \leq r$ and $1 \leq i \leq n_j - 1$, noting that $\eta_{j-1,i} = \varepsilon'_{j,i} u^{\Omega_j^\sharp(\beta_{j,i,k})_{k=1}^j - bn_1 n_2 \cdots n_{j-1} (n_j - i)}$. Here, we may assume by a nonsingular change of coordinates that $(g_1 \circ \tau_m)_{proper} = (1 + \bar{u})$ for brevity of the notation in the standard resolution of the singular point $(v, 1 + u) = (0, 0)$ of $V((g_j \circ \tau_m)_{proper})$ for $2 \leq j \leq r$.

Fact[3] For any positive integer $r \geq 2$, observe by (c) of Sublemma 12.4 that g_r is irreducible in $\mathbb{C}\{y, z\}$ if and only if g_1 is irreducible in $\mathbb{C}\{y, z\}$ and $(g_r \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, 1 + \bar{u}\}$, noting that g_1 is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$.

Fact[4] Note that $\gcd(n_1, \beta_{1,0,1}) = 1$ and let j be an arbitrary positive integer with $j \geq 1$. Using (d3) of Sublemma 12.4 or Theorem 3.6, then we get the following:

- (4a) If $\beta_{1,0,1} > 1$, then $g_j \in$ the type [1] under τ_m .
- (4b) If $\beta_{1,0,1} = 1$, then $g_j \in$ the type [0] under τ_m .
- (4c) If $\beta_{1,0,1} \geq 1$, then $z^\delta y g_j \in$ the type [1] under τ_m , whether δ is either 1 or 0.

Now, for the induction proof, it is enough to consider two cases, respectively:

Case(1) $r = 2$, and Case(2) $r \geq 2$.

Case(1): Let $r = 2$. As we have seen in (12.4.2), we may write for brevity of the notation that $(g_0 \circ \tau_m)_{proper} = (1 + \varepsilon_{1,0} u)$ and $(g_1 \circ \tau_m)_{proper} = (1 + \varepsilon_{1,0} \bar{u})$ with $\varepsilon_{1,0} = 1$.

The proof for [A] First to prove [A], by (12.0.2) or Sublemma 12.4, $(g_2 \circ \tau_m)_{total}$ in $\mathbb{C}\{v, 1 + \bar{u}\}$ may be written in the form

$$\begin{aligned} (12.0.4) \quad (g_2 \circ \tau_m)_{total} &= v^{n_1 \beta_{1,0,1} n_2} u^{bn_1 n_2} (g_2 \circ \tau_m)_{proper} \quad \text{with} \\ (g_2 \circ \tau_m)_{proper} &= (g_1 \circ \tau_m)_{proper}^{n_2} + \varepsilon'_{2,0} v^{\Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2} u^{\Omega_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - bn_1 n_2} \\ &\quad + c_2 \sum_{i=1}^{n_2-1} \varepsilon'_{2,i} v^{\Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - n_1 \beta_{1,0,1} (n_2 - i)} u^{\Omega_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - bn_1 (n_2 - i)} (g_1 \circ \tau_m)_{proper}^i, \end{aligned}$$

where $\Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - n_1 \beta_{1,0,1} (n_2 - i) > 0$ and $\Omega_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - bn_1 (n_2 - i) > 0$

and $\varepsilon'_{2,i} = \varepsilon_{2,i} \circ \tau_m(v, u)$ is a unit in $\mathbb{C}\{v, 1 + \bar{u}\}$ for $0 \leq i < n_i$.

By (d2) of The 4-th Cond⁽⁰⁾ in the assumption of the theorem,

$$(12.0.4^*) \quad \Delta_2(\beta_{2,i,k})_{k=1}^2 > (n_2 - i)n_1\beta_{1,0,1} \quad \text{for } 0 \leq i < n_1.$$

Now, we are going to prove the necessity and sufficiency of the condition in [A].

Since $\Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) = \Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2})$ for notation, then it is clear by (12.0.4) and (12.0.4*) that $(g_2 \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$ under the standard resolution τ_m if and only if $\frac{\Delta_2(\beta_{2,i,k})_{k=1}^2}{n_2 - i} > \frac{\Delta_2(\beta_{2,0,k})_{k=1}^2}{n_2}$ for $0 < i < n_2$ because $\gcd(n_2, \Delta_2(\beta_{2,0,k})_{k=1}^2) = 1$. Note that $v(g_2 \circ \tau_m)_{proper} \in$ the type[1].

Therefore, it can be easily proved by Fact[3] or by (c) of Sublemma 12.4 that

$$(12.0.5) \quad \begin{aligned} &g_2 \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ with } yg_2 \in \text{the type}[2] \text{ under the standard resolution} \\ &\iff g_1 \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ and } (g_2 \circ \tau_m)_{proper} \text{ is irreducible in } \mathbb{C}\{v, 1 + \bar{u}\} \\ &\quad \text{with } v(g_2 \circ \tau_m)_{proper} \in \text{the type}[1] \text{ under the standard resolution} \\ &\iff \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \text{ for } 0 < i < n_1 \text{ and } \frac{\Delta_2(\beta_{2,i,k})_{k=1}^2}{n_2 - i} > \frac{\Delta_2(\beta_{2,0,k})_{k=1}^2}{n_2} \text{ for } 0 < i < n_2. \end{aligned}$$

Thus, we proved [A] for $r = 2$.

The proof for [B] Next to prove the condition in [B], let g_2 be irreducible in $\mathbb{C}\{y, z\}$ with $yg_2 \in$ the type[2] under the standard resolution. If $r = 2$, then by Fact[4] or Sublemma 12.4 we get the following:

$$(12.0.6) \quad \begin{aligned} &(i) \text{ If } \beta_{1,0,1} > 1, \text{ then } g_2 \in \text{the type } [1] \text{ under } \tau_m. \\ &(ii) \text{ If } \beta_{1,0,1} = 1, \text{ then } g_2 \in \text{the type } [0] \text{ under } \tau_m. \\ &(iii) \text{ If } \beta_{1,0,1} \geq 1, \text{ then } z^\delta yg_2 \in \text{the type } [1] \text{ under } \tau_m \text{ whether } \delta \text{ is 1 or zero.} \end{aligned}$$

Let $V_2 = \{(y, z) : g_2(y, z) = 0\}$ and $W_2 = \{(y, z) : yg_2(y, z) = 0\}$ be analytic varieties at the origin in $\mathbb{C}\{y, z\}$, respectively. At $(v, 1 + \bar{u}) = (0, 0)$, $\tau_m^{-1}(V_2)$ and $\tau_m^{-1}(W_2)$ can be written as follows:

$$(12.0.7) \quad \begin{aligned} \tau_m^{-1}(V_2) &= \{(g_2 \circ \tau_m)_{total} = v^{n_1\beta_{1,0,1}n_2} u^{bn_1n_2} (g_2 \circ \tau_m)_{proper} = 0\}, \\ \tau_m^{-1}(W_2) &= \{((yg_2) \circ \tau_m)_{total} = v^{n_1\beta_{1,0,1}n_2+n_1} u^{bn_1n_2+a} (g_2 \circ \tau_m)_{proper} = 0\}, \end{aligned}$$

such that

$$\begin{aligned} (g_2 \circ \tau_m)_{proper} &= (1 + \bar{u})^{n_2} + \varepsilon'_{2,0} v^{\Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2} \times u^{\Omega_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - bn_1n_2} \\ &= (1 + \bar{u})^{n_2} + \eta_1 v^{\Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2} \\ &= h_1 \quad \text{by following the notation in Sublemma 12.5,} \end{aligned}$$

where $\varepsilon'_{2,0}$ and $\eta_1 = \varepsilon'_{2,0} u^{\Omega_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - bn_1n_2}$ are units in $\mathbb{C}\{v, 1 + \bar{u}\}$.

Since $(g_2 \circ \tau_m)_{proper}$ and $(\bar{u} + 1)^{n_2} + v^{\Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2}$ have the same analytic type of the singularity at $(\bar{u} + 1, v) = (0, 0)$, we may start with assuming that they are the same equations. Let $V(\phi) = \{(v, \bar{u} + 1) : \phi(v, \bar{u} + 1) = 0\}$ be an analytic variety at $(v, \bar{u} + 1) = (0, 0)$ defined by

$$(12.0.8) \quad \begin{aligned} \phi &= \phi(v, \bar{u} + 1) = v^{\gamma_1} (g_2 \circ \tau_m)_{proper} = v^{\gamma_1} h_1 \\ \text{such that } \begin{cases} \gamma_1 = 1, & \text{if } \Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2 = 1, \\ \gamma_1 = 0, & \text{if } \Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1\beta_{1,0,1}n_2 \geq 2. \end{cases} \end{aligned}$$

Let $Z_1 = \{(v, \bar{u} + 1) : v h_1 = v(g_2 \circ \tau_m)_{proper} = 0\}$ be another analytic variety at $(v, \bar{u} + 1) = (0, 0)$. Since $(g_2 \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$, then note that $Z_1 = \tau_m^{-1}(V_2) = \tau_m^{-1}(W_2)$ have the same two irreducible components at $(v, \bar{u} + 1) = (0, 0)$ as reduced variety, and so they have the homeomorphic resolution, using the composition of the same number of successive blow-ups at $(v, \bar{u} + 1) = (0, 0)$.

Now, we apply consequences of Theorem 3.6 to the proof of [B]. First of all, it is clear that the equation of $V(\phi) = V(v^{\gamma_1} h_1)$ of (12.0.8) satisfies the same kind of assumption as in Theorem 3.6, which can be represented as follows:

h_1 of (12.0.7) satisfies the same assumption as g_1 of (3.6.1) does in Theorem 3.6. Let $V(h_1) = \{(v, u+1) : h_1(v, \bar{u}+1) = 0\}$, $V(f) = \{(v, \bar{u}+1) : f(y, z) = 0\}$ and $V(\phi) = \{(v, \bar{u}+1) : v^{\gamma_1} h(v, \bar{u}+1) = 0\}$ be analytic varieties at $(v, \bar{u}+1) = (0, 0)$ in \mathbb{C}^2 , each of which is written respectively as follows: For convenience of notation, substitute g_0 of (12.4.7) by h_1 , for an application of Theorem 3.6 or Sublemma 12.4.

$$(12.0.9) \quad \begin{aligned} h_1 &= (1 + \bar{u})^{n_2} + \eta_1 v^{k_2} \quad \text{with } k_2 = \Delta_2^\#(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2, \\ f &= h_1^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(2)} v^\alpha (1 + \bar{u})^\beta \quad \text{with } n_2 \alpha + k_2 \beta > n_2 k_2 d = n_2 k_2, \\ F &= v^{\delta_1} (\bar{u} + 1)^{\delta_2} f, \\ \phi &= v^{\gamma_1} h_1 \end{aligned}$$

satisfying the properties (i), (ii), (iii), (iv) and (v):

- (i) $\gcd(n_2, k_2) = 1$ and $d = 1$.
- (ii) If $k_2 = 1$, then $\gamma = 1$, and if $k_2 \geq 2$, then $\gamma = 0$.
- (iii) η_1 is assumed to be one in $\mathbb{C}\{v, \bar{u} + 1\}$, and the $c_{\alpha, \beta}^{(2)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_2 \alpha + k_2 \beta > n_2 k_2$, if exist.
- (iv) Each δ_i is either a positive integer or 0 for $i = 1, 2$.
- (v) If $k_2 = 1$, assume additionally that $\delta_1 > 0$, whether or not δ_2 is zero.

So, we have the same kind of conclusion as we have seen in Theorem 3.6, up to change of notations.

h_1 of (12.0.7) satisfies the same conclusion as g_1 of (3.6.1) does in Theorem 3.6. As we have seen in Theorem 3.6, let ω_s be the composition of s iterations of blow-ups which is needed to get the standard resolution for the singular point at $(v, \bar{u} + 1) = (0, 0)$ of $V(\phi)$, by using the same method as we have done with τ_m , either in the beginning of the proof of Theorem 3.6 or in (12.4.8) of Sublemma 12.4, because $\phi = v^{\gamma_1} (g_2 \circ \tau_m)_{proper} = v^{\gamma_1} h_1$ satisfies the same kind of properties as $y^\gamma g_1$ does. Therefore, by the conclusion of Theorem 3.6, there is nothing to prove for the conclusion of Sublemma 12.4.

As far as the above ω_s is concerned, we can always use one of the local coordinates for each blow-up of ω_s in the sense of Definition 2.11, whenever it is necessary.

Whether $k_2 = \Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2$ is either greater than one or equal to one at $(v, \bar{u} + 1) = (0, 0)$, then $\phi \in$ the type [1] and $vh_1 \in$ the type [1] under ω_s by Fact[4] or Theorem 3.6. Also, the m -th exceptional curve of the first kind among $\tau_m^{-1}(0, 0)$, denoted by $E_m = \{v = 0\}$, can be viewed as one of two irreducible components of $V(vh_1)$ at $(v, \bar{u} + 1) = (0, 0)$. Thus, both V_2 and W_2 of (12.0.7) can have the same standard resolution of the singular point $(y, z) = (0, 0)$, using the composition $\tau_m \circ \omega_s$ of a finite number $(m + s)$ of successive blow-ups at $(y, z) = (0, 0)$.

Since $\phi \in$ the type [1] and $vh_1 \in$ the type [1] at $(v, \bar{u} + 1) = (0, 0)$ under ω_s where $Z_1 = \{vh_1 = 0\} = \tau_m^{-1}(V_2) = \tau_m^{-1}(W_2)$, then by (12.0.6) or Fact[4] again, we get the following:

- (12.0.10) (i) If $\beta_{1,0,1} > 1$, then $g_2 \in$ the type [2] under $\tau_m \circ \omega_s$.
- (ii) If $\beta_{1,0,1} = 1$, then $g_2 \in$ the type [1] under $\tau_m \circ \omega_s$.
- (iii) If $\beta_{1,0,1} \geq 1$, then $z^\delta y g_2 \in$ the type [2] under $\tau_m \circ \omega_s$, whether δ is 1 or 0.

Thus, we proved [B] for $r = 2$.

Case(2): Let $r \geq 2$. For the proof of the theorem, by the induction assumption, suppose we have shown that [A] and [B] of the theorem are true on the integer $r \geq 2$.

In order to prove the theorem on the integer $(r + 1)$, it suffices to consider the defining equation for g_{r+1} with the additive assumptions:

$$(12.0.11) \quad \begin{aligned} g_{r+1} &= g_r^{n_{r+1}} + \varepsilon_{r+1,0} y^{\beta_{r+1,0,1}} z^{\beta_{r+1,0,2}} g_1^{\beta_{r+1,0,3}} \cdots g_{r-1}^{\beta_{r+1,0,r+1}} \\ &\quad + c_{r+1} \sum_{i=1}^{n_{r+1}-1} \varepsilon_{r+1,i} y^{\beta_{r+1,i,1}} z^{\beta_{r+1,i,2}} g_1^{\beta_{r+1,i,3}} \cdots g_{r-1}^{\beta_{r+1,i,r+1}} g_r^i, \\ \Delta_{j+1}^\#(\beta_{j+1,i,k})_{k=1}^{j+1} &> n_1 \beta_{1,0,1} n_2 n_3 \cdots n_r (n_{r+1} - i) \quad \text{on } g_{r+1}, \end{aligned}$$

where

(i) g_1, g_2, \dots, g_r satisfies the same properties and notations in the assumptions and conclusions of the theorem by induction assumption,

(ii) $X_{r+1} = \{n_{r+1}\} \cup \{\beta_{r+1,i,1} : 0 \leq i < n_{r+1}\} \cup \{\beta_{r+1,i,2} : 0 \leq i < n_3\} \cup \dots \cup \{\beta_{r+1,i,r+1} : 0 \leq i < n_{r+1}\} \subset N_0$ with $n_{r+1} \geq 2$, and $\varepsilon_{r+1,i} = \varepsilon_{r+1,i}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $0 \leq i < n_{r+1}$, and the c_{j+1} are nonzero complex numbers,

(iii) $\Delta_{r+1}(t_k)_{k=1}^{r+1} = t_{r+1}\Delta_r(\beta_{r,0,k})_{k=1}^r + n_r\Delta_r(t_k)_{k=1}^r$ for each $(t_k)_{k=1}^{r+1} \in N_0^{r+1}$,

(iv) $\Delta_{r+1}(\beta_{r+1,i,k})_{k=1}^{r+1} > (n_{r+1}-i)n_r\Delta_r(\beta_{r,0,k})_{k=1}^r$ for $0 \leq i < n_{r+1}$.

Now, it is needed to show that two statements [A] and [B] are true for the integer $(r+1)$, respectively. In the beginning of the proof, it was already known by Sublemma 12.7 that if g_j is irreducible in $\mathbb{C}\{y, z\}$ for any $j \geq 1$, then g_1 is irreducible in $\mathbb{C}\{y, z\}$.

The proof for [A] First, we are going to prove the necessity and sufficiency of the condition in [A], assuming that $\frac{\beta_{1,i,1}}{n_1-i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$. Using the composition τ_m again of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(y^\gamma g_1)$ in the beginning of the proof, then $h_r = (g_{r+1} \circ \tau_m)_{proper}$ can be written in the form

$$(12.0.12) \quad h_r = h_{r-1}^{s_r} + \eta_{r,0} v^{\gamma_{r,0,1}} (1 + \bar{u})^{\gamma_{r,0,2}} h_1^{\gamma_{r,0,3}} \dots h_{r-2}^{\gamma_{r,0,r}}, \\ + c_{r+1} \sum_{i=1}^{s_r-1} \eta_{r,i} v^{\gamma_{r,i,1}} (1 + \bar{u})^{\gamma_{r,i,2}} h_1^{\gamma_{r,i,3}} \dots h_{r-2}^{\gamma_{r,i,r}} h_{r-1}^i,$$

where

(i) each $h_j = (g_{j+1} \circ \tau_m)_{proper}$ is in $\mathbb{C}\{v, 1 + \bar{u}\}$ for $j = 1, 2, \dots, r$, which has been already represented by Sublemma 12.5,

(ii) $(g_{r+1} \circ \tau_m)_{total} = v^{n_1\beta_{1,0,1}n_2 \dots n_r} u^{bn_1n_2 \dots n_r} (g_{r+1} \circ \tau_m)_{proper}$,

(iii) $\eta_{r,i} = \varepsilon'_{r+1,i} u^{\Omega_{r+1}^\sharp(\beta_{r+1,i,k})_{k=1}^{r+1} - bn_1n_2 \dots n_{r+1}}$ is a unit in $\mathbb{C}\{v, \bar{u} + 1\}$, satisfying the same notations and consequences as in Sublemma 12.5,

(iv) $s_r = n_{r+1} \geq 2$, $\gamma_{r,i,1} = \Delta_{r+1}^\sharp(\beta_{r+1,i,k})_{k=1}^{r+1} - n_1\beta_{1,0,1}n_2n_3 \dots n_r(n_{r+1} - i) > 0$, $\gamma_{r,i,2} = \beta_{r+1,i,3}, \gamma_{r,i,3} = \beta_{r+1,i,4}, \dots, \gamma_{r,i,r} = \beta_{r+1,i,r+1}$.

Because blow-ups process preserves irreducibility of plane curve singularity, note by Fact[3] that g_{r+1} is irreducible in $\mathbb{C}\{y, z\}$ if and only if g_1 is irreducible in $\mathbb{C}\{y, z\}$ and $h_r = (g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$. To prove the aim in [A], we may assume that $\frac{\beta_{1,i,1}}{n_1-i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$ because g_1 is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\frac{\beta_{1,i,1}}{n_1-i} > \frac{\beta_{1,0,1}}{n_1} > 0$ for $0 < i < n_1$. First of all, whether or not h_r is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$, we proved by Sublemma 12.5 that h_r satisfies the same kind of assumptions as g_r does in this theorem, up to change of notations. So, by induction assumption on the integer r and by following the same notations as in Sublemma 12.5, then it is easy to get the following: Note by The (5 α)-th Cond⁽¹⁾ of Sublemma 12.5 that $\gcd(s_q, \Xi_q(\gamma_{q,0,k})_{k=1}^q) = \gcd(n_{q+1}, \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1}) = 1$ for $q = 1, 2, \dots, r$.

(12.0.13) h_r is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$

with $v^\gamma h_r \in$ the type $[r]$ under the standard resolution

$$\iff \frac{\gamma_{1,i,1}}{s_1-i} > \frac{\gamma_{1,0,1}}{s_1} \quad \text{for } 0 < i < s_1, \quad \text{and} \\ \frac{\Xi_{j-1}(\gamma_{j-1,i,k})_{k=1}^{j-1}}{s_{j-1}-i} > \frac{\Xi_{j-1}(\gamma_{j-1,0,k})_{k=1}^{j-1}}{s_{j-1}} \quad \text{for each } j=2,3, \dots, r+1 \text{ and for } 0 < i < s_{j-1}.$$

Therefore, we get by (12.0.13) and by an equality in (12.5.5 α .2) or (12.5.6.2) of The (5 α)-th Cond⁽¹⁾ of Sublemma 12.5 that

(12.0.14) $h_r = (g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$

with $v^\gamma h_r \in$ the type $[r]$ under the standard resolution

$$\iff \frac{\Delta_j(\beta_{j,i,k})_{k=1}^j}{n_j-i} > \frac{\Delta_j(\beta_{j,0,k})_{k=1}^j}{n_j} \quad \text{for each } j = 2, 3, \dots, r+1 \text{ and for } 0 < i < n_j.$$

Note by Fact[3] or by (c) of Sublemma 12.4 that

$$\begin{aligned}
(12.0.15) \quad & g_{r+1} \text{ is irreducible in } \mathbb{C}\{y, z\} \\
& \text{with } y^\gamma g_{r+1} \in \text{the type}[r+1] \text{ under the standard resolution} \\
\iff & \frac{\beta_{1,i,1}}{n_1 - i} > \frac{\beta_{1,0,1}}{n_1} > 0 \text{ for } 0 < i < n_1 \text{ and } h_r = (g_{r+1} \circ \tau_m)_{proper} \text{ is irreducible} \\
& \text{in } \mathbb{C}\{v, \bar{u} + 1\} \text{ with } v^\gamma h_r \in \text{the type}[r] \text{ under the standard resolution.}
\end{aligned}$$

Thus, we can prove by (12.0.14) and (12.0.15) and by the induction assumption that

$$\begin{aligned}
(12.0.16) \quad & g_{r+1} \text{ is irreducible in } \mathbb{C}\{y, z\} \\
& \text{with } y^\gamma g_{r+1} \in \text{the type}[r+1] \text{ under the standard resolution} \\
\iff & \frac{\Delta_j(\beta_{j,i,k})_{k=1}^j}{n_j - i} > \frac{\Delta_j(\beta_{j,0,k})_{k=1}^j}{n_j} \quad \text{for each } j = 1, 2, \dots, r+1 \text{ and for } 0 < i < n_j. \\
\iff & g_1, g_2, \dots, g_r \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \frac{\Delta_{r+1}(\beta_{r+1,i,k})_{k=1}^{r+1}}{n_{r+1} - i} > \frac{\Delta_{r+1}(\beta_{r+1,0,k})_{k=1}^{r+1}}{n_{r+1}}.
\end{aligned}$$

Therefore, we proved [A].

The proof for [B] Next to prove [B], let g_{r+1} be irreducible in $\mathbb{C}\{y, z\}$. If g_j is irreducible in $\mathbb{C}\{y, z\}$ for any $j \geq 1$, note that $\gcd(n_1, \beta_{1,0,1}) = 1$ by (d) of Sublemma 12.2 and Theorem 3.7. By Fact[4] or Sublemma 12.4 again, we have the following:

$$\begin{aligned}
(12.0.17) \quad & \text{(i) If } \beta_{1,0,1} > 1, \text{ then } g_{r+1} \in \text{the type}[1] \text{ under } \tau_m. \\
& \text{(ii) If } \beta_{1,0,1} = 1, \text{ then } g_{r+1} \in \text{the type}[0] \text{ under } \tau_m. \\
& \text{(iii) If } \beta_{1,0,1} \geq 1, \text{ then } z^\delta y g_{r+1} \in \text{the type}[1] \text{ under } \tau_m \text{ whether } \delta \text{ is 1 or 0.}
\end{aligned}$$

Let $V_{r+1} = \{(y, z) : g_{r+1}(y, z) = 0\}$ and $W_{r+1} = \{(y, z) : y g_{r+1}(y, z) = 0\}$ be analytic varieties at the origin in $\mathbb{C}\{y, z\}$, respectively. Let τ_m be again the composition of a finite number m of successive blow-ups which is needed to get the standard resolution of the singular point of $V(yg_1)$.

At $(v, 1 + \bar{u}) = (0, 0)$, $\tau_m^{-1}(V_{r+1})$ and $\tau_m^{-1}(W_{r+1})$ can be written as follows:

$$\begin{aligned}
(12.0.18) \quad & \tau_m^{-1}(V_{r+1}) = \{(g_{r+1} \circ \tau_m)_{total} = v^{n_1 \beta_{1,0,1} n_2 \cdots n_{r+1}} u^{bn_1 n_2 \cdots n_r} (g_{r+1} \circ \tau_m)_{proper} = 0\} \quad \text{and} \\
& \tau_m^{-1}(W_{r+1}) = \{(yg_{r+1}) \circ \tau_m)_{total} = v^{n_1 \beta_{1,0,1} n_2 \cdots n_{r+1} + n_1} u^{bn_1 n_2 \cdots n_r + a} (g_{r+1} \circ \tau_m)_{proper} = 0\},
\end{aligned}$$

noting by (12.0.12) that $h_r = (g_{r+1} \circ \tau_m)_{proper}$ with

$$\begin{aligned}
(12.0.19) \quad & h_r = h_{r-1}^{s_r} + \eta_{r,0} v^{\gamma_{r,0,1}} (1 + \bar{u})^{\gamma_{r,0,2}} h_1^{\gamma_{r,0,3}} \cdots h_{r-2}^{\gamma_{r,0,r}} \\
& + c_{r+1} \sum_{i=1}^{s_r-1} \eta_{r,i} v^{\gamma_{r,i,1}} (1 + \bar{u})^{\gamma_{r,i,2}} h_1^{\gamma_{r,i,3}} \cdots h_{r-2}^{\gamma_{r,i,r}} h_{r-1}^i,
\end{aligned}$$

where $\eta_{r,i} = \varepsilon'_{r+1,i} u^{\Omega_{r+1}^\sharp(\beta_{r+1,i,k})_{k=1}^{r+1} - bn_1 n_2 \cdots n_{r+1}}$ is a unit in $\mathbb{C}\{v, \bar{u} + 1\}$.

Let $V(\phi_r) = \{(v, \bar{u} + 1) : \phi_r(v, \bar{u} + 1) = 0\}$ be an analytic variety at $(v, \bar{u} + 1) = (0, 0)$ defined by

$$\begin{aligned}
(12.0.20) \quad & \phi_r = \phi_r(v, \bar{u} + 1) = v^{\gamma_1} h_r \quad \text{with} \quad h_r = (g_{r+1} \circ \tau_m)_{proper} \\
& \text{such that} \quad \begin{cases} \gamma_1 = 1 & \text{if } \Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2 = 1, \\ \gamma_1 = 0 & \text{if } \Delta_2^\sharp(\beta_{2,0,1}, \beta_{2,0,2}) - n_1 \beta_{1,0,1} n_2 \geq 2. \end{cases}
\end{aligned}$$

Let $Z_r = \{(v, \bar{u} + 1) : v h_r(v, \bar{u} + 1) = 0\}$ be an analytic variety at $(v, \bar{u} + 1) = (0, 0)$. Since $(g_{r+1} \circ \tau_m)_{proper}$ is irreducible in $\mathbb{C}\{v, \bar{u} + 1\}$, then note that $Z_r = \tau_m^{-1}(V_{r+1}) = \tau_m^{-1}(W_{r+1})$

have the same two irreducible components at $(v, \bar{u} + 1) = (0, 0)$ as reduced variety, and so they have the same standard resolution of the singular point $(v, \bar{u} + 1) = (0, 0)$. Let ω_s be the composition of s iterations of blow-ups which is needed to get the standard resolution of the singular point $(v, \bar{u} + 1) = (0, 0)$ of $V(\phi_r)$. Since we proved by Sublemma 12.5 that $V(h_r)$ satisfies the same kind of properties and notations as $V(g_r)$ does in the assumption of this theorem, then by the induction assumption on the integer r , both $V(\phi_r)$ and Z_r belong to the type $[r]$ under ω_s . That is, both $\tau_m^{-1}(V_{r+1})$ and $\tau_m^{-1}(W_{r+1})$ belong to the type $[r]$ under ω_s at $(v, \bar{u} + 1) = (0, 0)$, as reduced variety.

Now, the m -th exceptional curve of the first kind among $\tau_m^{-1}(0, 0)$, denoted by $E_m = \{v = 0\}$, can be viewed as one of two irreducible components of Z_r at $(v, \bar{u} + 1) = (0, 0)$. Then, both V_{r+1} and W_{r+1} can have the same standard resolution $\tau_m \circ \omega_s$ of the singular point $(y, z) = (0, 0)$, using the composition $\tau_m \circ \omega_s$ of a finite number $(m + s)$ of successive blow-ups at $(y, z) = (0, 0)$.

Since $Z_r \in$ the type $[r]$ under ω_s , then by (12.0.17) or Sublemma 12.4 again, it is trivial to get the following: Note that δ is an integer.

(12.0.21)

(i) If $\beta_{1,0,1} > 1$, then $g_{r+1} \in$ the type $[r + 1]$ under $\tau_m \circ \omega_s$.

(ii) If $\beta_{1,0,1} = 1$, then $g_{r+1} \in$ the type $[r]$ under $\tau_m \circ \omega_s$.

(iii) If $\beta_{1,0,1} \geq 1$, then $z^\delta y g_{r+1} \in$ the type $[r]$ under $\tau_m \circ \omega_s$, whether δ is zero or not.

Thus, the proof for $[B]$ is done. Therefore, we completed the proof of theorem. \square

Corollary 12.6.

Assumptions Suppose that the assumptions of Theorem 12.0 hold. Let r be an arbitrary positive integer with $r \geq 1$.

Conclusions Then, we get the following:

For any $r \geq 1$, $g_r = g_r(y, z)$ can be written in the form

$$(12.6.1) \quad g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,0,1}})^{n_2 n_3 \cdots n_r} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(r)} y^\alpha z^\beta \quad \text{with } \varepsilon_1 = 1 \text{ and} \\ \text{with } n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_r,$$

where a unit $\varepsilon_1 = \varepsilon_1(y, z)$ may be analytically assumed to be one in $\mathbb{C}\{y, z\}$, and the $c_{\alpha, \beta}^{(r)}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + \beta_{1,0,1} \beta > n_1 \beta_{1,0,1} n_2 n_3 \cdots n_r$. \square

Proof of Corollary. There is nothing to prove by Sublemma 12.2. \square

§14. How to compute the divisor of the total transform of irreducible plane curve singularities defined by the quasi-Puiseux convergent power series of the recursive type in $\mathbb{C}\{y, z\}$ under the standard resolution

Theorem 14.0. Assumptions *Let r be an arbitrary positive integer.*

[A] *Let $g_{r+1} \in \mathbb{C}\{y, z\}$ be a generalized semi-quasi-Puiseux germ of the recursive $(r+1)$ -type which is defined by Definition 12.0.0:*

$$\begin{aligned} \text{Let } \{X_k : k = 1, 2, \dots, r+1\} \quad & \text{with } X_k \subset N_0, \\ \{g_k : k = 1, 2, \dots, r+1\} \quad & \text{with } g_k \in \mathbb{C}\{y, z\}, \\ \{\Delta_k : N_0^k \rightarrow N_0 \text{ is an integer-valued function for } k = 1, 2, \dots, r+1\} \end{aligned}$$

be three sequences, satisfying four conditions for each k :

*Such conditions are denoted by **The 1-th Cond**⁽⁰⁾, ..., **The 4-th Cond**⁽⁰⁾ for brevity.*

The 1st Cond⁽⁰⁾ *The family $\{X_\ell : \ell = 1, 2, \dots, r+1\}$ with $X_\ell \subset N_0$ is as follows:*

- (1)(1a) $X_1 = \{n_1\} \cup \{\beta_{1,i,1} : 0 \leq i < n_1\}$ with $n_1 \geq 2$ and $\beta_{1,0,1} \geq 1$ where $X_1 \subset N$.
(1b) $X_j = \{n_j\} \cup \{\beta_{j,i,1} : 0 \leq i < n_j\} \cup \{\beta_{j,i,2} : 0 \leq i < n_j\} \cup \dots \cup \{\beta_{j,i,j} : 0 \leq i < n_j\}$
with $n_j \geq 2$ where $j = 2, 3, \dots, r+1$.
For each $j = 2, 3, \dots, r+1$, assume that at least one of $\beta_{j,0,1}, \beta_{j,0,2}, \dots, \beta_{j,0,j}$ is nonzero.

The 2nd Cond⁽⁰⁾ *For each $j = 1, 2, \dots, r+1$, let $g_j = g_j(y, z, c_j)$ be in $\mathbb{C}\{y, z\}$ where all the c_j are complex numbers, each of which is defined by the following way:*

- (2)(2a) $g_1 = z^{n_1} + \varepsilon_{1,0} y^{\beta_{1,0,1}} + c_1 \sum_{i=1}^{n_1-1} \varepsilon_{1,i} y^{\beta_{1,i,1}} z^i$ with $\varepsilon_{1,0} = 1$.
(2b) $g_j = g_{j-1}^{n_j} + \varepsilon_{j,0} y^{\beta_{j,0,1}} z^{\beta_{j,0,2}} g_1^{\beta_{j,0,3}} \dots g_{j-2}^{\beta_{j,0,j}} + c_j \sum_{i=1}^{n_j-1} \varepsilon_{j,i} y^{\beta_{j,i,1}} z^{\beta_{j,i,2}} g_1^{\beta_{j,i,3}} \dots g_{j-2}^{\beta_{j,i,j}} g_{j-1}^i$,
where $j = 2, 3, \dots, r+1$.

Note that each $\varepsilon_{j,i} = \varepsilon_{j,i}(y, z)$ is a unit in $\mathbb{C}\{y, z\}$ for $1 \leq j \leq r+1$ and $0 \leq i < n_j$, if exists. As far as analytic equivalence of isolated plane curve singularities defined by all g_j , $1 \leq j \leq r+1$, is concerned, then we may assume that $\varepsilon_{1,0}$ is equal to one by a suitable nonsingular change of coordinates at the origin in \mathbb{C}^2 .

The 3rd Cond⁽⁰⁾ *Let $\{\Delta_k : N_0^k \rightarrow N_0 : k = 1, 2, \dots, r+1\}$ be a sequence such that each Δ_k is an integer-valued function defined by the following way:*

- (3) (3a) $\Delta_1(t) = t$ for each $t \in N_0$.
(3b) $\Delta_j(t_k)_{k=1}^j = t_j \Delta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1} + n_{j-1} \Delta_{j-1}(t_k)_{k=1}^{j-1}$ for each $(t_k)_{k=1}^j \in N_0^j$.

The 4-th Cond⁽⁰⁾ *Then, the following inequalities hold: Note that $r+1 \geq 2$.*

- (4)(4a) $\Delta_1(\beta_{1,i,1}) = \beta_{1,i,1} > 0$ for $0 \leq i < n_1$.
(4b) $\Delta_j(\beta_{j,i,k})_{k=1}^j > (n_j - i) n_{j-1} \Delta_{j-1}(\beta_{j-1,0,k})_{k=1}^{j-1}$ for $0 \leq i < n_j$ where $j = 2, \dots, r+1$.

[B] *Let $g_r \in \mathbb{C}\{y, z\}$ be a generalized semi-quasi-Puiseux convergent power series of the recursive r -type as in [A]. There are an additional condition, denoted by **The 5-th Cond**⁽⁰⁾.*

The 5-th Cond⁽⁰⁾ *For each $q = 1, 2, \dots, r$, the following inequalities hold:*

- (5)(5a) $\gcd(n_q, \Delta_q(\beta_{q,0,k})_{k=1}^q) = 1$ for $1 \leq q \leq r$.
(5b) $\frac{\Delta_q(\beta_{q,i,k})_{k=1}^q}{n_q - i} > \frac{\Delta_q(\beta_{q,0,k})_{k=1}^q}{n_q}$ for $0 < i < n_q$.

*In addition, for each $j = 1, 2, \dots, r+1$, let $(0, 0)$ be the singular point of analytic varieties $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ and $V(yg_j) = \{(y, z) : yg_j(y, z) = 0\}$ except for $V(g_1)$ with $\beta_{1,0,1} = 1$, satisfying the following two properties (*1) and (*2):*

- (*1) g_r is irreducible in $\mathbb{C}\{y, z\}$ for $r \geq 1$.
(*2) g_{r+1} may not be irreducible in $\mathbb{C}\{y, z\}$, but g_{r+1} satisfies
 $\Delta_{r+1}(\beta_{r+1,i,k})_{k=1}^{r+1} > (n_{r+1} - i) n_r \Delta_r(\beta_{r,0,k})_{k=1}^r$ for $0 \leq i < n_j$ where $j = 2, \dots, r+1$.

Remark 14.0.1. *It was already proved by Theorem 12.0 and the above assumptions that the following are true:*

g_r is irreducible in $\mathbb{C}\{y, z\}$

$$\iff g_1, \dots, g_{r-1} \text{ are irreducible in } \mathbb{C}\{y, z\} \text{ and } \frac{\Delta_r(\beta_{r,i,k})_{k=1}^r}{n_r - i} > \frac{\Delta_r(\beta_{r,0,k})_{k=1}^r}{n_r} \text{ for } 0 < i < n_r$$

$$\iff \frac{\Delta_j(\beta_{j,i,k})_{k=1}^j}{n_j - i} > \frac{\Delta_j(\beta_{j,0,k})_{k=1}^j}{n_j} \text{ for each } j = 1, 2, \dots, r \text{ and for } 0 < i < n_j.$$

Conclusions

(1) Let $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ and $V(yg_j) = \{(y, z) : yg_j(y, z) = 0\}$ be analytic varieties at the origin in \mathbb{C}^2 for $1 \leq j \leq r$. For each $j = 1, 2, \dots, r$, we may assume that $\tau_{\lambda_j} = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ is the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point $(0, 0)$ of either $V(g_j)$ or $V(yg_j)$. If $r = 1$ and $\beta_{1,0,1} = 1$, then $V(g_1)$ has no singularity at the origin, but $V(yg_1)$ has an isolated singular point at the origin.

(2) Let $\tau_{\lambda_j}^{-1}(0, 0) = \cup_{i=1}^{\lambda_j} E_i$ for each fixed $j = 1, 2, \dots, r$, where each E_i is called an exceptional curve of the first kind. We may assume by Proposition 14.1 that $(g_j \circ \tau_{\lambda_j})_{\text{divisor}}$ and $(g_{j+1} \circ \tau_{\lambda_j})_{\text{divisor}}$ are the divisors of $g_j \circ \tau_{\lambda_j}$ and $g_{j+1} \circ \tau_{\lambda_j}$, respectively, which can be defined by

$$(14.0.2) \quad \begin{aligned} (g_j \circ \tau_{\lambda_j})_{\text{divisor}} &= V^{(\lambda_j)}(g_j) + \sum_{i=1}^{\lambda_j} e_{j,i} E_i, \\ (g_{j+1} \circ \tau_{\lambda_j})_{\text{divisor}} &= V^{(\lambda_j)}(g_{j+1}) + \sum_{i=1}^{\lambda_j} e_{j+1,i} E_i, \end{aligned}$$

where $V^{(\lambda_j)}(g_j)$ and $V^{(\lambda_j)}(g_{j+1})$ are the proper transforms of $V(g_j)$ and $V(g_{j+1})$ under τ_{λ_j} , respectively.

Then, we have the following:

(2a) $e_{j+1,i} = n_{j+1} e_{j,i}$ for $i = 1, 2, \dots, \lambda_j$.

(2b) Let $d_j = n_{j+1} n_{j+2} \dots n_r$ for $1 \leq j < r$, and $d_r = 1$. In particular, we have

$$(14.0.3) \quad e_{r,\lambda_j} = n_j d_j \Delta_j(\beta_{j,0,k})_{k=1}^j \quad \text{with} \quad e_{r,\lambda_1} = n_1 d_1 \beta_{1,0,1}.$$

(3) As far as the standard resolution of the singular point of $V(g_r)$ is concerned, each E_i of λ_r exceptional curves of the first kind has at most three distinct intersection points with other exceptional curves and the proper transform under τ_{λ_r} :

(3a) If $\beta_{1,0,1} = 1$, then $g_r \in$ the type $[r-1]$ under τ_{λ_r} in the sense of Definition 2.5 such that $E_{\lambda_2}, E_{\lambda_3}, \dots, E_{\lambda_r}$ are the only $(r-1)$ exceptional curves, each of which has three distinct intersection points with other exceptional curves and the proper transform.

(3b) If $\beta_{1,0,1} \geq 2$, then $g_r \in$ the type $[r]$ under τ_{λ_r} in the sense of Definition 2.5 such that $E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_r}$ are the only r exceptional curves, each of which has three distinct intersection points with other exceptional curves and the proper transform. \square

§14.1. In preparation for the proof of Theorem 14.0

Let $\{g_j : j = 1, 2, \dots, r+1\}$ with $g_j \in \mathbb{C}\{y, z\}$ be defined as in Theorem 14.0. For the proof of Theorem 14.0, first of all, it suffices to prepare the proposition (Proposition 14.1) with its proof, by the following observation:

Observation.

(1) For an easy proof of the theorem, Proposition 14.1 is the restatement of Theorem 14.0.

(2) The proof of Proposition 14.1 will be by induction on the integer j , which says that τ_{λ_j} of Theorem 14.0 is the composition of a finite number λ_j of successive blow-ups, which is needed to get the standard resolution of the singular point of $V(g_j)$ or $V(yg_j)$.

(3) Proposition 14.1 is a generalization of Sublemma 12.4 of Theorem 12.0, because it is clear that if the integer j of Proposition 14.1 is equal to an integer one, then Sublemma 12.4 and Proposition 14.1 are the same statement.

(4) For the induction proof of Proposition 14.1, we will prepare three statements, denoted by Sublemma 14.2, Sublemma 14.3 and Proposition 14.1 in this section. After then, these three statement will be proved in §14.2, respectively. If then, we can finish the proof of this theorem by Proposition 14.1, without any additional assumption.

Proposition 14.1(Theorem 14.0). Assumptions Suppose that the assumptions and notations of Theorem 14.0 hold. Note by Theorem 12.0 and the above assumptions of Theorem 14.0 that the following are true:

$$(14.1.1) \quad g_r \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ with } yg_r \in \text{the type}[r] \text{ under} \\ \text{the standard resolution, but } g_{r+1} \text{ may not be irreducible in } \mathbb{C}\{y, z\}.$$

Conclusions For each $j = 1, 2, \dots, r$, let $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ and $V(yg_j) = \{(y, z) : yg_j(y, z) = 0\}$ be analytic varieties at the origin in \mathbb{C}^2 .

Let $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point of $V(g_j)$ or $V(yg_j)$. If $j \geq 2$ or $\beta_{1,1} \geq 2$, then τ_{λ_j} can be chosen with the same local coordinate system, in order to study the resolution of the singularity of both $V(g_j)$ and $V(yg_j)$, noting that both $V(g_j)$ and $V(yg_j)$ have the singular point at the origin. But if $j = 1$ and $\beta_{1,1} = 1$, then $V(g_1)$ has no singularity at the origin, but note that $V(yg_1)$ still has the singular point at the origin.

(a) For any $s = 1, 2, \dots, \lambda_j - 1$, $V^{(s)}(yg_\ell)$ with $j \leq \ell \leq r$ has one and only one quasisingular point on $\tau_s^{-1}(0, 0)$ in the sense of Definition 2.6.

(a1) Then, we can use just one coordinate patch of the local coordinates for each blow-up π_i of

$$(14.1.2) \quad \tau_{\lambda_j} = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{\lambda_j}$$

with $1 \leq i \leq \lambda_j$ in the sense of Definition 2.11, to study $V^{(s)}(yg_\ell)$.

(a2) Both $V^{(s)}(g_{r+1})$ and $V^{(s)}(yg_{r+1})$ have the same quasisingular point on $\tau_s^{-1}(0, 0)$ as $V^{(s)}(yg_j)$ does on $\tau_s^{-1}(0, 0)$ for $1 \leq s \leq \lambda_j - 1$.

Then, we can use a common one coordinate patch of the local coordinates for each blow-up π_i of τ_{λ_j} with $1 \leq \lambda_j$ in the sense of Definition 2.13, in order to study both $V^{(s)}(g_\ell)$ and $V^{(s)}(yg_\ell)$ with $j \leq \ell \leq r + 1$.

(b) Suppose that $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ satisfies the same assumptions and notations as in (a). In more detail, τ_{λ_j} can be written in the form

$$(14.1.3) \quad \tau_{\lambda_j} = \mu_{1,m_1} \circ \mu_{2,m_2} \circ \dots \circ \mu_{j,m_j} \quad \text{with } m_1 = \lambda_1, \\ M^{(\lambda_j)} \xrightarrow{\mu_{j,m_j}} M^{(\lambda_{j-1})} \xrightarrow{\mu_{j-1,m_{j-1}}} M^{(\lambda_{j-2})} \rightarrow \dots \rightarrow M^{(\lambda_1)} \xrightarrow{\mu_{1,m_1}} M^{(\lambda_0)} = \mathbb{C}^2,$$

where

$$(14.1.4) \quad \lambda_j = m_1 + m_2 + \dots + m_j \text{ for } 1 \leq j \leq r, \\ \mu_{1,m_1} = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{\lambda_1} \text{ with } \lambda_1 = m_1, \\ \mu_{j,m_j} = \pi_{\lambda_{j-1}+1} \circ \pi_{\lambda_{j-1}+2} \circ \dots \circ \pi_{\lambda_j} \text{ with } \lambda_{j-1} + m_j = \lambda_j, \quad 2 \leq j \leq r,$$

satisfying the following properties:

(b1) $\tau_{\lambda_1} = \mu_{1,\lambda_1}$ is the composition of a finite number λ_1 of successive blow-ups which is needed to get the standard resolution of the singular point of either $V(yg_1)$ or $V(g_1)$.

(b2) μ_{j,m_j} is the composition of a finite number m_j of successive blow-ups which is needed to get the standard resolution of one and only one quasisingular point of $V^{(\lambda_{j-1})}(g_j)$ with $2 \leq j \leq r$. That is, $\tau_{\lambda_j} = \mu_{1,m_1} \circ \mu_{2,m_2} \circ \cdots \circ \mu_{j,m_j}$.

Now, we are going to compute the representation of the local defining equation for $V^{(\lambda_j)}(g_{j+\ell})$, denoted by $(g_{j+\ell} \circ \tau_{\lambda_j})_{proper}$, which is called the proper transform of $V(g_{j+\ell})$ at $(y, z) = (0, 0)$ under τ_{λ_j} , where $j + \ell \leq r + 1$ with $1 \leq j \leq r$ and $\ell \geq 0$.

Just for notation, let $(v_{\lambda_j}, u_{\lambda_j})$ be a common one coordinate patch of the same local coordinates for $M^{(\lambda_j)}$, that is, also the λ_j -th common coordinate patch such that $\pi_{\lambda_j} : M^{(\lambda_j)} \rightarrow M^{(\lambda_{j-1})}$ is the λ_j -th blow-up of τ_{λ_j} at a quasisingular point of $V^{(\lambda_{j-1})}(g_j)$ in the sense of (a2). Again, let $(v_{\lambda_{j-1}}, u_{\lambda_{j-1}})$ be a common one coordinate patch of the same local coordinates for $M^{(\lambda_{j-1})}$ such that $\pi_{\lambda_{j-1}} : M^{(\lambda_{j-1})} \rightarrow M^{(\lambda_{j-1}-1)}$ is the λ_{j-1} -th blow-up of $\tau_{\lambda_{j-1}}$ at a quasisingular point of $V^{(\lambda_{j-1}-1)}(g_j)$ in the sense of (a2).

Being viewed as an analytic map by Sublemma 12.4 and Theorem 3.6, along $v_{\lambda_j} = 0$, $\mu_{j,m_j} : M^{(\lambda_j)} \rightarrow M^{(\lambda_{j-1})}$ can be written in the form

$$(14.1.5) \quad \begin{aligned} \mu_{j,m_j}(v_{\lambda_j}, u_{\lambda_j}) &= (v_{\lambda_{j-1}}, 1 + \varepsilon_{j-1,1} u_{\lambda_{j-1}}) \\ &= (v_{\lambda_j}^{p_j} u_{\lambda_j}^{a_j}, v_{\lambda_j}^{q_j} u_{\lambda_j}^{b_j}), \end{aligned}$$

where

- (i) for $j = 1$, $p_1 = n_1$ and $q_1 = \beta_{1,1} = \Delta_1(\beta_{1,1})$ with $(v_{\lambda_0}, 1 + \varepsilon_{0,1} u_{\lambda_0}) = (y, z)$,
- (ii) for $j \geq 2$, $p_j = n_j$ and $q_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} > 0$, $\varepsilon_{j-1,1} = \varepsilon_{j-1,1}(v_{\lambda_{j-1}}, 1 + \varepsilon_{j-1,1} u_{\lambda_{j-1}})$ is a unit in $\mathbb{C}\{v_{\lambda_{j-1}}, 1 + \varepsilon_{j-1,1} u_{\lambda_{j-1}}\}$, and for brevity of notations $\varepsilon_{j-1,1}$ may be defined by one, using a nonsingular change of coordinates from $\mathbb{C}\{v_{\lambda_{j-1}}, 1 + \varepsilon_{j-1,1} u_{\lambda_{j-1}}\}$ to itself, independently of analytic representation of $V^{(\lambda_{j-1})}(g_i)$ for all $i \geq j - 1$,
- (iii) $a_j > 0$ and $b_j \geq 0$ are some nonnegative integers with $a_j q_j - b_j p_j = 1$,
- (iv) $E_{\lambda_j} = \{v_{\lambda_j} = 0\}$ is defined by the λ_j -th exceptional curve of the first kind under τ_{λ_j} , for notation.

For simplicity of the notations in (14.1.5), we may use the following, if necessary:

$$(14.1.6) \quad \bar{v}_j = v_{\lambda_j}, \quad \bar{u}_j = u_{\lambda_j}, \quad \bar{\varepsilon}_{j,k} = \varepsilon_{j,k} \quad \text{for } 1 \leq j \leq r \text{ and } 1 \leq k \leq r - j + 2.$$

Note that $\bar{\varepsilon}_{j,k} = \bar{\varepsilon}_{j,k}(\bar{v}_j, 1 + \bar{\varepsilon}_{j,1} \bar{u}_j)$ and $\varepsilon_{j,k} = \varepsilon_{j,k}(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j})$ are the same units in $\mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}\}$ for $1 \leq j \leq r$.

(c) By induction on the integer $j = 1, 2, \dots, r$, at $(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}) = (0, 0)$ along $v_{\lambda_j} = 0$, $(g_j \circ \tau_{\lambda_j})_{total}$ with $(g_j \circ \tau_{\lambda_j})_{proper}$ is written as follows:

For each $\ell = 1, 2, \dots, r - j$,

$$(14.1.7) \quad \begin{aligned} (g_j \circ \tau_{\lambda_j})_{total} &= v_{\lambda_j}^{e_{j,\lambda_j}} (g_j \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\ (g_j \circ \tau_{\lambda_j})_{proper} &= 1 + \varepsilon_{j,1} u_{\lambda_j} \quad \text{with } \varepsilon_{j,1} = 1, \\ (g_{j+1} \circ \tau_{\lambda_j})_{total} &= v_{\lambda_j}^{s_1^{(j)} e_{j,\lambda_j}} (g_{j+1} \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\ (g_{j+1} \circ \tau_{\lambda_j})_{proper} &= (1 + \varepsilon_{j,1} u_{\lambda_j})^{s_1^{(j)}} + \sum_{i=0}^{s_1^{(j)}-1} \eta_{1i}^{(j)} v_{\lambda_j}^{\gamma_{1i1}^{(j)}} (1 + \varepsilon_{j,1} u_{\lambda_j})^i, \\ (g_{j+\ell} \circ \tau_{\lambda_j})_{total} &= v_{\lambda_j}^{s_\ell^{(j)} s_{\ell-1}^{(j)} \cdots s_1^{(j)} e_{j,\lambda_j}} (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\ (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} &= (g_{j+\ell-1} \circ \tau_{\lambda_j})_{proper}^{s_\ell^{(j)}} \\ &\quad + \sum_{i=0}^{s_\ell^{(j)}-1} \eta_{\ell,i}^{(j)} v_{\lambda_j}^{\gamma_{\ell,i,1}^{(j)}} \left\{ \prod_{k=2}^{\ell} (g_{j+k-2} \circ \tau_{\lambda_j})_{proper}^{\gamma_{\ell,i,k}^{(j)}} \right\} (g_{j+\ell-1} \circ \tau_{\lambda_j})_{proper}^i, \end{aligned}$$

(14.1.8) where for each $\ell=1,2, \dots, r-j$,

$$(c0) \quad e_{j,\lambda_j} = n_j \Delta_j(\beta_{j,0,k})_{k=1}^j,$$

$$(c1) \quad s_1^{(j)} = n_{j+1},$$

$$\gamma_{1,i,1}^{(j)} = \Delta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} - n_{j+1} n_j \Delta_j(\beta_{j,i,k})_{k=1}^j > 0,$$

$$(c,\ell+1) \quad s_{\ell+1}^{(j)} = n_{j+\ell+1},$$

$$\begin{aligned} \gamma_{\ell+1,i,1}^{(j)} &= \Delta_{j+1}(\beta_{j+\ell+1,i,k})_{k=1}^{j+1} + \{\beta_{j+\ell+1,i,j+2} + n_{j+1} \beta_{j+\ell+1,i,j+3} + \dots \\ &\quad + n_{j+1} n_{j+2} \dots n_{j+\ell-1} \beta_{j+\ell+1,i,j+\ell+1} - (\prod_{k=1}^{\ell+1} n_{j+k})\} n_j \Delta_j(\beta_{j,i,k})_{k=1}^j > 0, \end{aligned}$$

$$\gamma_{\ell+1,i,2}^{(j)} = \beta_{j+\ell+1,i,j+2}, \gamma_{\ell+1,i,3}^{(j)} = \beta_{j+\ell+1,i,j+3}, \dots, \gamma_{\ell+1,i,\ell+1}^{(j)} = \beta_{j+\ell+1,i,j+\ell+1},$$

and $\eta_{\ell+1,i}^{(j)} = \eta_{\ell+1,i}^{(j)}(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j})$ is a unit in $\mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}\}$ for $1 \leq \ell \leq r-j$ and $0 \leq i \leq s_\ell$.

For brevity of notation, $\varepsilon_{j,1}$ may be defined by one, using a nonsingular change of coordinates from $\mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}\}$ to itself, independently of analytic representation of $V^{(\lambda_j)}(g_i)$ for all $i \geq j$.

(d) Let $\tau_{\lambda_j}^{-1}(0,0) = \cup_{i=1}^{\lambda_j} E_i$ where each E_i is an exceptional curve of the first kind. For $0 \leq \ell \leq r-j+1$, let $(g_{j+\ell} \circ \tau_{\lambda_j})_{\text{divisor}}$ be the divisor of $g_{j+\ell} \circ \tau_{\lambda_j}$ defined by

$$(14.1.9) \quad (g_{j+\ell} \circ \tau_{\lambda_j})_{\text{divisor}} = V^{(\lambda_j)}(g_{j+\ell}) + \sum_{i=1}^{\lambda_j} e_{j+\ell,i} E_i,$$

where $V^{(\lambda_j)}(g_{j+\ell})$ is the proper transform of $V(g_{j+\ell})$ under τ_{λ_j} and $\lambda_j = m_1 + m_2 + \dots + m_j$ as we have seen in (14.1.4).

Then, we have the following:

(d1) For each $\ell = 1, 2, \dots, r-j+1$,

$$(14.1.10) \quad e_{j+\ell,i} = n_{j+\ell} n_{j+\ell-1} \dots n_{j+1} e_{j,i} \quad \text{for } 1 \leq i \leq \lambda_j.$$

In particular, $e_{j,\lambda_j} = n_j \Delta_j(\beta_{j,k})_{k=1}^j$ by (c0) of (14.1.7).

(d2) For any $\ell = 0, 1, 2, \dots, r-j+1$, where E_{λ_j} is defined by $\{v_{\lambda_j} = 0\}$,

$$(14.1.11) \quad V^{(\lambda_j)}(g_{j+\ell}) \cap (\cup_{i=1}^{\lambda_j} E_i) = V^{(\lambda_j)}(g_j) \cap E_{\lambda_j} = \{(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}) = (0,0)\}$$

(d3) We get the following in the sense of Definition 2.6: Note that $\gcd(n_r, \Delta_r(\beta_{r,k})_{k=1}^r) = 1$ because g_r is irreducible in $\mathbb{C}\{y, z\}$ by assumption.

If $\beta_{1,1} \geq 2$, then $g_k \in$ the type $[j]$ under τ_{λ_j} for any $k = j, j+1, \dots, r+1$.

If $\beta_{1,1} = 1$, then $g_k \in$ the type $[j-1]$ under τ_{λ_j} for any $k = j, j+1, \dots, r+1$.

Whether $\beta_{1,1} \geq 2$ or $\beta_{1,1} = 1$, note that $yg_k \in$ the type $[j]$ under τ_{λ_j} for any $k \geq j$. \square

Remark 14.1.1. Moreover, $(y^{\beta_{j+\ell,i,1}} z^{\beta_{j+\ell,i,2}} g_1^{\beta_{j+\ell,i,3}} g_2^{\beta_{j+\ell,i,4}} \dots g_{j+\ell-2}^{\beta_{j+\ell,i,j+\ell}} g_{j+\ell-1}^i) \circ \tau_{\lambda_j}(v_j, u_j)$ can be viewed as

$$(14.1.12) \quad \varepsilon_{j,\ell+1} v_{\lambda_j}^{\Delta_{j+1}^{(j+\ell,k)}_{k=1}^{j+1}} \left(\prod_{k=2}^{\ell} (g_{j+k-2} \circ \tau_{\lambda_j})_{\text{proper}}^{\beta_{j+\ell,i,j+k}} \right) (g_{j+\ell} \circ \tau_{\lambda_j})_{\text{proper}}^i,$$

where (i) the $\beta_{j+\ell,j+k}$ are nonnegative integers for $1 \leq k \leq \ell$,

(ii) for brevity of notation, we write $\Delta_{j+1}^{(j+\ell,k)}_{k=1}^{j+1} = \Delta_{j+1}(\beta_{j+\ell,k})_{k=1}^{j+1} + (\beta_{j+\ell,j+2} + n_{j+1} \beta_{j+\ell,j+3} + \dots + n_{j+1} n_{j+2} \dots n_{j+\ell-2} \beta_{j+\ell,j+\ell}) n_j \Delta_j(\beta_{j,k})_{k=1}^j$,

(iii) $\varepsilon_{j,\ell+1} = \varepsilon_{j,\ell+1}(v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j})$ is a unit in $\mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j}\}$.

Using $e_{j,\lambda_j} = n_j \Delta_j(\beta_{j,0,k})_{k=1}^j$ in (c0) of (14.1.8) and the notation in (ii) of (14.1.12), for each $\ell = 1, 2, \dots, r-j+1$, $\gamma_{\ell,i,1}^{(j)}$ in (c, ℓ) of (14.1.8) can be rewritten in the form

$$\begin{aligned}
 (14.1.13) \quad \gamma_{\ell,i,1}^{(j)} &= \Delta_{j+1}(\beta_{j+\ell,i,k})_{k=1}^{j+1} + \{\beta_{j+\ell,i,j+2} + n_{j+1}\beta_{j+\ell,i,j+3} + n_{j+1}n_{j+2}\beta_{j+\ell,i,j+4} + \dots \\
 &\quad + n_{j+1}n_{j+2} \dots n_{j+\ell-2}\beta_{j+\ell,i,j+\ell} - n_{j+1}n_{j+2} \dots n_{j+\ell-1}n_{j+\ell}\}e_{j,\lambda_j} \\
 &= \Delta_{j+1}^\sharp(\beta_{j+\ell,i,k})_{k=1}^{j+1} - n_{j+1}n_{j+2} \dots n_{j+\ell-1}n_{j+\ell}e_{j,\lambda_j} \\
 &= \Delta_{j+1}^\sharp(\beta_{j+\ell,i,k})_{k=1}^{j+1} - e_{j+\ell,\lambda_j} > 0 \quad \text{by (14.1.10).}
 \end{aligned}$$

Sublemma 14.2. Assumptions *Let j be an arbitrary positive integer with $1 \leq j \leq r$. Suppose that the assumptions and notations of Proposition 14.1 hold. Note by Theorem 12.0 and the above assumptions of Proposition 14.1 that the following is true:*

(14.2.1) g_r is irreducible in $\mathbb{C}\{y, z\}$ with $yg_r \in \text{the type}[r]$ under the standard resolution, but g_{r+1} may not be irreducible in $\mathbb{C}\{y, z\}$.

Conclusions *For each j , let $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point of an analytic variety defined by $V(g_j)$ or $V(yg_j)$.*

*Then, we can construct three sequences, denoted by **Sequences**[I]^(j):*

$$\begin{aligned}
 (14.2.2) \quad \{Y_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\} &\quad \text{with } Y_\ell^{(j)} \subset N_0, \\
 \{h_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\} &\quad \text{with } h_\ell^{(j)} = (g_{j+\ell} \circ \tau_{\lambda_j})_{\text{proper}} \in \mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j}\}, \\
 \{\Xi_\ell^{(j)} : N_0^\ell \rightarrow N_0 \text{ is an integer-valued function for } \ell = 1, 2, \dots, r-j+1\}, \\
 &\quad \text{satisfying the following five conditions for each } k :
 \end{aligned}$$

Five conditions are denoted by **The 1st Cond**^(j), \dots , **The 5-th Cond**^(j) of **Sequences**[I]^(j).

The 1st Cond^(j): *Let $\{Y_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\}$ with $Y_\ell^{(j)} \subset N_0$ be defined by (14.2.3).*

$$\begin{aligned}
 (14.2.3)(1a) \quad Y_1^{(j)} &= \{s_1^{(j)}\} \cup \{\gamma_{1,i,1}^{(j)} : 0 \leq i < s_1^{(j)}\} \quad \text{with } s_1^{(j)} \geq 2 \text{ and } \gamma_{1,0,1}^{(j)} \geq 1 \text{ where } Y_1^{(j)} \subset N, \\
 (1b) \quad Y_\ell^{(j)} &= \{s_\ell^{(j)}\} \cup \{\gamma_{\ell,i,1}^{(j)} : 0 \leq i < s_\ell^{(j)}\} \cup \{\gamma_{\ell,i,2}^{(j)} : 0 \leq i < s_\ell^{(j)}\} \cup \dots \cup \{\gamma_{\ell,i,\ell}^{(j)} : 0 \leq i < s_\ell^{(j)}\} \\
 &\quad \text{with } s_\ell^{(j)} \geq 2 \text{ where } j = 1, 2, \dots, r,
 \end{aligned}$$

such that for each $j = 1, 2, \dots, r$, assume that at least one of $\gamma_{\ell,0,1}^{(j)}, \gamma_{\ell,0,2}^{(j)}, \dots, \gamma_{\ell,0,\ell}^{(j)}$ is nonzero by Sublemma 12.1 and

such that each of (14.2.3) satisfies either (a) or (b) of (14.2.4):

$$\begin{aligned}
 (14.2.4)(a) \quad &\text{if } j = 1, \text{ then the first family } \{Y_\ell^{(1)} : \ell = 1, 2, \dots, r\} \text{ is defined as follows:} \\
 (a0) \quad &e_{1,\lambda_1} = n_1 \Delta_1(\beta_{1,0,1}) = n_1 \beta_{1,0,1}, \\
 (a1) \quad &s_1^{(1)} = n_2 \geq 2, \gamma_{1,i,1}^{(1)} = \Delta_2^\sharp(\beta_{2,i,1}, \beta_{2,i,2}) - (n_2 - i)n_1 \beta_{1,0,1} > 0 \text{ for } 0 \leq i < n_2, \\
 (a2) \quad &s_\ell^{(1)} = n_{\ell+1} \geq 2, \gamma_{\ell,i,1}^{(1)} = \Delta_{\ell+1}^\sharp(\beta_{\ell+1,i,k})_{k=1}^{\ell+1} - (n_{\ell+1} - i)n_\ell \dots n_2 n_1 \beta_{1,0,1} > 0, \text{ and} \\
 &\gamma_{\ell,i,2}^{(1)} = \beta_{\ell+1,i,3}, \gamma_{\ell,i,3}^{(1)} = \beta_{\ell+1,i,4}, \dots, \gamma_{\ell,i,\ell}^{(1)} = \beta_{\ell+1,i,\ell+1} \text{ for } \ell \geq 2 \text{ and } 0 \leq i < n_\ell, \\
 &\text{noting that } \gamma_{1,0,1}^{(1)}, \gamma_{2,0,1}^{(1)}, \dots, \gamma_{\ell,0,1}^{(1)} \text{ are positive by Sublemma 12.1.}
 \end{aligned}$$

(14.2.4)(b) *for each $j = 2, 3, \dots, r$, the family $\{Y_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\}$ is defined as follows:*

$$\begin{aligned}
 (b0) \quad &e_{j,\lambda_j} = n_j \Delta_j(\beta_{j,0,k})_{k=1}^j, \\
 (b1) \quad &s_1^{(j)} = s_2^{(j-1)} = s_3^{(j-2)} = \dots = s_j^{(1)} = n_{j+1} \geq 2, \gamma_{1,i,1}^{(j)} = \Xi_2^{(j-1)\sharp}(\gamma_{2,i,1}^{(j-1)}, \gamma_{2,i,2}^{(j-1)}) - \\
 &\quad (s_2^{(j-1)} - i)s_1^{(j-1)}\gamma_{1,0,1}^{(j-1)} > 0 \text{ for } 0 \leq i < s_2^{(j-1)}, \\
 (b2) \quad &s_\ell^{(j)} = s_{\ell+1}^{(j-1)} = s_{\ell+2}^{(j-2)} = \dots = s_j^{(\ell)} = n_{j+\ell}. \\
 &\gamma_{\ell,i,1}^{(j)} = \Xi_{\ell+1}^{(j-1)\sharp}(\gamma_{\ell+1,i,k}^{(j-1)})_{k=1}^{\ell+1} - (s_{\ell+1}^{(j-1)} - i)s_\ell^{(j-1)} \dots s_2^{(j-1)} s_1^{(j-1)} \gamma_{1,0,1}^{(j-1)} > 0, \gamma_{\ell,i,2}^{(j)} = \\
 &\gamma_{\ell+1,i,3}^{(j-1)}, \gamma_{\ell,i,3}^{(j)} = \gamma_{\ell+1,i,4}^{(j-1)}, \gamma_{\ell,i,4}^{(j)} = \gamma_{\ell+1,i,5}^{(j-1)}, \dots, \gamma_{\ell,i,\ell}^{(j)} = \gamma_{\ell+1,i,\ell+1}^{(j-1)} \text{ for } 0 \leq i < s_{\ell+1}^{(j-1)},
 \end{aligned}$$

The 2-th Cond^(j): For $\ell = 1, 2, \dots, r-j+1$, let $h_\ell^{(j)} = (g_{j+\ell} \circ \tau_{\lambda_j})_{proper}$ be defined by

$$\begin{aligned}
(14.2.5) \quad & (g_j \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{e_{j,\lambda_j}} (g_j \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\
& (g_j \circ \tau_{\lambda_j})_{proper} = 1 + \varepsilon_{j,1} u_{\lambda_j} \quad \text{with } \varepsilon_{j,1} = 1, \\
& (g_{j+1} \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{s_1^{(j)} e_{j,\lambda_j}} (g_{j+1} \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\
& (g_{j+1} \circ \tau_{\lambda_j})_{proper} = (1 + \varepsilon_{j,1} u_{\lambda_j})^{s_1^{(j)}} + \sum_{i=0}^{s_1^{(j)}-1} \eta_{1i}^{(j)} v_{\lambda_j}^{\gamma_{1i}^{(j)}} (1 + \varepsilon_{j,1} u_{\lambda_j})^i, \\
& (g_{j+\ell} \circ \tau_{\lambda_j})_{total} = v^{s_\ell^{(j)} s_{\ell-1}^{(j)} \cdots s_1^{(j)} e_{j,\lambda_j}} (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\
& (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} = (g_{j+\ell-1} \circ \tau_{\lambda_j})_{proper}^{s_\ell^{(j)}} \\
& \quad + \sum_{i=0}^{s_\ell^{(j)}-1} \eta_{\ell,i}^{(j)} v_{\lambda_j}^{\gamma_{\ell,i,1}^{(j)}} \left\{ \prod_{k=2}^{\ell} (g_{j+k-2} \circ \tau_{\lambda_j})_{proper}^{\gamma_{\ell,i,k}^{(j)}} \right\} (g_{j+\ell-1} \circ \tau_{\lambda_j})_{proper}^i,
\end{aligned}$$

where $\varepsilon_{j,1} = \varepsilon_{j,1}(v_{\lambda_j}, u_{\lambda_j})$ is a unit in $\mathbb{C}\{v_{\lambda_j}, u_{\lambda_j}\}$, and also $\eta_{\ell,i}^{(j)} = \eta_{\ell,i(\ell)}^{(j)}(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j})$ is a unit in $\mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}\}$ for $\ell = 1, 2, \dots, r-j+1$ and $i = i(\ell) = 0, 1, \dots, s_\ell^{(j)} - 1$. For brevity of notation, $\varepsilon_{j,1}$ may be defined by one, using a nonsingular change of coordinates from $\mathbb{C}\{v_{\lambda_j}, u_{\lambda_j}\}$ to itself, independently of analytic representation of $V^{(\lambda_j)}(g_i)$ for all $i \geq j$.

The 3-th Cond^(j). Let $\{\Xi_\ell^{(j)} : N_0^\ell \rightarrow N_0, \ell = 1, 2, \dots, r-j+1\}$ be a sequence such that each $\Xi_k^{(j)}$ is an integer-valued function defined by the following: Note that $2 \leq \ell \leq r-j+1$.

$$\begin{aligned}
(14.2.6) \quad & (3a) \quad \Xi_1^{(j)}(t) = t \text{ for each } t \in N_0. \\
& (3b) \quad \Xi_\ell^{(j)}(t_k)_{k=1}^\ell = t_\ell \Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1} + s_{\ell-1}^{(j)} \Xi_{\ell-1}^{(j)}(t_k)_{k=1}^{\ell-1} \text{ for each } (t_k)_{k=1}^\ell \in N_0^\ell.
\end{aligned}$$

Moreover, for brevity of notation, let $\{\Xi_\ell^{(j)\sharp} : N_0^\ell \rightarrow N_0, \ell = 2, 3, \dots, r-j+1\}$ be a sequence such that each $\Xi_\ell^{(j)\sharp}$ is an integer-valued function defined by the following:

$$\begin{aligned}
(14.2.6-1) \quad & \Xi_2^{(j)\sharp}(\gamma_{2,i,k}^{(j)})_{k=1}^2 = \Xi_2^{(j)}(\gamma_{2,i,k}^{(j)})_{k=1}^2 \quad \text{for } 0 \leq i < s_2^{(j)}, \\
& \Xi_\ell^{(j)\sharp}(\gamma_{\ell,i,k}^{(j)})_{k=1}^\ell = \Xi_2^{(j)}(\gamma_{\ell,i,k}^{(j)})_{k=1}^2 + s_1^{(j)} \gamma_{1,0,1}^{(j)} \gamma_{\ell,i,3}^{(j)} + s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} \gamma_{\ell,i,4}^{(j)} \\
& \quad + s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} s_3^{(j)} \gamma_{\ell,i,5}^{(j)} + \cdots + s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} \cdots s_{\ell-2}^{(j)} \gamma_{\ell,i,\ell}^{(j)} \quad \text{for } 0 \leq i < s_\ell^{(j)}.
\end{aligned}$$

The (4 α)-th Cond^(j). Then, the following equalities hold: Note that $r \geq 2$.

(a) For $j = 1$, the following equality holds with respect to $\{Y_\ell^{(1)} : \ell = 1, 2, \dots, r\}$:

$$\begin{aligned}
(14.2.7)(14.2.7-1) \quad & \Xi_1^{(1)}(\gamma_{1,i,1}^{(1)}) = \gamma_{1,i,1}^{(1)} = \Delta_2(\beta_{2,i,1}, \beta_{2,i,2}) - (n_2 - i)n_1\beta_{1,0,1} > 0, \\
& \Xi_\ell^{(1)}(\gamma_{\ell,i,k}^{(1)})_{k=1}^\ell - (s_\ell^{(1)} - i)s_{\ell-1}^{(1)} \Xi_{\ell-1}^{(1)}(\gamma_{\ell-1,0,k}^{(1)})_{k=1}^{\ell-1} \quad \text{for } 2 \leq \ell \leq r \\
& \quad = \Delta_{\ell+1}(\beta_{\ell+1,i,k})_{k=1}^{\ell+1} - (n_{\ell+1} - i)n_\ell \Delta_\ell(\beta_{\ell,0,k})_{k=1}^\ell > 0.
\end{aligned}$$

(b) For each $j = 2, 3, \dots, r$, the following equality holds with respect to the family $\{Y_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\}$:

$$\begin{aligned}
(14.2.7)(14.2.7-2) \quad & \Xi_1^{(j)}(\gamma_{1,i,1}^{(j)}) = \gamma_{1,i,1}^{(j)} = \Xi_2^{(j-1)}(\gamma_{2,i,1}^{(j-1)}, \gamma_{2,i,2}^{(j-1)}) - (s_2^{(j-1)} - i)s_1^{(j-1)} \gamma_{1,0,1}^{(j-1)} > 0, \\
& \Xi_\ell^{(j)}(\gamma_{\ell,i,k}^{(j)})_{k=1}^\ell - (s_\ell^{(j)} - i)s_{\ell-1}^{(j)} \Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1} \quad \text{for } 2 \leq \ell \leq r-j+1 \\
& \quad = \Xi_{\ell+1}^{(j-1)}(\gamma_{\ell+1,i,k}^{(j-1)})_{k=1}^{\ell+1} - (s_{\ell+1}^{(j-1)} - i)s_\ell^{(j-1)} \Xi_\ell^{(j-1)}(\gamma_{\ell,0,k}^{(j-1)})_{k=1}^\ell > 0.
\end{aligned}$$

The 4-th Cond^(j). The following inequalities hold: Note that $r \geq 2$.

$$(14.2.8) \quad (4a) \quad \Xi_1^{(j)}(\gamma_{1,i,1}^{(j)}) = \gamma_{1,i,1}^{(j)} > 0.$$

$$(4b) \quad \Xi_\ell^{(j)}(\gamma_{\ell,i,k}^{(j)})_{k=1}^\ell > (s_\ell^{(j)} - i)s_{\ell-1}^{(j)}\Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,i,k}^{(j)})_{k=1}^{\ell-1} \quad \text{for } 2 \leq \ell \leq r-j+1.$$

The 5-th Cond^(j). For $\ell=1,2, \dots, r-j+1$, the following equalities hold: Note that $r \geq 2$.

$$(14.2.9)(14.2.9-1) \quad \gcd(s_\ell^{(1)}, \Xi_\ell^{(1)}(\gamma_{\ell,0,k}^{(1)})_{k=1}^\ell) = \gcd(n_{\ell+1}, \Delta_{\ell+1}(\beta_{\ell+1,0,k})_{k=1}^{\ell+1}), \quad \text{for } 1 \leq \ell \leq r,$$

$$\gcd(s_\ell^{(j)}, \Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell) = \gcd(s_{\ell+1}^{(j-1)}, \Xi_{\ell+1}^{(j-1)}(\gamma_{\ell+1,0,k}^{(j-1)})_{k=1}^{\ell+1}) \quad \text{for } j \geq 2 \text{ with } j + \ell \leq r + 1.$$

The 5-th Cond^(j). The following inequalities hold: Note that $r \geq 2$.

$$(14.2.9)(14.2.9-2) \quad \frac{\gamma_{1,i,1}^{(j)}}{s_1^{(j)} - i} > \frac{\gamma_{1,0,1}^{(j)}}{s_1^{(j)}} > 0 \quad \text{for } 0 < i < s_1^{(j)}.$$

$$\frac{\Xi_\ell^{(j)}(\gamma_{\ell,i,k}^{(j)})_{k=1}^\ell}{s_\ell^{(j)} - i} > \frac{\Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell}{s_\ell^{(j)}} > s_{\ell-1}^{(j)}\Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1} \quad \text{for } 0 < i < s_\ell^{(j)}.$$

Remark for The 5-th Cond^(j). For notation, the above inequality in (14.2.9-2) may be equivalently rewritten as follows: For each $\ell = 1, 2, \dots, r-j+1$,

$$\frac{\Xi_\ell^{(j)}(\gamma_{\ell,i,k}^{(j)})_{k=1}^\ell - (s_\ell^{(j)} - i)s_{\ell-1}^{(j)}\Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1}}{s_\ell^{(j)} - i}$$

$$> \frac{\Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell - s_\ell^{(j)}s_{\ell-1}^{(j)}\Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1}}{s_\ell^{(j)}} > 0 \quad \text{for } 0 < i < s_\ell^{(j)}.$$

Remark 14.2.1.1. The 5-th Cond⁽⁰⁾ in the assumptions of Theorem 14.0 For each $\ell = 1, 2, \dots, r$, the following inequalities hold:

$$(5)(5a) \quad \gcd(n_\ell, \Delta_\ell(\beta_{\ell,0,k})_{k=1}^\ell) = 1 \quad \text{for } 1 \leq \ell \leq r.$$

$$(5b) \quad \frac{\Delta_\ell(\beta_{\ell,i,k})_{k=1}^\ell}{n_\ell - i} > \frac{\Delta_\ell(\beta_{\ell,0,k})_{k=1}^\ell}{n_\ell} \quad \text{for } 0 < i < n_\ell.$$

Remark 14.2.1.2. For brevity of the proof of **Sublemma 14.2**, suppose that **The 1st Cond^(j)**, **The 2nd Cond^(j)** and **The 3rd Cond^(j)** have proved in the conclusions of this sublemma. Then, for the remaining proof, it suffices to show that **The (4 α)-th Cond^(j)** is true, because of the following two facts:

Fact(1): If **The 4 α -th Cond^(j)** is true, then it is clear that **The 4-th Cond^(j)** is true.

Fact(2): If **The 4 α -th Cond^(j)** is true, then we can easily prove by **The 1-th Cond^(j)** that **The 5-th Cond^(j)** is true, by using the following elementary computation:

Case(A): $j = 1$, and Case(B): $j \geq 2$.

Case(A). Let $j = 1$.

$$(i) \quad \gcd(s_1^{(1)}, \gamma_{1,0,1}) = \gcd(n_2, \Delta_2(\beta_{2,0,1}, \beta_{2,0,2}) - n_2 n_1 \beta_{1,0,1}) = \gcd(n_2, \Delta_2(\beta_{2,0,1}, \beta_{2,0,2})) = 1.$$

$$(ii) \quad \text{For each } q = 2, 3, \dots, r-1, s_q^{(1)} = n_{q+1}, \text{ and by The (4}\alpha\text{)-th Cond}^{(j)} \text{ we have}$$

$$(14.2.10-1) \quad \gcd(s_q^{(1)}, \Xi_q(\gamma_{q,0,k})_{k=1}^q) = \gcd(s_q^{(1)}, \Xi_q(\gamma_{q,0,k})_{k=1}^q - s_q^{(1)}s_{q-1}^{(1)}\Xi_{q-1}(\gamma_{q-1,0,k})_{k=1}^{q-1})$$

$$= \gcd(n_{q+1}, \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1} - n_{q+1}n_q\Delta_q(\beta_{q,0,k})_{k=1}^q) \quad \text{by (12.5.4}\alpha\text{)}$$

$$= \gcd(n_{q+1}, \Delta_{q+1}(\beta_{q+1,0,k})_{k=1}^{q+1}) = 1. \quad \square$$

Case(B). Let $j \geq 2$.

$$(i) \gcd(s_1^{(j)}, \gamma_{1,0,1}^{(j)}) = \gcd(s_2^{(j-1)}, \Xi_2^{(j-1)}(\gamma_{2,0,k}^{(j-1)})_{k=1}^2 - s_2^{(j-1)} s_1^{(j-1)} \gamma_{1,0,1}^{(j-1)}) \\ = \gcd(s_2^{(j-1)}, \Xi_2^{(j-1)}(\gamma_{2,0,k}^{(j-1)})_{k=1}^2) = 1.$$

(ii) For each $\ell = 2, 3, \dots, r-1$, $s_\ell^{(j)} = s_{\ell+1}^{(j-1)}$, and by The (4 α)-th Cond⁽ⁱ⁾ we have

$$(14.2.10-2) \quad \gcd(s_\ell^{(j)}, \Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell) \\ = \gcd(s_\ell^{(j)}, \Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell - s_\ell^{(j)} s_{\ell-1}^{(j)} \Xi_{\ell-1}^{(j)}(\gamma_{\ell-1,0,k}^{(j)})_{k=1}^{\ell-1}) \\ = \gcd(s_{\ell+1}^{(j-1)}, \Xi_{\ell+1}^{(j-1)}(\gamma_{\ell+1,0,k}^{(j-1)})_{k=1}^{\ell+1} - s_{\ell+1}^{(j-1)} s_\ell^{(j-1)} \Xi_\ell^{(j-1)}(\gamma_{\ell,0,k}^{(j-1)})_{k=1}^\ell) \\ = \dots \\ = \gcd(n_{j+\ell}, \Delta_{j+\ell}(\beta_{j+\ell,0,k})_{k=1}^{j+\ell} - n_{j+\ell} n_{j+\ell-1} \Delta_{j+\ell-1}(\beta_{j+\ell-1,0,k})_{k=1}^{j+\ell-1}) \\ = \gcd(n_{j+\ell}, \Delta_{j+\ell}(\beta_{j+\ell,0,k})_{k=1}^{j+\ell}) = 1 \quad \text{by (14.2.10-1)} \quad \square$$

Sublemma 14.3. Assumptions Let r be an arbitrary integer with $r \geq 2$. Let $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point of $V(g_j)$ or $V(yg_j)$. Suppose that the assumptions of Sublemma 14.2 hold.

Conclusions The aim is to prove the following by induction on the integer $j > 0$.

(1) As we have seen in the conclusion of Proposition 14.1, $\tau_{\lambda_j} = \mu_{1,m_1} \circ \mu_{2,m_2} \circ \dots \circ \mu_{j,m_j}$ can be represented as the composition of a finite number $\lambda_j = m_1 + m_2 + \dots + m_j$ of successive blow-ups which is needed to get the standard resolution of the isolated singular point of $V(yg_j)$ or $V(g_j)$, satisfying the following with the desired properties and notations:

(i) $\tau_{\lambda_1} = \mu_{1,\lambda_1}$ is the composition of a finite number λ_1 of successive blow-ups which is needed to get the standard resolution of the singular point of either $V(yg_1)$ or $V(g_1)$.

(ii) μ_{s,m_s} is the composition of a finite number m_s of successive blow-ups which is needed to get the standard resolution of one and only one quasisingular point of $V^{(\lambda_{s-1})}(g_s)$ with $2 \leq s \leq j \leq r$. That is, $\tau_{\lambda_s} = \mu_{1,m_1} \circ \mu_{2,m_2} \circ \dots \circ \mu_{s,m_s}$.

(2) Whenever the family $\{(g_{j+\ell} \circ \tau_{\lambda_j})_{proper} : \ell = 1, 2, \dots, r-j+1\}$ with $(g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \in \mathbb{C}\{1 + \varepsilon_{j,1} u_{\lambda_j}, v_{\lambda_j}\}$ satisfies five conditions in the assumptions of Sublemma 14.2, denoted by **The 1-th Cond**⁽ⁱ⁾, ..., **The 5-th Cond**⁽ⁱ⁾, then without assuming irreducibility of $(g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \in \mathbb{C}\{1 + \varepsilon_{j,1} u_{\lambda_j}, v_{\lambda_j}\}$ it was already proved by either Sublemma 12.2 of Theorem 12.0 or Remark 14.2.1 that we get the following:

For any $\ell = 1, 2, \dots, r-j+1$, $(g_{j+\ell} \circ \tau_{\lambda_j})_{total}$ with $(g_{j+\ell} \circ \tau_{\lambda_j})_{proper}$ can be written in the form

$$(14.3.1) \quad (g_j \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{e_{j,\lambda_j}} (g_j \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\ (g_j \circ \tau_{\lambda_j})_{proper} = (1 + \varepsilon_{j,1} u_{\lambda_j}) \\ (g_{j+\ell} \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{s_\ell^{(j)} s_{\ell-1}^{(j)} \dots s_2^{(j)} s_1^{(j)} e_{j,\lambda_j}} (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \quad \text{with} \\ (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} = \{(1 + \varepsilon_{j,1} u_{\lambda_j})^{s_1^{(j)}} + v_{\lambda_j}^{\gamma_{1,0,1}^{(j)}}\}^{s_2^{(j)} s_3^{(j)} \dots s_\ell^{(j)}} + \sum_{\alpha, \beta \geq 0} B_{\ell, \alpha, \beta}^{(j)} v_{\lambda_j}^\alpha (1 + \varepsilon_{j,1} u_{\lambda_j})^\beta,$$

where a unit $\varepsilon_{j,1} = \varepsilon_{j,1}(1 + \varepsilon_{j,1} u_{\lambda_j}, v_{\lambda_j})$ may be analytically assumed to be one in $\mathbb{C}\{1 + \varepsilon_{j,1} u_{\lambda_j}, v_{\lambda_j}\}$, and the $B_{\ell, \alpha, \beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $s_1^{(j)} \alpha + \gamma_{1,0,1}^{(j)} \beta > s_\ell^{(j)} s_{\ell-1}^{(j)} \dots s_1^{(j)} \gamma_{1,0,1}^{(j)}$.

(3) Let $\tau_{\lambda_j}^{-1}(0,0) = \cup_{i=1}^{\lambda_j} E_i$ where each E_i is an exceptional curve of the first kind. For $0 \leq \ell \leq r-j+1$, let $(g_j \circ \tau_{\lambda_j})_{\text{divisor}}$ and $(g_{j+\ell} \circ \tau_{\lambda_j})_{\text{divisor}}$ be the divisors of $g_j \circ \tau_{\lambda_j}$ and $g_{j+\ell} \circ \tau_{\lambda_j}$ under τ_{λ_j} , respectively, which are defined by

$$(14.3.2) \quad \begin{aligned} (g_j \circ \tau_{\lambda_j})_{\text{divisor}} &= V^{(\lambda_j)}(g_j) + \sum_{i=1}^{\lambda_j} e_{j,i} E_i, \\ (g_{j+\ell} \circ \tau_{\lambda_j})_{\text{divisor}} &= V^{(\lambda_j)}(g_{j+\ell}) + \sum_{i=1}^{\lambda_j} e_{j+\ell,i} E_i, \end{aligned}$$

where $V^{(\lambda_j)}(g_{j+\ell})$ is the proper transform of $V(g_{j+\ell})$ under τ_{λ_j} and $\lambda_j = m_1 + m_2 + \dots + m_j$ as we have seen in (14.1.4) of Proposition 14.1.

Then, we have the following:

$$(3a) \text{ For each } \ell = 1, 2, \dots, r-j+1, \\ (14.3.3) \quad e_{j+\ell,i} = n_{j+\ell} n_{j+\ell-1} \dots n_{j+1} e_{j,i} \quad \text{for } 1 \leq i \leq \lambda_j.$$

In particular, $e_{j,\lambda_j} = n_j \Delta_j(\beta_{j,0,k})_{k=1}^j$ by (a0) and (b0) of (14.2.4).

$$(3b) \text{ For any } \ell = 0, 1, 2, \dots, r-j+1, \text{ where } E_{\lambda_j} \text{ is defined by } \{v_{\lambda_j} = 0\}, \\ (14.3.4) \quad V^{(\lambda_j)}(g_{j+\ell}) \cap (\cup_{i=1}^{\lambda_j} E_i) = V^{(\lambda_j)}(g_j) \cap E_{\lambda_j} = \{(v_{\lambda_j}, 1 + \varepsilon_{j,1} u_{\lambda_j}) = (0, 0)\}.$$

(3c) We get the following in the sense of Definition 2.6.

If $\beta_{1,0,1} \geq 2$, then $g_k \in$ the type $[j]$ under τ_{λ_j} for any $k = j, j+1, \dots, r+1$.

If $\beta_{1,0,1} = 1$, then $g_k \in$ the type $[j-1]$ under τ_{λ_j} for any $k = j, j+1, \dots, r+1$.

Whether $\beta_{1,0,1} \geq 2$ or $\beta_{1,0,1} = 1$, note that $yg_k \in$ the type $[j]$ under τ_{λ_j} for any $k \geq j$. \square

§14.2. The proof of Proposition 14.1 with Sublemma 14.2 and Sublemma 14.3

In this section, Proposition 14.1 with Sublemma 14.2 and Sublemma 14.3 will be proved by induction on the integer j where for $1 \leq j \leq r$, each $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ of Proposition 14.1 is defined to be the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point of $V(g_j)$ or $V(yg_j)$. So, for the proof of Proposition 14.1, it is enough to consider two cases, respectively:

Case(I) $j = 1$, and Case(II) $2 \leq j \leq r$.

Case(I): Let $j = 1$. As far as τ_{λ_1} is concerned, there is nothing to prove for Proposition 14.1, because if $j = 1$ for τ_{λ_j} then the proof of Proposition 14.1 with Sublemma 14.2 and Sublemma 14.3 was already done by Sublemma 12.4 and Sublemma 12.5 of Theorem 12.0.

Case(II): We are going to prove that Proposition 14.1 with Sublemma 14.2 and Sublemma 14.3 is true for $2 \leq j \leq r$. By the induction proof on the positive integer j , suppose we have shown by Case(I) that Sublemma 14.2, Sublemma 14.3 and Proposition 14.1 are true for $1 \leq j < r$.

Then, it suffices to show that the following three statements are true, respectively.

(1) **Statement 14.4:** Firstly, we prove that Sublemma 14.2 on the integer $j+1$ is true.

(2) **Statement 14.5:** Secondly, we prove that Sublemma 14.3 on the integer $j+1$ is true.

(3) **Statement 14.6:** Thirdly, we prove that Proposition 14.1 on the integer $j+1$ is true.

(1) Statement 14.4 with proof.

In order to finish the proof of sublemma 14.2 on the integer $j+1$, it suffices to prove the following, called Statement 14.4.

In preparation for the proof of Sublemma 14.2 on the integer $(j+1)$, first of all it is needed to show that the conclusion for Sublemma 14.2 on the integer (j) implies the assumption of Sublemma 12.5 up to the change of notations, noting that instead of an equality of (12.5.0) in the assumption of Sublemma 12.5, we need one and only one equality, being defined by $\gcd(s_1^{(j)}, \gamma_{1,0,1}^{(j)}) = 1$ of (14.2.9), which was already proved in **The 5-th Cond**⁽ⁱ⁾ of Sublemma 14.2 on the integer (j) .

Statement 14.4. Assumptions *Let j be an arbitrary positive integer with $1 \leq j \leq r$. Suppose that the assumptions and notations of Proposition 14.1 hold. Note by Theorem 12.0 and the above assumptions of Proposition 14.1 that the following is true:*

$$(14.4.0) \quad g_r \text{ is irreducible in } \mathbb{C}\{y, z\} \text{ with } yg_r \in \text{the type}[r] \text{ under} \\ \text{the standard resolution, but } g_{r+1} \text{ may not be irreducible in } \mathbb{C}\{y, z\}.$$

*By induction assumption on the integer j , suppose that the statement on the integer j in the conclusion of Sublemma 14.2 is true. In other words, we may assume by Sublemma 14.2 that there are three sequences of (14.2.2) as we have seen, denoted by **Sequences**[I]^(j):*

$$(14.4.1) \quad \{Y_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\} \quad \text{with } Y_\ell^{(j)} \subset N_0, \\ \{h_\ell^{(j)} : \ell = 1, 2, \dots, r-j+1\} \quad \text{with } h_\ell^{(j)} = (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \in \mathbb{C}\{v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j}\}, \\ \{\Xi_\ell^{(j)} : N_0^\ell \rightarrow N_0 \text{ is an integer-valued function for } \ell = 1, 2, \dots, r-j+1\}, \\ \text{satisfying five conditions in the conclusion of Sublemma 14.2.2 :}$$

*Five conditions are denoted by **The 1st Cond**^(j), ..., **The 5-th Cond**^(j) of **Sequences**[I]^(j).*

Conclusions *Then, we can prove that the statement on the integer $(j+1)$ in the conclusion of Sublemma 14.2 is true.*

Equivalently, for each j , let $\tau_{\lambda_{j+1}} : M^{(\lambda_{j+1})} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ_{j+1} of successive blow-ups which is needed to get the standard resolution of the singular point of an analytic variety defined by $V(g_{j+1})$ or $V(yg_{j+1})$.

*Then, we can construct three sequences, denoted by **Sequences**[I]^(j+1):*

$$(14.4.2) \quad \{Y_\ell^{(j+1)} : \ell = 1, 2, \dots, r-j\} \quad \text{with } Y_\ell^{(j+1)} \subset N_0, \\ \{h_\ell^{(j+1)} : \ell = 1, 2, \dots, r-j\} \quad \text{with } h_\ell^{(j+1)} = (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} \in \mathbb{C}\{v_{\lambda_{j+1}}, 1 + \varepsilon_{j+1,1}u_{\lambda_{j+1}}\}, \\ \{\Xi_\ell^{(j+1)} : N_0^\ell \rightarrow N_0 \text{ is an integer-valued function for } \ell = 1, 2, \dots, r-j\}, \\ \text{satisfying the following five conditions for each } j:$$

*Five conditions are denoted by **The 1st Cond**^(j+1), ..., **The 5-th Cond**^(j+1) of **Sequences**[I]^(j+1).*

The 1st Cond^(j+1): *Let $\{Y_\ell^{(j+1)} : \ell = 1, 2, \dots, r-j\}$ with $Y_\ell^{(j+1)} \subset N_0$ be defined by (14.2.3).*

$$(14.4.3)(1a) \quad Y_1^{(j+1)} = \{s_1^{(j+1)}\} \cup \{\gamma_{1,i,1}^{(j+1)} : 0 \leq i < s_1^{(j+1)}\} \quad \text{with } s_1^{(j+1)} \geq 2 \text{ and } \gamma_{1,0,1}^{(j+1)} \geq 1 \text{ where} \\ Y_1^{(j+1)} \subset N, \\ (1b) \quad Y_\ell^{(j+1)} = \{s_\ell^{(j+1)}\} \cup \{\gamma_{\ell,i,1}^{(j+1)} : 0 \leq i < s_\ell^{(j+1)}\} \cup \{\gamma_{\ell,i,2}^{(j+1)} : 0 \leq i < s_\ell^{(j+1)}\} \cup \dots \cup \{\gamma_{\ell,i,\ell}^{(j+1)} : \\ 0 \leq i < s_\ell^{(j+1)}\} \quad \text{with } s_\ell^{(j+1)} \geq 2 \text{ where } j = 1, 2, \dots, r,$$

such that for each $j = 1, 2, \dots, r$, assume that at least one of $\gamma_{\ell,0,1}^{(j+1)}, \gamma_{\ell,0,2}^{(j+1)}, \dots, \gamma_{\ell,0,\ell}^{(j+1)}$ is nonzero by Sublemma 12.1 and

such that each of (14.4.3) satisfies (a) of (14.4.4):

$$(14.4.4)(a) \quad \text{for each } j+1 = 2, 3, \dots, r, \text{ the family } \{Y_\ell^{(j+1)} : \ell = 1, 2, \dots, r-j\} \text{ is defined as follows:}$$

$$(a0) \quad e_{j+1,\lambda_{j+1}} = n_{j+1} \Delta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}, \\ (a1) \quad s_1^{(j+1)} = s_2^{(j)} = n_{j+2} \geq 2, \quad \gamma_{1,i,1}^{(j+1)} = \Xi_2^{(j)\#}(\gamma_{2,i,1}^{(j)}, \gamma_{2,i,2}^{(j)}) - (s_2^{(j)} - i)s_1^{(j)}\gamma_{1,0,1}^{(j)} > 0 \text{ for} \\ 0 \leq i < s_2^{(j)}, \\ (a2) \quad s_\ell^{(j+1)} = s_{\ell+1}^{(j)} = n_{j+1+\ell}. \\ \gamma_{\ell,i,1}^{(j+1)} = \Xi_{\ell+1}^{(j)\#}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{\ell+1} - (s_{\ell+1}^{(j)} - i)s_\ell^{(j)} \dots s_2^{(j)} s_1^{(j)} \gamma_{1,0,1}^{(j)} > 0, \quad \gamma_{\ell,i,2}^{(j+1)} = \gamma_{\ell+1,i,3}^{(j)}, \\ \gamma_{\ell,i,3}^{(j+1)} = \gamma_{\ell+1,i,4}^{(j)}, \quad \gamma_{\ell,i,4}^{(j+1)} = \gamma_{\ell+1,i,5}^{(j)} \dots, \quad \gamma_{\ell,i,\ell}^{(j+1)} = \gamma_{\ell+1,i,\ell+1}^{(j)} \text{ for } 0 \leq i < s_{\ell+1}^{(j)},$$

The 2-th Cond^(j+1): For $\ell = 1, 2, \dots, r-j$, let $h_\ell^{(j+1)} = (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper}$ be defined by

$$\begin{aligned}
(14.4.5) \quad & (g_{j+1} \circ \tau_{\lambda_{j+1}})_{total} = v_{\lambda_{j+1}}^{e_{j+1, \lambda_{j+1}}} (g_{j+1} \circ \tau_{\lambda_{j+1}})_{proper} \quad \text{with} \\
& (g_{j+1} \circ \tau_{\lambda_{j+1}})_{proper} = 1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}} \quad \text{with } \varepsilon_{j+1,1} = 1, \\
& (g_{j+2} \circ \tau_{\lambda_{j+1}})_{total} = v_{\lambda_{j+1}}^{s_1^{(j+1)} e_{j+1, \lambda_{j+1}}} (g_{j+2} \circ \tau_{\lambda_{j+1}})_{proper} \quad \text{with} \\
& (g_{j+2} \circ \tau_{\lambda_{j+1}})_{proper} = (1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})^{s_1^{(j+1)}} + \sum_{i=0}^{s_1^{(j+1)}-1} \eta_{1,i}^{(j+1)} v_{\lambda_{j+1}}^{\gamma_{1,i,1}^{(j+1)}} (1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})^i, \\
& (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{total} = v_{\lambda_{j+1}}^{s_\ell^{(j+1)} s_{\ell-1}^{(j+1)} \dots s_1^{(j+1)} e_{j+1, \lambda_{j+1}}} (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} \quad \text{with} \\
& (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} = (g_{j+1+\ell-1} \circ \tau_{\lambda_{j+1}})_{proper}^{s_\ell^{(j+1)}} \\
& + \sum_{i=0}^{s_\ell^{(j+1)}-1} \eta_{\ell,i}^{(j+1)} v_{\lambda_{j+1}}^{\gamma_{\ell,i,1}^{(j+1)}} \left\{ \prod_{k=2}^{\ell} (g_{j+1+k-2} \circ \tau_{\lambda_{j+1}})_{proper}^{\gamma_{\ell,i,k}^{(j+1)}} \right\} (g_{j+1+\ell-1} \circ \tau_{\lambda_{j+1}})_{proper}^i,
\end{aligned}$$

where $\varepsilon_{j+1,1} = \varepsilon_{j+1,1}(v_{\lambda_{j+1}}, u_{\lambda_{j+1}})$ is a unit in $\mathbb{C}\{v_{\lambda_{j+1}}, u_{\lambda_{j+1}}\}$, and also $\eta_{\ell,i}^{(j+1)} = \eta_{\ell,i(i)}^{(j+1)}(v_{\lambda_{j+1}}, 1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})$ is a unit in $\mathbb{C}\{v_{\lambda_{j+1}}, 1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}}\}$ for $\ell = 1, 2, \dots, r-j+1$ and $i = i(\ell) = 0, 1, \dots, s_\ell^{(j+1)} - 1$. For brevity of notation, $\varepsilon_{j+1,1}$ may be defined by one, using a nonsingular change of coordinates from $\mathbb{C}\{v_{\lambda_{j+1}}, u_{\lambda_{j+1}}\}$ to itself, independently of analytic representation of $V^{(\lambda_{j+1})}(g_i)$ for all $i \geq j+1$.

The 3-th Cond^(j+1). Let $\{\Xi_\ell^{(j+1)} : N_0^\ell \rightarrow N_0, \ell = 1, 2, \dots, r-j\}$ be a sequence such that each $\Xi_k^{(j+1)}$ is an integer-valued function defined by the following: Note that $2 \leq \ell \leq r-j$.

$$\begin{aligned}
(14.4.6) \quad & (3a) \quad \Xi_1^{(j+1)}(t) = t \text{ for each } t \in N_0. \\
& (3b) \quad \Xi_\ell^{(j+1)}(t_k)_{k=1}^\ell = t_\ell \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,0,k}^{(j+1)})_{k=1}^{\ell-1} + s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(t_k)_{k=1}^{\ell-1} \text{ for each } (t_k)_{k=1}^\ell \in N_0^\ell.
\end{aligned}$$

Moreover, for brevity of notation, let $\{\Xi_\ell^{(j+1)\sharp} : N_0^\ell \rightarrow N_0, \ell = 2, 3, \dots, r-j\}$ be a sequence such that each $\Xi_\ell^{(j+1)\sharp}$ is an integer-valued function defined by the following:

$$\begin{aligned}
(14.4.6-1) \quad & \Xi_2^{(j+1)\sharp}(\gamma_{2,i,k}^{(j+1)})_{k=1}^2 = \Xi_2^{(j+1)}(\gamma_{2,i,k}^{(j+1)})_{k=1}^2 \quad \text{for } 0 \leq i < s_2^{(j+1)}, \\
& \Xi_\ell^{(j+1)\sharp}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^\ell = \Xi_\ell^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^2 + s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} \gamma_{\ell,i,3}^{(j+1)} + s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} s_2^{(j+1)} \gamma_{\ell,i,4}^{(j+1)} \\
& + s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} s_2^{(j+1)} s_3^{(j+1)} \gamma_{\ell,i,5}^{(j+1)} + \dots + s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} s_2^{(j+1)} \dots s_{\ell-2}^{(j+1)} \gamma_{\ell,i,\ell}^{(j+1)} \quad \text{for } 0 \leq i < s_\ell^{(j+1)}.
\end{aligned}$$

The (4 α)-th Cond^(j+1). Then, the following equalities hold: Note that $r \geq 2$.

For each $j+1 = 2, 3, \dots, r$, the following equality holds with respect to the family $\{Y_\ell^{(j+1)} : \ell = 1, 2, \dots, r-j\}$:

$$\begin{aligned}
(14.4.7) \quad & \Xi_1^{(j+1)}(\gamma_{1,i,1}^{(j+1)}) = \gamma_{1,i,1}^{(j+1)} = \Xi_2^{(j)}(\gamma_{2,i,1}^{(j)}, \gamma_{2,i,2}^{(j)}) - (s_2^{(j)} - i) s_1^{(j)} \gamma_{1,0,1}^{(j)} > 0, \\
& \Xi_\ell^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^\ell - (s_\ell^{(j+1)} - i) s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,0,k}^{(j+1)})_{k=1}^{\ell-1} \quad \text{for } 2 \leq \ell \leq r-j \\
& = \Xi_{\ell+1}^{(j)}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{\ell+1} - (s_{\ell+1}^{(j)} - i) s_\ell^{(j)} \Xi_\ell^{(j)}(\gamma_{\ell,0,k}^{(j)})_{k=1}^\ell > 0.
\end{aligned}$$

The 4-th Cond^(j+1). The following inequalities hold: Note that $r \geq 2$.

$$\begin{aligned}
(14.4.8) \quad & (4a) \quad \Xi_1^{(j+1)}(\gamma_{1,i,1}^{(j+1)}) = \gamma_{1,i,1}^{(j+1)} > 0. \\
& (4b) \quad \Xi_\ell^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^\ell > (s_\ell^{(j+1)} - i) s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,i,k}^{(j+1)})_{k=1}^{\ell-1} \quad \text{for } 2 \leq \ell \leq r-j.
\end{aligned}$$

The 5-th Cond^(j+1). For $\ell=1,2, \dots, r-j$, the following equalities hold: Note that $r \geq 2$.

$$(14.4.9-1) \quad \gcd(s_\ell^{(1)}, \Xi_\ell^{(1)}(\gamma_{\ell,0,k}^{(1)})_{k=1}^\ell) = \gcd(n_{\ell+1}, \Delta_{\ell+1}(\beta_{\ell+1,0,k})_{k=1}^{\ell+1}), \quad \text{for } 1 \leq \ell \leq r,$$

$$\gcd(s_\ell^{(j+1)}, \Xi_\ell^{(j+1)}(\gamma_{\ell,0,k}^{(j+1)})_{k=1}^\ell) = \gcd(s_{\ell+1}^{(j)}, \Xi_{\ell+1}^{(j)}(\gamma_{\ell+1,0,k}^{(j)})_{k=1}^{\ell+1}) \quad \text{for } j \geq 1 \text{ with } j + \ell \leq r.$$

The 5-th Cond^(j+1). The following inequalities hold: Note that $r \geq 2$.

$$(14.4.9-2) \quad \frac{\gamma_{1,i,1}^{(j+1)}}{s_1^{(j+1)} - i} > \frac{\gamma_{1,0,1}^{(j+1)}}{s_1^{(j+1)}} > 0 \quad \text{for } 0 < i < s_1^{(j+1)}.$$

$$\frac{\Xi_\ell^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^\ell}{s_\ell^{(j+1)} - i} > \frac{\Xi_\ell^{(j+1)}(\gamma_{\ell,0,k}^{(j+1)})_{k=1}^\ell}{s_\ell^{(j+1)}} > s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,0,k}^{(j+1)})_{k=1}^{\ell-1} \quad \text{for } 0 < i < s_\ell^{(j+1)}.$$

Remark for The 5-th Cond^(j+1) For notation, the above inequality in (14.4.9-2) may be equivalently rewritten as follows: For each $\ell = 1, 2, \dots, r-j$,

$$\frac{\Xi_\ell^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^\ell - (s_\ell^{(j+1)} - i) s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,0,k}^{(j+1)})_{k=1}^{\ell-1}}{s_\ell^{(j+1)} - i}$$

$$> \frac{\Xi_\ell^{(j+1)}(\gamma_{\ell,0,k}^{(j+1)})_{k=1}^\ell - s_\ell^{(j+1)} s_{\ell-1}^{(j+1)} \Xi_{\ell-1}^{(j+1)}(\gamma_{\ell-1,0,k}^{(j+1)})_{k=1}^{\ell-1}}{s_\ell^{(j+1)}} > 0 \quad \text{for } 0 < i < s_\ell^{(j+1)}.$$

Proof of Statement 14.4. For the proof, it suffices to find the method how to apply Sublemma 12.4 and Sublemma 12.5 of Theorem 12.0 to the proof of this statement. In order to prove that the assumption of Sublemma of 12.5 and that of this statement are same up to the change of notations, because the assumption of Sublemma of 12.5 is the assumption of Theorem 12.0, which can be rewritten by five conditions in Definition 12.0.0 and the assumption of this statement is the same as the conclusion on Sublemma on the integer j .

Therefore, as an application of Sublemma 12.5, there is nothing to prove for Sublemma 14.2 on the integer $j+1$, except for proving the following equality:

$$(14.4.1) \quad e_{j+1,\lambda_{j+1}} = n_{j+1} \Delta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}.$$

To prove that an equality in (14.4.1) is true, we are going to apply Sublemma 12.2 and Corollary 3.8 to this statement, by using the following observations:

Let $\tau_{\lambda_j} : M^{(\lambda_j)} \rightarrow \mathbb{C}^2$ be the composition of a finite number λ_j of successive blow-ups which is needed to get the standard resolution of the singular point of $V(g_j)$ or $V(yg_j)$, satisfying the desired properties and notations in Sublemma 14.2 and Sublemma 14.3 on the integer (j) .

By applying Sublemma 12.2 of Theorem 12.0 to Sublemma 14.2 on the integer (j) , for any $\ell = 1, 2, \dots, r-j+1$, $(g_{j+\ell} \circ \tau_{\lambda_j})_{total}$ with $(g_{j+\ell} \circ \tau_{\lambda_j})_{proper}$ can be written in the form

$$(14.4.2) \quad (g_j \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{e_{j,\lambda_j}} (g_j \circ \tau_{\lambda_j})_{proper} \quad \text{with}$$

$$(g_j \circ \tau_{\lambda_j})_{proper} = (1 + \varepsilon_{j,1} u_{\lambda_j}),$$

$$(g_{j+\ell} \circ \tau_{\lambda_j})_{total} = v_{\lambda_j}^{s_\ell^{(j)} s_{\ell-1}^{(j)} \dots s_1^{(j)} e_{j,\lambda_j}} (g_{j+\ell} \circ \tau_{\lambda_j})_{proper} \quad \text{with}$$

$$(g_{j+\ell} \circ \tau_{\lambda_j})_{proper} = \{(1 + \varepsilon_{j,1} u_{\lambda_j})^{s_1^{(j)}} + v_{\lambda_j}^{\gamma_{1,1}^{(j)}}\}^{s_2^{(j)} s_3^{(j)} \dots s_\ell^{(j)}}$$

$$+ \sum_{\alpha, \beta \geq 0} B_{\ell, \alpha, \beta}^{(j)} v_{\lambda_j}^\alpha (1 + \varepsilon_{j,1} u_{\lambda_j})^\beta,$$

where a unit $\varepsilon_{j,1} = \varepsilon_{j,1}(u_{\lambda_j}, v_{\lambda_j})$ may be analytically assumed to be one in $\mathbb{C}\{1 + \varepsilon_{j,1}u_{\lambda_j}, v_{\lambda_j}\}$, and the $B_{\ell,\alpha,\beta}^{(j)}$ are nonzero complex numbers for some nonnegative integers α and β such that $s_1^{(j)}\alpha + \gamma_{1,0,1}^{(j)}\beta > s_\ell^{(j)}s_{\ell-1}^{(j)} \cdots s_1^{(j)}\gamma_{1,0,1}^{(j)}$.

For the construction of the statement of the conclusion, let $V(G_0^{(j+1)}) = \{(v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j}) : G_0^{(j+1)} = v_{\lambda_j}^\gamma g_0^{(j+1)} = 0\}$ be another analytic variety with isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(14.4.3) \quad \begin{aligned} g_0^{(j+1)} &= (1 + \varepsilon_{j,1}u_{\lambda_j})^{s_1^{(j)}} + v_{\lambda_j}^{\gamma_{1,0,1}^{(j)}}, \\ G_0^{(j+1)} &= v_{\lambda_j}^\gamma g_0^{(j+1)}, \end{aligned}$$

satisfying the properties (i) and (ii):

- (i) If $\gamma_{1,0,1}^{(j)} = 1$, then $\gamma = 1$.
- (ii) If $\gamma_{1,0,1}^{(j)} \geq 2$, then $\gamma = 0$.

Let $\mu_{j+1,m_{j+1}}$ be the composition of a finite number m_{j+1} of successive blow-ups which is needed to get the standard resolution of the singular point of $V(G_0^{(j+1)})$. Then, as compared with the above $\mu_{j+1,m_{j+1}}$, exactly the same $\mu_{j+1,m_{j+1}}$ can be also used for the standard resolution of one and only one quasisingular point of $V^{(\lambda_j)}(g_{j+1})$ being defined by $(g_{j+1} \circ \tau_{\lambda_j})_{proper}$ of (14.4.2), as far as the blow-ups process is concerned. That is, $\tau_{\lambda_{j+1}} = \mu_{1,m_1} \circ \mu_{2,m_2} \circ \cdots \circ \mu_{j+1,m_{j+1}}$ is the composition of a finite number $\lambda_{j+1} = m_1 + m_2 + \cdots + m_{j+1}$ of successive blow-ups which is needed to get the standard resolution of the isolated singular point of $V(yg_{j+1})$ or $V(g_{j+1})$. Recall that for each $j = 1, 2, \dots, r+1$, $V(g_j) = \{(y, z) : g_j(y, z) = 0\}$ is an analytic variety at the origin in \mathbb{C}^2 defined by **The 2-th Cond⁽⁰⁾**, as we have seen in the assumption of the theorem.

(b) For simplicity of notations, let $(v_{\lambda_{j+1}}, u_{\lambda_{j+1}})$ be the common one of the local coordinates for the λ_{j+1} -th blow-up $\pi_{\lambda_{j+1}} : M^{(\lambda_{j+1})} \rightarrow M^{(\lambda_{j+1}-1)}$ of $\tau_{\lambda_{j+1}}$ at a quasisingular point of $V^{(m_{j+1}-1)}(G_0^{(j+1)})$. Being viewed as an analytic mapping, $\mu_{j+1,m_{j+1}} : M^{(\lambda_{j+1})} \rightarrow M^{(\lambda_j)}$ can be written in the form

$$(14.4.4) \quad \mu_{j+1,m_{j+1}}(v_{\lambda_{j+1}}, u_{\lambda_{j+1}}) = (v_{\lambda_j}, 1 + \varepsilon_{j,1}u_{\lambda_j}) = (v_{\lambda_{j+1}}^{s_1^{(j)}} u_{\lambda_{j+1}}^a, v_{\lambda_{j+1}}^{\gamma_{1,0,1}^{(j)}} u_{\lambda_{j+1}}^b),$$

where (i) $a > 0$ and $b \geq 0$ are some integers such that $a\gamma_{1,0,1}^{(j)} - bs_1^{(j)} = 1$,
(ii) $E_{\lambda_{j+1}} = \{v_{\lambda_{j+1}} = 0\}$ is defined by the λ_{j+1} -th exceptional curve of the first kind.

Now, apply Corollary 3.8 and $\mu_{j+1,m_{j+1}}(v_{\lambda_{j+1}}, v_{\lambda_{j+1}})$ in (14.4.4), to the local defining equation of $(g_{j+1} \circ \tau_{\lambda_j})_{total}$ of (14.4.2), and then by (14.4.2) and (14.1.8), we have

$$(14.4.5) \quad \begin{aligned} e_{j+1,\lambda_{j+1}} &= s_1^{(j)} s_1^{(j)} e_{j,\lambda_j} + s_1^{(j)} \gamma_{1,0,1}^{(j)} \\ &= n_{j+1} n_{j+1} n_j \Delta_j(\beta_{j,0,k})_{k=1}^j + n_{j+1} (\Delta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1} - n_{j+1} n_j \Delta_j(\beta_{j,0,k})_{k=1}^j) \\ &= n_{j+1} \Delta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}. \end{aligned}$$

Thus, the proof of this statement is done. \square

For the proof of this sublemma, it suffices to show that this three sequences satisfy the remaining two conditions **The 4 α -th Cond⁽¹⁾** and **The 5 α -th Cond⁽¹⁾** in Conclusions of this sublemma, because there is nothing to prove that the truth of **The 4 α -th Cond⁽¹⁾** implies that of **The 4-th Cond⁽¹⁾**. So, for the proof, firstly we will prove by **[I]** that **The 4 α -th Cond⁽¹⁾** is true, and secondly, by **[II]** that **The 5-th Cond⁽¹⁾** is true.

[I] For the proof of the truth of **The 4 α -th Cond⁽¹⁾**, it remains to prove the second inequality in (12.5.4 α) by using the following three steps: Let ℓ and q be an arbitrary positive integer such that $r-1 \geq \ell \geq q \geq 2$.

$$\begin{aligned} \text{Step(i)} \quad \Xi_q^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^q &= \Xi_{q+1}^{(j)}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{q+1} + (s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \\ &\quad \times \{\gamma_{\ell+1,i,q+2}^{(j)} + s_{q+1}^{(j)} \gamma_{\ell+1,i,q+3}^{(j)} + s_{q+1}^{(j)} s_{q+2}^{(j)} \gamma_{\ell+1,i,q+4}^{(j)} \\ &\quad + \cdots + s_{q+1}^{(j)} s_{q+2}^{(j)} \cdots s_{\ell-1}^{(j)} \gamma_{\ell+1,i,\ell+1}^{(j)} - s_{q+1}^{(j)} s_{q+2}^{(j)} \cdots s_{\ell}^{(j)} (s_{\ell+1}^{(j)} - i)\}. \end{aligned}$$

Step(ii) In particular, if $\ell = q$ then

$$\Xi_q^{(j+1)}(\gamma_{q,i,k}^{(j+1)})_{k=1}^q = \Xi_{q+1}^{(j)}(\gamma_{q+1,i,k}^{(j)})_{k=1}^{q+1} - (s_{\ell+1}^{(j)} - i)(s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \quad \text{from Step(i).}$$

$$\begin{aligned} \text{Step(iii)} \quad \Xi_q^{(j+1)}(\gamma_{q,i,k}^{(j+1)})_{k=1}^q &- (s_q^{(j+1)} - i) s_{q-1}^{(j+1)} \Xi_{q-1}^{(j+1)}(\gamma_{q-1,0,k}^{(j+1)})_{k=1}^{q-1} \\ &= \Xi_{q+1}^{(j)}(\gamma_{q+1,i,k}^{(j)})_{k=1}^{q+1} - (s_{q+1}^{(j)} - i) s_q^{(j)} \Xi_q^{(j)}(\gamma_{q,0,k}^{(j)})_{k=1}^q > 0 \quad \text{from Step(ii).} \end{aligned}$$

We will prove Step(i), Step(ii) and Step(iii) in order, by induction on the integer $q \geq 2$.

So, it is enough to consider two cases, respectively:

Case(I) $q = 2$, and Case(II) $q \geq 2$.

Case(I): Let $q = 2$. Note by **The 3-th Cond⁽¹⁾** that $\Xi_2^{(j+1)}(t_1, t_2) = t_2 \Xi_1^{(j+1)}(\gamma_{1,0,1}) + s_1^{(j+1)} \Xi_1^{(j+1)}(t_1) = t_2 \gamma_{1,0,1}^{(j+1)} + s_1^{(j+1)} t_1$ for each $(t_1, t_2) \in N_0^2$.

$$\begin{aligned} \text{Step(i)} \quad \Xi_2^{(j+1)}(\gamma_{\ell,i,1}^{(j+1)}, \gamma_{\ell,i,2}^{(j+1)}) &= s_1^{(j+1)} \gamma_{\ell,i,1}^{(j+1)} + \gamma_{1,0,1}^{(j+1)} \gamma_{\ell,i,2}^{(j+1)} \\ &= s_1^{(j+1)} \{\Xi_{\ell+1}^{(j)}(\gamma_{\ell+1,k}^{(j)})_{k=1}^{\ell+1} - (s_{\ell+2}^{(j)} - i) s_{\ell+1}^{(j)} \cdots s_2^{(j)} s_1^{(j)} \gamma_{1,0,1}^{(j)}\} \\ &\quad + \{\Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) - s_2^{(j)} s_1^{(j)} \gamma_{1,0,1}^{(j)}\} \gamma_{\ell+1,i,3}^{(j)} \quad \text{by (12.5.1)} \\ &= s_2^{(j)} \{\Xi_2^{(j)}(\gamma_{\ell+1,i,1}^{(j)}, \gamma_{\ell+1,i,2}^{(j)}) + s_1^{(j)} \gamma_{1,0,1}^{(j)} \gamma_{\ell+1,i,3}^{(j)} + s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} \gamma_{\ell+1,i,4}^{(j)} + \cdots \\ &\quad + s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} \cdots s_{\ell-1}^{(j)} \gamma_{\ell+1,i,\ell+1}^{(j)} - s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)} \cdots s_{\ell}^{(j)} (s_{\ell+1}^{(j)} - i)\} \\ &\quad + \{\Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) - s_2^{(j)} s_1^{(j)} \gamma_{1,0,1}^{(j)}\} \gamma_{\ell+1,i,3}^{(j)} \quad \text{by (12.1.1)} \\ &= \{\Xi_3^{(j)}(\gamma_{\ell+1,i,1}^{(j)}, \gamma_{\ell+1,i,2}^{(j)}, \gamma_{\ell+1,i,3}^{(j)}) + (s_2^{(j)})^2 (s_1^{(j)}) \gamma_{1,0,1}^{(j)} \{\gamma_{\ell+1,i,4}^{(j)} + s_3^{(j)} \gamma_{\ell+1,i,5}^{(j)} + s_3^{(j)} s_4^{(j)} \gamma_{\ell+1,i,6}^{(j)} \\ &\quad + \cdots + s_3^{(j)} s_4^{(j)} \cdots s_{\ell-1}^{(j)} \gamma_{\ell+1,i,\ell+1}^{(j)} - s_3^{(j)} s_4^{(j)} \cdots s_{\ell}^{(j)} (s_{\ell+1}^{(j)} - i)\}\}, \end{aligned}$$

by the definition of $\{\Xi_3^{(j)}(\gamma_{\ell+1,i,1}^{(j)}, \gamma_{\ell+1,i,2}^{(j)}, \gamma_{\ell+1,i,3}^{(j)})\}$ only, which implies the proof of Step(i).

Step(ii) In particular, if $\ell = 2$ then an equation of Step(i) gives

$$\Xi_2^{(j+1)}(\gamma_{2,i,1}^{(j+1)}, \gamma_{2,i,2}^{(j+1)}) = \Xi_3^{(j)}(\gamma_{3,i,k}^{(j)})_{k=1}^3 - (s_2^{(j)})^2 (s_1^{(j)}) \gamma_{1,0,1}^{(j)} ((s_3^{(j)} - i)).$$

Thus, the proof of Step(ii) is done.

Step(iii) To prove that $\Xi_2^{(j+1)}(\gamma_{2,i,1}^{(j+1)}, \gamma_{2,i,2}^{(j+1)}) - (s_2^{(j+1)} - i) s_1^{(j+1)} \Xi_1^{(j+1)}(\gamma_{1,0,1}^{(j+1)}) = \Xi_3^{(j)}((\gamma_{3,i,k}^{(j)})_{k=1}^3) - (s_3^{(j)} - i) s_2^{(j)} \Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) > 0$, first note by (12.5.1.1) that

$$\begin{aligned} (s_2^{(j+1)} - i) s_1^{(j+1)} \Xi_1^{(j+1)}(\gamma_{1,0,1}^{(j+1)}) &= (s_2^{(j+1)} - i) s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} \\ &= (s_2^{(j+1)} - i) s_1^{(j+1)} \{\Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) - s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)}\}. \end{aligned}$$

$$\begin{aligned} \text{Then,} \quad \Xi_2^{(j+1)}(\gamma_{2,i,1}^{(j+1)}, \gamma_{2,i,2}^{(j+1)}) &- (s_2^{(j+1)} - i) s_1^{(j+1)} \gamma_{1,0,1}^{(j+1)} \\ &= \Xi_3^{(j)}(\gamma_{3,i,k}^{(j)})_{k=1}^3 - (s_2^{(j)})^2 (s_1^{(j)}) \gamma_{1,0,1}^{(j)} ((s_3^{(j)} - i) - (s_3^{(j)} - i) s_2^{(j)} \{\Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) - s_1^{(j)} \gamma_{1,0,1}^{(j)} s_2^{(j)}\}) \\ &= \Xi_3^{(j)}((\gamma_{3,i,k}^{(j)})_{k=1}^3) - (s_3^{(j)} - i) s_2^{(j)} \Xi_2^{(j)}(\gamma_{2,0,1}^{(j)}, \gamma_{2,0,2}^{(j)}) > 0, \end{aligned}$$

by Step(ii) and by **The 4-th Cond**⁽⁰⁾ in the assumption of Theorem 12.0, which implies the proof of Step(iii).

Thus, if $q = 2$, then we proved that the second inequality in (12.5.4 α) holds.

Case(II): Let $q \geq 2$. By the induction proof, suppose we have shown that all the equalities of Step(i), Step(ii) and Step(iii) are true on the integer $q \leq r - 1$ with $r - 1 \geq \ell \geq q$.

Then, it is enough to prove Step(i), Step(ii) and Step(iii) in order, on the integer $(q+1) \leq \ell$ as follows:

Step(i) Note that $s_q^{(j+1)} = s_{q+1}^{(j)}$ and $\gamma_{\ell,i,q+1}^{(j+1)} = \gamma_{\ell+1,i,q+2}^{(j)}$.

$$\begin{aligned} & \Xi_{q+1}^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^{q+1} \\ &= s_q^{(j+1)} \Xi_q^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^q + \gamma_{\ell,i,q+1}^{(j+1)} \Xi_q^{(j+1)}(\gamma_{q,0,k}^{(j+1)})_{k=1}^q \quad \text{by definition of } \Xi_{q+1}^{(j+1)} \\ &= s_{q+1}^{(j)} \{ \Xi_{q+1}^{(j)}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{q+1} + (s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \times [\gamma_{\ell+1,i,q+2}^{(j)} + s_{q+1}^{(j)} \gamma_{\ell+1,i,q+3}^{(j)} \\ & \quad + s_{q+1}^{(j)} s_{q+2}^{(j)} \gamma_{\ell+1,i,q+4}^{(j)} + \cdots + s_{q+1}^{(j)} s_{q+2}^{(j)} \cdots s_{\ell-1}^{(j)} \gamma_{\ell+1,i,\ell+1}^{(j)} - s_{q+1}^{(j)} s_{q+2}^{(j)} \cdots s_{\ell}^{(j)} (s_{\ell+1}^{(j)} - i)] \} \\ & \quad + \gamma_{\ell+1,i,q+2}^{(j)} \{ \Xi_{q+1}^{(j)}(\gamma_{q+1,0,k}^{(j)})_{k=1}^{q+1} - (s_{q+1}^{(j)}) (s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \}, \end{aligned}$$

by Step(i), Step(ii) and Step(iii) on the induction integer q because $s_q^{(j+1)} = s_{q+1}^{(j)}$ and $\gamma_{\ell,i,q}^{(j+1)} = \gamma_{\ell,i,q+1}^{(j)}$.

Then, the definition of $\Xi_{q+2}^{(j)}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{q+2}$ implies the following:

$$\begin{aligned} \Xi_{q+1}^{(j+1)}(\gamma_{\ell,i,k}^{(j+1)})_{k=1}^{q+1} &= \Xi_{q+2}^{(j)}(\gamma_{\ell+1,i,k}^{(j)})_{k=1}^{q+2} + \{ (s_{q+1}^{(j)})^2 (s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \} \\ & \quad \times \{ \gamma_{\ell+1,i,q+3}^{(j)} + s_{q+2}^{(j)} \gamma_{\ell+1,i,q+4}^{(j)} + s_{q+2}^{(j)} s_{q+3}^{(j)} \gamma_{\ell+1,i,q+5}^{(j)} + \cdots \\ & \quad + s_{q+2}^{(j)} s_{q+3}^{(j)} \cdots s_{\ell-1}^{(j)} \gamma_{\ell+1,i,\ell+1}^{(j)} - s_{q+2}^{(j)} s_{q+3}^{(j)} \cdots s_{\ell}^{(j)} (s_{\ell+1}^{(j)} - i) \}, \end{aligned}$$

which implies the proof of Step(i).

Step(ii) In particular, if $\ell = q + 1$, then $\ell + 1 = q + 2 < q + 3$ and so

$$\Xi_{q+1}^{(j+1)}(\gamma_{q+1,i,k}^{(j+1)})_{k=1}^{q+1} = \Xi_{q+2}^{(j)}(\gamma_{q+2,i,k}^{(j)})_{k=1}^{q+2} + \{ (s_{q+2}^{(j)} - i) (s_{q+1}^{(j)})^2 (s_q^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \}$$

by Step(i) on the integer $q + 1$, which implies the proof of Step(ii) on the integer $q + 1$.

Step(iii) To prove that the equality in (12.5.4 α) is true, we have

$$\begin{aligned} & \Xi_{q+1}^{(j+1)}(\gamma_{q+1,i,k}^{(j+1)})_{k=1}^{q+1} - (s_{q+1}^{(j+1)} - i) s_q^{(j+1)} \Xi_q^{(j+1)}(\gamma_{q,0,k}^{(j+1)})_{k=1}^q \\ &= \Xi_{q+2}^{(j)}(\gamma_{q+2,i,k}^{(j)})_{k=1}^{q+2} + \{ (s_{q+2}^{(j)} - i) (s_{q+1}^{(j)})^2 (s_q^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \} \\ & \quad - ((s_{q+2}^{(j)} - i) s_{q+1}^{(j)} \{ \Xi_{q+1}^{(j)}(\gamma_{q+1,0,k}^{(j)})_{k=1}^{q+1} - (s_{q+1}^{(j)}) (s_q^{(j)})^2 (s_{q-1}^{(j)})^2 \cdots (s_2^{(j)})^2 s_1^{(j)} \gamma_{1,0,1}^{(j)} \} \\ &= \Xi_{q+2}^{(j)}(\gamma_{q+2,i,k}^{(j)})_{k=1}^{q+2} - (s_{q+2}^{(j)} - i) (s_{q+1}^{(j)}) \Xi_{q+1}^{(j)}(\gamma_{q+1,0,k}^{(j)})_{k=1}^{q+1}, \end{aligned}$$

by Step(ii) on the integer q and $q + 1$, and by **The 4-th Cond**⁽⁰⁾ of Theorem 12.0, which implies the proof of Step(iii).

Thus, we proved that the second inequality in (12.5.3) is true, and so we can finish the proof of the truth of **The 4 α -th Cond**⁽¹⁾.

(2) Statement 14.5 with proof. Suppose that the assumptions of Sublemma 14.2 are true and that the proof of Sublemma 14.2 is done.

In order to prove the proof of Statement 14.5, it suffices to prove the following sublemma, called Sublemma 14.5.1.

Sublemma 14.5.1. Assumptions Under the same assumptions and notations of Proposition 14.1, suppose we have shown by Case(I) and by the induction assumption on the positive integer j that Sublemma 14.2, Sublemma 14.3 and Proposition 14.1 are true whenever j is

arbitrary positive integer with $1 \leq j < r$. Then, we showed by proofs of Statement 14.4 and Statement 14.5 that Sublemma 14.2 is true.

Conclusions Since the assumptions of Sublemma 14.2 and the conclusions of Sublemma 14.2 have the same kind of conditions and notations, then by the same method as in Sublemma 12.2 of Theorem 12.0, it is clear that the conclusions of Sublemma 14.2 have the same kind of representations as in the proof of Statement 14.4 with (14.4.2), up to change of notations only, as follows.

Whenever the family $\{(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} : \ell = 1, 2, \dots, r-j\}$ with $(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} \in \mathbb{C}\{1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}}, v_{\lambda_{j+1}}\}$ satisfies four conditions in the conclusions of Sublemma 14.2, denoted by **The 1-th Cond^(j+1)**, ..., **The 4-th Cond^(j+1)**, then without assuming irreducibility of $(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} \in \mathbb{C}\{1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}}, v_{\lambda_{j+1}}\}$ the conclusions of Sublemma 14.2 have the following representations:

For any $\ell = 1, 2, \dots, r-j$, $(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{total}$ with $(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper}$ can be written in the form

$$(14.5.1) \quad (g_{j+1} \circ \tau_{\lambda_{j+1}})_{total} = v_{\lambda_{j+1}}^{e_{j+1, \lambda_{j+1}}} (g_{j+1} \circ \tau_{\lambda_{j+1}})_{proper} \quad \text{with}$$

$$(g_{j+1} \circ \tau_{\lambda_{j+1}})_{proper} = (1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})$$

$$(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{total} = v_{\lambda_{j+1}}^{s_{\ell}^{(j+1)} s_{\ell-1}^{(j+1)} \dots s_2^{(j+1)} s_1^{(j+1)} e_{j+1, \lambda_{j+1}}} (g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} \quad \text{with}$$

$$(g_{j+1+\ell} \circ \tau_{\lambda_{j+1}})_{proper} = \{(1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})^{s_1^{(j+1)}} + v_{\lambda_{j+1}}^{\gamma_{1,1}^{(j+1)}}\} s_2^{(j+1)} s_3^{(j+1)} \dots s_{\ell}^{(j+1)}$$

$$+ \sum_{\alpha, \beta \geq 0} B_{\ell, \alpha, \beta}^{(j+1)} v_{\lambda_{j+1}}^{\alpha} (1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}})^{\beta},$$

where a unit $\varepsilon_{j+1,1} = \varepsilon_{j+1,1}(1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}}, v_{\lambda_{j+1}})$ may be analytically assumed to be one in $\mathbb{C}\{1 + \varepsilon_{j+1,1} u_{\lambda_{j+1}}, v_{\lambda_{j+1}}\}$, and the $B_{\ell, \alpha, \beta}^{(j+1)}$ are nonzero complex numbers for some nonnegative integers α and β such that $s_1^{(j+1)} \alpha + \gamma_{1,1}^{(j+1)} \beta > s_{\ell}^{(j+1)} s_{\ell-1}^{(j+1)} \dots s_1^{(j+1)} \gamma_{1,1}^{(j+1)}$. \square

As an application of Theorem 3.8, there is nothing to prove for the remaining for Sublemma 14.5.1.

(3) Statement 14.6 with proof. In order to prove Statement 14.6, it suffices to prove the following sublemma, called Sublemma 14.6.1.

Sublemma 14.6.1. Assumptions Under the same assumptions and notations of Proposition 14.1, suppose we have shown by Case(I) and by the induction assumption on the positive integer j that Sublemma 14.2, Sublemma 14.3 and Proposition 14.1 are true whenever j is arbitrary positive integer with $1 \leq j < r$. Then, we showed by proofs of Statement 14.4 and Statement 14.5 that Sublemma 14.2 and Sublemma 14.3 on the integer $j+1$ are true.

Conclusions To prove that Proposition 14.1 is true on the integer $(j+1)$, it remains to show that either the equalities of (14.1.8) on the integer $(j+1)$ or the following equalities are true for $1 \leq \ell \leq r-j$:

$$(14.6.1) \quad (0) \quad e_{j+1, \lambda_{j+1}} = n_{j+1} \Delta_{j+1} (\beta_{j+1, k})_{k=1}^{j+1},$$

$$(1) \quad s_1^{(j+1)} = n_{j+2},$$

$$\gamma_{1,1}^{(j+1)} = \Delta_{j+2} (\beta_{j+2, k})_{k=1}^{j+2} - n_{j+2} n_{j+1} \Delta_{j+1} (\beta_{j+1, k})_{k=1}^{j+1} > 0,$$

$$(2) \quad s_2^{(j+1)} = n_{j+3},$$

$$\gamma_{2,1}^{(j+1)} = \Delta_{j+2} (\beta_{j+3, k})_{k=1}^{j+2} + \{\beta_{j+3, j+3} - n_{j+3} n_{j+2}\} n_{j+1} \Delta_{j+1} (\beta_{j+1, k})_{k=1}^{j+1} > 0,$$

$$\gamma_{2,2}^{(j+1)} = \beta_{j+3, j+3},$$

$$(\ell) \quad s_{\ell}^{(j+1)} = n_{j+1+\ell},$$

$$\gamma_{\ell,1}^{(j+1)} = \Delta_{j+2} (\beta_{j+1+\ell, k})_{k=1}^{j+2} + \{\beta_{j+1+\ell, j+3} + n_{j+2} \beta_{j+1+\ell, j+4} + \dots$$

$$\dots + n_{j+2} n_{j+3} \dots n_{j-1+\ell} \beta_{j+1+\ell, j+1+\ell} - \prod_{k=2}^{1+\ell} n_{j+k}\} n_{j+1} \Delta_{j+1} (\beta_{j+1, k})_{k=1}^{j+1} > 0,$$

$$\gamma_{\ell,2}^{(j+1)} = \beta_{j+1+\ell, j+3}, \gamma_{\ell,3}^{(j+1)} = \beta_{j+1+\ell, j+4}, \dots, \gamma_{\ell, \ell}^{(j+1)} = \beta_{j+1+\ell, j+1+\ell}.$$

Proof of Sublemma 14.6.1. Since it was shown by Statement 14.4 that $e_{j+1, \lambda_{j+1}} = n_{j+1} \Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1}$, then for the proof of Sublemma 14.6.1 it suffices to show that the following equalities are true:

- (a) $s_q^{(j+1)} = n_{j+1+q}$ for each $q = 1, 2, \dots, r-j$.
- (b) $\gamma_{q, k}^{(j+1)} = \beta_{j+1+q, j+1+k}$ for any $q = 2, 3, \dots, r-j$, and any $k = 2, 3, \dots, q$.
- (c) $\gamma_{q, 1}^{(j+1)} = \Delta_{j+2}(\beta_{j+1+q, k})_{k=1}^{j+2} + \{\beta_{j+1+q, j+3} + n_{j+2}\beta_{j+1+q, j+4} + \dots + n_{j+2}n_{j+3} \dots n_{j+q-1}\beta_{j+1+q, j+1+q} - n_{j+2}n_{j+3} \dots n_{j+q}n_{j+1+q}\}n_{j+1}\Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1}$ for any $q = 1, 2, \dots, r-j$.

Now, we prove (a), (b) and (c), respectively.

(a) Since it is clear by Statement 14.4 that $s_q^{(j+1)} = s_{q+1}^{(j)}$ and by (14.1.8) that $s_{q+1}^{(j)} = n_{j+1+q}$ for each $q = 1, 2, \dots, r-j$, then the proof of (a) is done.

(b) Since it is clear by Statement 14.4 that $\gamma_{q, k}^{(j+1)} = \gamma_{q+1, k+1}^{(j)}$ and by (14.1.8) that $\gamma_{q+1, k+1}^{(j)} = \beta_{j+1+q, j+1+k}$ for any $q = 2, 3, \dots, r-j$, and any $k = 2, 3, \dots, q$, then the proof of (b) is done.

(c) In preparation for the proof of (c), we use (c1), (c2), (c3) and (c4).

$$(c1) \quad \gamma_{q, 1}^{(j+1)} = \Xi_{q+1}^{(j)}(\gamma_{q+1, k}^{(j)})_{k=1}^{q+1} - s_{q+1}^{(j)}s_q^{(j)}s_{q-1}^{(j)} \dots s_1^{(j)}\gamma_{1, 1}^{(j)} \quad \text{by Statement 14.4.}$$

$$(c2) \quad \Xi_{q+1}^{(j)}(\gamma_{q+1, k}^{(j)})_{k=1}^{q+1} = \Xi_2^{(j)}(\gamma_{q+1, k}^{(j)})_{k=1}^2 + s_1^{(j)}\gamma_{1, 1}^{(j)}\{\gamma_{q+1, 3}^{(j)} + s_2^{(j)}\gamma_{q+1, 4}^{(j)} + s_2^{(j)}s_3^{(j)}\gamma_{q+1, 5}^{(j)} + \dots + s_2^{(j)}s_3^{(j)} \dots s_{q+1-2}^{(j)}\gamma_{q+1, q+1}^{(j)}\} \quad \text{by (14.2.6-1).}$$

$$(c3) \quad \Xi_2^{(j)}(\gamma_{q+1, k}^{(j)})_{k=1}^2 = s_1^{(j)}\gamma_{q+1, 1}^{(j)} + \gamma_{q+1, 2}^{(j)}\gamma_{1, 1}^{(j)} \quad \text{by (14.2.6).}$$

(c4) For brevity of notation, put

$$D = \gamma_{q+1, 3}^{(j)} + s_2^{(j)}\gamma_{q+1, 4}^{(j)} + s_2^{(j)}s_3^{(j)}\gamma_{q+1, 5}^{(j)} + \dots + s_2^{(j)}s_3^{(j)} \dots s_{q+1-2}^{(j)}\gamma_{q+1, q+1}^{(j)} - s_2^{(j)}s_3^{(j)} \dots s_{q+1}^{(j)}.$$

Then, $D = \beta_{j+1+q, j+3} + n_{j+2}\beta_{j+1+q, j+4} + \dots + n_{j+2}n_{j+3} \dots n_{j+q-1}\beta_{j+1+q, j+1+q} - n_{j+2}n_{j+3} \dots n_{j+q}n_{j+1+q}$ by (14.1.8).

Also, applying D to $\gamma_{q+1, 1}^{(j)}$ of (14.1.8), $\gamma_{q+1, 1}^{(j)}$ can be rewritten as follows:

$$(14.6.2) \quad \begin{aligned} \gamma_{q+1, 1}^{(j)} &= \Delta_{j+1}(\beta_{j+q+1, k})_{k=1}^{j+1} + \{\beta_{j+q+1, j+2} + n_{j+1}\beta_{j+q+1, j+3} \\ &\quad + n_{j+1}n_{j+2}\beta_{j+q+1, j+4} + \dots + n_{j+1}n_{j+2} \dots n_{j+q-1}\beta_{j+q+1, j+q+1} \\ &\quad - n_{j+1}n_{j+2} \dots n_{j+q}n_{j+q+1}\}n_j\Delta_j(\beta_{j, k})_{k=1}^j \\ &= \Delta_{j+1}(\beta_{j+q+1, k})_{k=1}^{j+1} + \{\beta_{j+q+1, j+2} + n_{j+1}D\}n_j\Delta_j(\beta_{j, k})_{k=1}^j. \end{aligned}$$

Now, the truth of (c) can be just proved by the following computation:

$$\begin{aligned} \gamma_{q, 1}^{(j+1)} &= s_1^{(j)}\gamma_{q+1, 1}^{(j)} + \gamma_{q+1, 2}^{(j)}\gamma_{1, 1}^{(j)} + s_1^{(j)}\gamma_{1, 1}^{(j)}D \quad \text{by (c1), (c2), (c3), (c4),} \\ &= s_1^{(j)}\gamma_{q+1, 1}^{(j)} + (\gamma_{q+1, 2}^{(j)} + s_1^{(j)}D)\gamma_{1, 1}^{(j)} \\ &= n_{j+1}\{\Delta_{j+1}(\beta_{j+q+1, k})_{k=1}^{j+1} + (\beta_{j+q+1, j+2} + n_{j+1}D)n_j\Delta_j(\beta_{j, k})_{k=1}^j\} \\ &\quad + (\beta_{j+q+1, j+2} + n_{j+1}D)(\Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1} - n_{j+1}n_j\Delta_j(\beta_{j, k})_{k=1}^j) \quad \text{by (14.6.2),} \\ &= n_{j+1}\Delta_{j+1}(\beta_{j+q+1, k})_{k=1}^{j+1} + \beta_{j+q+1, j+2}\Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1} + Dn_{j+1}\Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1} \\ &= \Delta_{j+2}(\beta_{j+q+1, k})_{k=1}^{j+2} + Dn_{j+1}\Delta_{j+1}(\beta_{j+1, k})_{k=1}^{j+1}. \end{aligned}$$

Thus, we proved that Sublemma 14.6.1 is true, and so the proof of Statement 14.6 is finished. Therefore, the proof for Case(II) is done, and then we finished the proof of Proposition 14.1, completely. \square

Proof of Theorem 14.0 or Proposition 14.1. In preparation for the proof of this theorem, it suffices to show that Proposition 14.1 is true, whenever j is arbitrary with $1 \leq j \leq r$. First of all, as far as τ_{λ_1} is concerned, there is nothing to prove for Proposition 14.1, because if $j = 1$ for τ_{λ_j} then the proof of Proposition 14.1 was already done by Sublemma 12.4 and Sublemma 12.5 of Theorem 12.0. Also, for $2 \leq j \leq r$, the proof of Proposition 14.1 is done by Case(I) and Case(II). Thus, the proof of theorem is completely finished. \square

Chapter IX: The division algorithm for the W-polys

§15. The division algorithm for W-polys

Notation 15.0.0. Instead of Weierstrass polynomials, the Weierstrass preparation theorem, and the Weierstrass division theorem, we write W -polys, the WPT and the WDT respectively, for brevity of notation. Recall the well-known theorems.

Definition 15.0. Let $\mathbb{C}\{z_1, z_2, \dots, z_n\}$ or ${}_n\mathcal{O}_0$ be the ring of convergent power series at the origin in \mathbb{C}^n and $\mathbb{C}\{z_1, z_2, \dots, z_m\}[z_{m+1}, \dots, z_n]$ or ${}_m\mathcal{O}[z_{m+1}, \dots, z_n]$ be a polynomial ring in $z_{m+1}, z_{m+2}, \dots, z_n$ with coefficients from the ring ${}_m\mathcal{O}_0$.

(i) If $f \in {}_n\mathcal{O}_0$ and $f(0, \dots, 0, z_n)$ is not identically zero as a function of z_n in a neighborhood of $0 \in \mathbb{C}$, then f is regular at $0 \in \mathbb{C}^n$. Then, it is said that f is regular of order ν in z_n at $0 \in \mathbb{C}^n$ if there is the least integer ν such that $\partial^\nu f / \partial z_n^\nu$ is nonzero at the origin.

(ii) If $f \in {}_n\mathcal{O}_0$ is not identically zero, then the multiplicity of f at the origin in \mathbb{C}^n is defined by the least integer ν such that some partial derivative of f of order ν is nonzero at the origin.

(iii) If $h \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree ν in z_n , then h has the form $h = a_0 z_n^\nu + a_1 z_n^{\nu-1} + \dots + a_n$ where for each i , $a_i \in {}_{n-1}\mathcal{O}_0$ and a_0 is a unit in ${}_{n-1}\mathcal{O}_0$. Also, if $h \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\nu > 0$ in z_n , then h has the form $h = z_n^\nu + a_1 z_n^{\nu-1} + \dots + a_n$ where for each i , a_i is a nonunit in ${}_{n-1}\mathcal{O}_0$.

(iv) If $h \in {}_{n-1}\mathcal{O}[z_n]$ is a Weierstrass polynomial of degree $\nu > 0$ in z_n , which has the multiplicity $\mu > 0$ at the origin in \mathbb{C}^n , then h is said to be a Weierstrass polynomial of degree $\nu > 0$ in z_n with the multiplicity $\mu > 0$ at $0 \in \mathbb{C}^n$.

Theorem 15.1 (The WPT and The WDT).

(I)(The WPT): If $f \in {}_n\mathcal{O}_0$ is regular of order ν in z_n , then there is a unique W -poly $h \in {}_{n-1}\mathcal{O}[z_n]$ of degree ν in z_n such that $f = uh$ for some unit $u \in {}_n\mathcal{O}_0$.

(II)(The WDT): If $h \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree ν in z_n , then any $f \in {}_n\mathcal{O}_0$ can be written uniquely in the form $f = gh + r$ where $g \in {}_n\mathcal{O}_0$ and $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n . Also, if $f \in {}_{n-1}\mathcal{O}[z_n]$, then $g \in {}_{n-1}\mathcal{O}[z_n]$.

Theorem 15.2 (The WDT for the W-polys).

(1) Let $h \in {}_{n-1}\mathcal{O}[z_n]$ be a W -poly of degree $\nu > 0$ in z_n .

(1a) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a W -poly of degree $\mu \geq \nu$ in z_n . Then, f can be written uniquely in the form

$$(15.2.1.1) \quad f = gh + r,$$

where if $\mu > \nu$ then $g \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\mu - \nu > 0$ in z_n and if $\mu = \nu$ then g is equal to one, and if $\mu \geq \nu$ then $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n with $r(0, \dots, 0, z_n)$ identically zero.

(1b) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a W -poly of degree $\mu \geq \nu$ in z_n , and ℓ be a positive integer with $\ell\nu \leq \mu < (\ell + 1)\nu$. Then, f can be written uniquely in the form

$$(15.2.1.2) \quad f = \sum_{i=0}^{\ell} r_i h^i \quad \text{with} \quad h^0 = 1,$$

where if $\mu \geq \ell\nu$ then for each $i = 0, 1, \dots, \ell - 1$, $r_i \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n with $r_i(0, \dots, 0, z_n)$ identically zero, and if $\mu = \ell\nu$ then r_ℓ is equal to one and if $\mu > \ell\nu$ then $r_\ell \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\mu - \ell\nu < \nu$ in z_n .

(1c) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a polynomial of degree $\mu \geq \nu$ in z_n with $f(0, \dots, 0, z_n)$ identically zero. Then, f can be written uniquely in the form

$$(15.2.2.1) \quad f = gh + r$$

such that if $\mu > \nu$ then $g \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $\mu - \nu > 0$ in z_n and if $\mu = \nu$ then $g \in {}_{n-1}\mathcal{O}_0$ is a nonunit, and such that if $\mu \geq \nu$ then $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n where $g(0, \dots, 0, z_n)$ and $r(0, \dots, 0, z_n)$ are identically zero.

(1d) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a polynomial of degree $\mu \geq \nu$ in z_n with $f(0, \dots, 0, z_n)$ identically zero, and ℓ be a positive integer with $\ell\nu \leq \mu < (\ell+1)\nu$. Then, f can be written uniquely in the form

$$(15.2.2.2) \quad f = \sum_{i=0}^{\ell} r_i h^i \quad \text{with} \quad h^0 = 1,$$

where if $\mu \geq \ell\nu$ then for each $i = 0, 1, \dots, \ell-1$, $r_i \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n with $r_i(0, \dots, 0, z_n)$ identically zero, and if $\mu = \ell\nu$ then $r_\ell \in {}_{n-1}\mathcal{O}_0$ is a nonunit and if $\mu > \ell\nu$ then $r_\ell \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $\mu - \ell\nu < \nu$ in z_n with $r_\ell(0, \dots, 0, z_n)$ identically zero.

(2) Let $h \in {}_{n-1}\mathcal{O}[z_n]$ be a W -poly of degree $\nu > 0$ in z_n with the multiplicity $\nu > 0$ at $0 \in \mathbb{C}^n$.

(2a) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a W -poly of degree $\mu \geq \nu$ in z_n with the multiplicity $\mu \geq \nu$ at $0 \in \mathbb{C}^n$. Then, the above representation $f = \sum_{i=0}^{\ell} r_i h^i$ of (1b) satisfies the property that for each $i = 0, 1, \dots, \ell-1$, r_i has a multiplicity $\geq \mu - i\nu$ at $0 \in \mathbb{C}^n$ and that if $\mu = \ell\nu$ then r_ℓ is equal to one and if $\mu > \ell\nu$ then $r_\ell \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\mu - \ell\nu$ in z_n with the multiplicity $\mu - \ell\nu$ at $0 \in \mathbb{C}^n$.

(2b) Let $f \in {}_{n-1}\mathcal{O}[z_n]$ be a polynomial of degree $\mu > \nu$ in z_n with $f(0, \dots, 0, z_n)$ identically zero. If f has a multiplicity $m \geq \mu > \nu$ at $0 \in \mathbb{C}^n$, then the representation $f = \sum_{i=0}^{\ell} r_i h^i$ of (1d) satisfies the property that for each $i = 0, 1, \dots, \ell$, r_i has a multiplicity $\geq m - i\nu$ at $0 \in \mathbb{C}^n$.

Proof of Theorem 15.2. Recall by Theorem 15.1 that if $h \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\nu > 0$ in z_n and $f \in {}_{n-1}\mathcal{O}[z_n]$, then any f can be written uniquely in the form

$$(15.2.3) \quad f = gh + r$$

where $g \in {}_{n-1}\mathcal{O}[z_n]$, and $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n .

(1)(1a) Observe that $h(0, \dots, 0, z_n) = z_n^\nu$ and $f(0, \dots, 0, z_n) = z_n^\mu$ because h and f are W -polys in z_n . So, (15.2.3) implies the following:

$$(15.2.4) \quad z_n^\mu = z_n^\nu g(0, \dots, 0, z_n) + r(0, \dots, 0, z_n).$$

Since $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n , then $r(0, \dots, 0, z_n)$ is identically zero, and so $g(0, \dots, 0, z_n) = z_n^{\mu-\nu}$. If $\mu > \nu$ then $g \in {}_{n-1}\mathcal{O}[z_n]$ is a W -poly of degree $\mu - \nu$ in z_n because f and h are W -polys in z_n , and if $\mu = \nu$ then g is equal to one.

(1b) By (15.2.3) and (1a), rewrite $f = g_1 h + r_0$ where $g_1 = g$ and $r_0 = r$. Assume that $\mu - \nu \geq \nu$, otherwise it was already done by (1a). Now, apply the WDT with a divisor h to g_1 , and then g_1 can be written uniquely in the form $g_1 = g_2 h + r_1$ where $r_1 \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n with $r_1(0, \dots, 0, z_n)$ identically zero and $g_2 \in {}_{n-1}\mathcal{O}[z_n]$ is either a W -poly of degree $\mu - 2\nu > 0$ in z_n , or equal to one if $\mu = 2\nu$. Again, assume that $\mu - 2\nu \geq \nu$, otherwise it can be finished by the same method as we have done for $\mu - \nu < \nu$. Then apply the WDT with a divisor h to g_2 and so on, we can get the desired result. The uniqueness for the above representation follows immediately from the WDT.

(1c) Consider $f = gh + r$. By the assumption, $0 = g(0, \dots, 0, z_n)z_n^\nu + r(0, \dots, 0, z_n)$. Since $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ in z_n , then $r(0, \dots, 0, z_n) = g(0, \dots, 0, z_n) = 0$. So, if $\mu > \nu$ then $g \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $\mu - \nu > 0$ in z_n and if $\mu = \nu$ then $g \in {}_{n-1}\mathcal{O}_0$ is a nonunit because $f(0, \dots, 0, z_n)$ is zero.

(1d) It just follows by the similar technique as we have used in the proof of (1b) together with the result of (1c).

(2)(2a) By the WDT and (1a), f can be written uniquely with $f = gh + r$ as follows:

$$(15.2.5) \quad f = z_n^\mu + \sum_{i=1}^{\mu} a_i z_n^{\mu-i}, \quad h = z_n^\nu + \sum_{j=1}^{\nu} b_j z_n^{\nu-j} \quad \text{and} \quad g = z_n^{\mu-\nu} + \sum_{k=1}^{\mu-\nu} c_k z_n^{\mu-\nu-k},$$

where f , h and g are W -polys in z_n .

Since f and h are W -polys of degree $\mu \geq \nu$ in z_n with the multiplicity $\mu \geq \nu$ at $0 \in \mathbb{C}^n$, respectively, then each a_i is a nonunit in ${}_{n-1}\mathcal{O}_0$ with multiplicity $\geq i$ if exists, and also each b_j is a nonunit in ${}_{n-1}\mathcal{O}_0$ with multiplicity $\geq j$, if exists. If $\mu = \nu$ then g is equal to one by (1a) and so it is clear by (15.2.5) that r has a multiplicity $\geq \mu$ at $0 \in \mathbb{C}^n$, because $f = gh + r$.

We claim that if $\mu > \nu$, each coefficient $c_k \in {}_{n-1}\mathcal{O}_0$ of (15.2.5) has a multiplicity $\geq k$ if exists, and so g is a W -poly of degree $\mu - \nu$ in z_n with the multiplicity $\mu - \nu$ at $0 \in \mathbb{C}^n$.

For the proof of the claim, assume the contrary. Then there exists a nonzero coefficient $c_k \in {}_{n-1}\mathcal{O}_0$ with the multiplicity $< k$, and so let p be the smallest among the positive integers k such that $c_k \in {}_{n-1}\mathcal{O}_0$ has a multiplicity $< k$.

Now, rewrite g in the following form

$$(15.2.6) \quad g = \sum_1 + \sum_2 + \sum_3,$$

where $\sum_1 = z_n^{\mu-\nu} + c_1 z_n^{\mu-\nu-1} + \dots + c_{p-1} z_n^{\mu-\nu-(p-1)}$, $\sum_2 = c_p z_n^{\mu-\nu-p}$ and $\sum_3 = c_{p+1} z_n^{\mu-\nu-p-1} + c_{p+2} z_n^{\mu-\nu-p-2} + \dots + c_{\mu-\nu}$.

Consider hg as $hg = h\Sigma_1 + h\Sigma_2 + h\Sigma_3$. Then gh satisfies the following properties (2a-1), (2a-2) and (2a-3):

(2a-1) $h\Sigma_1$ has a multiplicity μ at $0 \in \mathbb{C}^n$ because for $1 \leq k \leq p-1$, c_k has a multiplicity $\geq k$ at $0 \in \mathbb{C}^{n-1}$ by definition of p .

(2a-2) In $h\Sigma_2$, there exists a nonzero term $c_p z_n^\nu z_n^{\mu-\nu-p} = c_p z_n^{\mu-p}$ with the multiplicity $< \mu$ because c_p has a multiplicity $< p$, and note that $h\Sigma_2 \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $\mu - p \geq \nu$ in z_n .

(2a-3) For any nonzero term $dz_n^q \in h\Sigma_3$ with a nonunit $d \in {}_{n-1}\mathcal{O}_0$, $dz_n^q \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $q \leq \nu + \mu - \nu - p - 1 < \mu - p$ in z_n .

Therefore, $c_p z_n^{\mu-p}$ is a nonzero term in $f = gh + r$ with multiplicity $< \mu$ by (2a-1), (2a-2) and (2a-3) because $r \in {}_{n-1}\mathcal{O}[z_n]$ is a polynomial of degree $< \nu$ and $\nu \leq \mu - p$. It would be impossible because f has a multiplicity μ at $0 \in \mathbb{C}^n$. Thus we proved the claim, and so r has a multiplicity $\geq \mu$ at $0 \in \mathbb{C}^n$ because hg has a multiplicity μ at $0 \in \mathbb{C}^n$.

Now, assume that $\mu - \nu \geq \nu$, otherwise it was done just before. Since g is a W -poly of degree $\mu - \nu$ in z_n with the multiplicity $\mu - \nu$ at $0 \in \mathbb{C}^n$, apply the WDT with a divisor h to g . Assuming by (15.2.3) that $f = gh + r$ as before, then g can be written uniquely in the form $g = g_2 h + r_1$, satisfying that either g_2 is a W -poly of degree $\mu - 2\nu > 0$ in z_n with the multiplicity $\mu - 2\nu > 0$ at $0 \in \mathbb{C}^n$ or g_2 is equal to one by the similar technique as we have used just before. Since $g_2 h$ has a multiplicity $\mu - \nu$ at $0 \in \mathbb{C}^n$, then r_1 has a multiplicity $\geq \mu - \nu$ at $0 \in \mathbb{C}^n$. Repeat the above process by (1b), and then it can be done.

(2b) It can be proved by the similar technique as in the proof of (2a) together with the result of (1d).

Thus, we finished the proof of Theorem 15.2. \square

Lemma 15.3. *Let $f = z^n + \sum_{i=0}^{n-1} c_i y^{\beta_i} z^i$ be a W -poly of degree $n \geq 2$ in z with the multiplicity $n \geq 2$ at $0 \in \mathbb{C}^n$ where for $0 \leq i \leq n-1$, each $c_i = c_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the β_i are positive integers. Assume that f may not be irreducible in ${}_2\mathcal{O}_0$. If c_{n-1} is not identically zero, then f can be rewritten by the WDT only, without using a nonsingular change of coordinates, as follows:*

$$(15.3.1) \quad f = (z + \frac{c_{n-1}}{n} y^{\beta_{n-1}})^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} (z + \frac{c_{n-1}}{n} y^{\beta_{n-1}})^i,$$

where a_{n-1} is identically zero and for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers.

Proof of Lemma 15.3. Apply the WDT to f with a divisor $z + \frac{c_{n-1}}{n} y^{\beta_{n-1}}$. Then, it is trivial by (1b) and (2a) of Theorem 15.2 that if $h = z + \frac{c_{n-1}}{n} y^{\beta_{n-1}}$ and $r_i = a_i y^{\alpha_i}$ then $\alpha_i \geq n - i$. \square

Throughout this section, replace $z + \frac{c_{n-1}}{n} y^{\beta_{n-1}}$ by z for brevity of notation, as we have done in Lemma 15.3.

Theorem 15.4 (The Division Algorithm for The W-polys).

Assumptions Let $f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be a W-poly of degree $n \geq 2$ in z where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Assume that f may not be irreducible in ${}_2\mathcal{O}_0$, and note that a_{n-1} is identically zero for convenience. Write $n = \prod_{k=1}^{\ell} n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

Conclusions For any j with $0 \leq j \leq \ell-1$, there exists a unique sequence of W-polys in z , $\{f_0 = z, f_1, \dots, f_j\}$ with $f_k \in \mathbb{C}\{y\}[z]$, satisfying the following notations and two properties: Let $f_{-1} = y$, $f_0 = z$ and for each $k = 1, 2, \dots, j$,

$$(15.4.1) \quad \begin{aligned} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \quad \text{and} \\ f &= f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i, \end{aligned}$$

- such that (i) $n = d_{j+1} \prod_{k=1}^j n_k$ with $n = d_1$,
(ii) for each $k \geq 1$, $f_k = f_k(y, z) \in \mathbb{C}\{y\}[z]$ is a W-poly of degree $\prod_{t=1}^k n_t$ in z ,
(iii) $f_k \in \mathbb{C}\{y\}[z, f_1, \dots, f_{k-1}]$ is a polynomial of degree n_k in f_{k-1} ,
(iv) $f \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ is a polynomial of degree d_{j+1} in f_j ,

considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at $0 \in \mathbb{C}^{j+2}$ if necessary, with two properties (1) and (2).

(1)(1a) Let k and i be fixed with $1 \leq k \leq j$ and $0 \leq i \leq n_k - 2$. If exists, then $R_{1,i} = R_{1,i}(y)$ is a nonunit in $\mathbb{C}\{y\}$ and for each $k \geq 2$, $R_{k,i} = R_{k,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \prod_{t=1}^{k-1} n_t$ in z with $R_{k,i}(0, z) = 0$.

(1b) Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. If $j = 0$, then $f_j = z$ and $S_{j+1,i} = a_i y^{\alpha_i}$, and for each $j \geq 1$, $S_{j+1,i} = S_{j+1,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \prod_{t=1}^j n_t$ in z with $S_{j+1,i}(0, z) = 0$.

(2) (2a) Let k and i be fixed with $1 \leq k \leq j$ and $0 \leq i \leq n_k - 2$. For any nonzero monomial $\prod_{t=1}^k f_{t-2}^{\delta_t}$ in $R_{k,i} = R_{k,i}(y, z, f_1, \dots, f_{k-2}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{k-2}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, k$.

(2b) Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. For any nonzero monomial $\prod_{t=1}^{j+1} f_{t-2}^{\gamma_t}$ in $S_{j+1,i} = S_{j+1,i}(y, z, f_1, \dots, f_{j-1}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$.

Remark 15.4.1. Under the same assumption as in Theorem 15.2, we have the following:

- (1) Suppose that f is a W-poly of degree $n \geq 2$ in z .
(i) By (15.4.1) and by (2a) of Theorem 15.2, $f_k \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}]$, and so f_k is a W-polynomial of degree n_k in f_{k-1} with coefficients from the ring $\mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}$.
(ii) By (15.4.1) and by (2b) of Theorem 15.2, $f \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$, and so f is a W-polynomial of degree d_{j+1} in f_j with coefficients from the ring $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$.

(2) Suppose that f is a W-poly of degree $n \geq 2$ in z with the multiplicity n at $0 \in \mathbb{C}^2$.

Then, at the same time, we can prove the following additional results:

- (i) For each k , $f_k(y, z)$ is a W-poly of degree $\prod_{t=1}^k n_t$ in z with the multiplicity $\prod_{t=1}^k n_t$ at $0 \in \mathbb{C}^2$.
(ii) As in (2a) of Theorem 15.2, $R_{k,i} \in \mathbb{C}\{y\}[z]$ of (15.4.1) has a multiplicity $\geq (n_k - i) \prod_{t=1}^{k-1} n_t$ at $0 \in \mathbb{C}^2$.
(iii) As in (2a) of Theorem 15.2, $S_{j+1,i} \in \mathbb{C}\{y\}[z]$ of (15.4.1) has a multiplicity $\geq (d_{j+1} - i) \prod_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

Corollary 15.4.1. *Instead of the assumption of $f = z^n + \sum_{i=0}^{n-2} R_i z^i$ as in Theorem 15.4, suppose that the R_i satisfy the following facts:*

For each $i = 1, 2, \dots, n-2$, any nonzero monomial in $R_i = R_i(y_1, y_2, \dots, y_m) \in {}_m\mathcal{O}_0$ can be written in the form

$$(*) \quad A_{\alpha_1, \alpha_2, \dots, \alpha_m} \prod_{k=1}^m y_k^{\alpha_k},$$

where $A_{\alpha_1, \alpha_2, \dots, \alpha_m}$ is a constant and α_s is a positive integer for some s with $1 \leq s \leq m$. Then, the division algorithm for a given W -poly $f \in {}_m\mathcal{O}[z]$ can be generalized.

Proof of Theorem 15.4. We are going to prove the theorem together with Remark 15.4.1 at the same time by induction on the nonnegative integer j . If $j = 0$, there is nothing to prove. For induction proof with $\ell \geq 2$, suppose we have shown that the theorem is true on the integer $j \leq \ell - 1$. Then, on the integer $j+1$ it suffices to show that given such a sequence $\{f_0, f_1, \dots, f_j\}$ as we have seen in (15.4.1), then f_{j+1} and f can be uniquely constructed as follows:

$$(15.4.2) \quad \begin{cases} f_{j+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i} f_j^i, \\ f &= f_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} S_{j+2,i} f_{j+1}^i, \end{cases}$$

where $d_{j+2} = \prod_{t=j+2}^{\ell} n_t$, and f_{j+1} with $R_{j+1,i}$ and f with $S_{j+2,i}$ satisfy the same kind of properties as f_j with $R_{j,i}$ and f with $S_{j+1,i}$ have done in (1) and (2) of the conclusion of the theorem, respectively, and also the properties in Remark 15.4.1.

For the proof of Theorem 15.4, it suffices to show that the following two sublemmas are true:

Sublemma 15.5. We prove that such a sequence $\{f_0, f_1, \dots, f_{j+1}\}$ for f can be constructed in the sense of (15.4.2).

Sublemma 15.6. We prove that a sequence $\{f_0, f_1, \dots, f_{j+1}\}$ for f in the sense of (15.4.2) constructed in Sublemma 14.5 must be unique.

First of all, in preparation for finding Sublemma 15.5, we need Sublemma 15.4.α:

Sublemma 15.4.α.

Assumptions Under the same induction assumption on the integer $j \leq \ell - 1$ with $\ell \geq 2$, suppose we have shown that the theorem is true on the integer $j \leq \ell - 1$, just as above.

Conclusions We show that $h_{j+1,1}$ and f can be constructed as follows:

$$(15.4.3) \quad \begin{cases} h_{j+1,1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(1)} f_j^i, \\ f &= h_{j+1,1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} T_{j+2,i} h_{j+1,1}^i, \end{cases}$$

where $h_{j+1,1} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\prod_{t=1}^{j+1} n_t$ in z with the multiplicity $\prod_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$, satisfying the following facts, Fact(A), Fact(B), Fact(C), Fact(D) and Fact(E).

Fact(A) For each $i = 0, 1, \dots, n_{j+1} - 2$, $R_{j+1,i}^{(1)} = R_{j+1,i}^{(1)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \prod_{t=1}^j n_t$ in z with $R_{j+1,i}^{(1)}(0, z) = 0$ and has a multiplicity $\geq (n_{j+1} - i) \prod_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

Fact(B) For each $i = 0, 1, \dots, n_{j+1} - 2$, and for any nonzero monomial $\prod_{t=1}^{j+1} f_t^{\delta_t}$ in $R_{j+1,i}^{(1)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $2 \leq t \leq j+1$.

Fact(C) For each $i = 0, 1, \dots, d_{j+2} - 1$, $T_{j+2,i} = T_{j+2,i}(y, z) = \sum a_{p,q} y^p z^q$ with a nonzero constant $a_{p,q}$ such that $p > 0$ and $q < \prod_{t=1}^{j+1} n_t$ and that $T_{j+2,i}(0, z) = 0$ and $T_{j+2,i}$ has a multiplicity $\geq (d_{j+2} - i) \prod_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$.

Moreover, consider y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} . Then, $T_{j+2,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ satisfies two facts Fact(D) and Fact(E).

Fact(D) For each $i = 0, 1, \dots, d_{j+2} - 1$, and for any nonzero monomial $\prod_{t=1}^{j+2} f_t^{\gamma_t}$ in $T_{j+2,i}$, $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $2 \leq t \leq j+2$.

Fact(E) In particular, if $i = d_{j+2} - 1$ for $T_{j+2,i}$ of Fact(D), then $\gamma_{j+2} \leq n_{j+1} - 2$. \square

Proof of Sublemma 15.4.α. We prove the above facts, respectively.

The Proofs of Fact(A) and Fact(B): We construct $R_{j+1,i}^{(1)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, which satisfies Fact(A) and Fact(B), by the following method:

As an example, for each fixed $i = 0, 1, \dots, n_{j+1} - 2$, and for any nonzero monomial $\Pi_{t=1}^{j+1} f_{t-2}^{\delta_t}$ in $R_{j+1,i}^{(1)}$, $\delta_1 > 0$ can be chosen sufficiently large such that $\delta_t < n_{t-1}$ for $2 \leq t \leq j+1$, which implies the following:

(i) $R_{j+1,i}^{(1)} = R_{j+1,i}^{(1)}(y, z)$ is a polynomial of degree $< \Pi_{t=1}^{j+1} n_t$ in z , because $\delta_2 + n_1 \delta_3 + n_1 n_2 \delta_4 + \dots + n_1 n_2 \dots n_{j-1} \delta_{j+1} \leq n_1 - 1 + n_1(n_2 - 1) + n_1 n_2(n_3 - 1) + \dots + n_1 n_2 \dots n_{j-1}(n_j - 1) < \Pi_{t=1}^j n_t$, using by the induction assumption on j that each $f_k(y, z)$ is a polynomial of degree $\Pi_{t=1}^k n_t$ in z for $0 \leq k \leq j-1$.

(ii) Also, it is clear that $R_{j+1,i}^{(1)}(0, z) = 0$ and $R_{j+1,i}^{(1)}(y, z)$ has the desired multiplicity by (2a) of Theorem 15.2, because $\delta_1 > 0$ can be chosen sufficiently large such that $h_{j+1,1} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^{j+1} n_t$ in z with the multiplicity $\Pi_{t=1}^{j+1} n_t$ at $(y, z) = (0, 0)$.

Thus, Fact(A) and Fact(B) are easily proved.

The Proof of Fact(C): Just, apply the WDT with a divisor $h_{j+1,1}$ to f as an element of $\mathbb{C}\{y\}[z]$, and so it is clear by (1b) and (2a) of Theorem 15.2.

The Proof of Fact(D): Recall that $f_j(y, z) \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^j n_t$ in z with the multiplicity $\Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$ by the induction assumption on j and that $T_{j+2,i}(y, z)$ is a polynomial of degree $< \Pi_{t=1}^{j+1} n_t$ in z by Fact (C). For each fixed $i = 0, 1, \dots, d_{j+2} - 1$, apply the WDT with a divisor f_j to $T_{j+2,i}$ and then by (1d) and (2b) of Theorem 15.2, $T_{j+2,i}$ can be written as follows:

$$(15.4.4) \quad T_{j+2,i} = \sum_{k_1=0}^{n_{j+1}-1} Q_{k_1} f_j^{k_1},$$

where for $0 \leq k_1 \leq n_{j+1} - 1$, each $Q_{k_1} \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{k=1}^j n_t$ in z with $Q_{k_1}(0, z) = 0$, if exists. Since $f_{j-1} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^{j-1} n_t$ in z with the multiplicity $\Pi_{t=1}^{j-1} n_t$ at $0 \in \mathbb{C}^2$, then for each fixed k_1 apply the WDT with a divisor f_{j-1} to Q_{k_1} again, and then by (1d) and (2b) of Theorem 15.2 again, Q_{k_1} may be written in the form

$$(15.4.5) \quad Q_{k_1} = \sum_{k_2=0}^{n_j-1} Q_{k_1, k_2} f_{j-1}^{k_2},$$

where for $0 \leq k_2 \leq n_j - 1$, each $Q_{k_1, k_2} \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{t=1}^{j-1} n_t$ in z with $Q_{k_1, k_2}(0, z)$ zero, if exists. Thus, continuing the same process as above, then Fact (D) can be easily proved by Theorem 15.2.

The Proof of Fact(E): Assume the contrary. By Fact(D), we get

$$(15.4.6) \quad \gamma'_{j+2} = n_{j+1} - 1 \text{ for some nonzero monomial } \Pi_{t=1}^{j+2} f_{t-2}^{\gamma'_t} \text{ in } T_{j+1, d_{j+2}-1},$$

where $\gamma'_1 > 0$ and $0 \leq \gamma'_t < n_{t-1}$ for $2 \leq t \leq j+2$.

To find a contradiction, note that for any nonzero monomial $y^\alpha z^\beta$ in $\Pi_{t=1}^{j+1} f_{t-2}^{\nu_t} \in \mathbb{C}\{y, z\}$ where $\nu_1 > 0$ and $\nu_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$, then

$$(15.4.7) \quad \begin{aligned} \beta &\leq \nu_2 + n_1 \nu_3 + n_1 n_2 \nu_4 + \dots + n_1 n_2 \dots n_{j-1} \nu_{j+1} \\ &\leq n_1 - 1 + n_1(n_2 - 1) + n_1 n_2(n_3 - 1) + \dots + n_1 n_2 \dots n_{j-1}(n_j - 1) \\ &= \Pi_{t=1}^j n_t - 1 < \Pi_{t=1}^j n_t = \text{the multiplicity of } f_j(y, z) \text{ at } 0 \in \mathbb{C}^2. \end{aligned}$$

So, for any positive integer p , we can get

$$(15.4.8) \quad (\Pi_{t=1}^{j+1} f_{t-2}^{\nu_t})^p = \sum a_{e_1, e_2, \dots, e_{j+2}} \Pi_{t=1}^{j+2} f_{t-2}^{e_t},$$

as a convergent power series in $\mathbb{C}\{y, z, f_1, \dots, f_j\}$ where each $a_{e_1 e_2 \dots e_{j+2}}$ is a nonzero complex number, and $e_1 > 0$, $e_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$ and $e_{j+2} < p$ if exists, using the same technique as in the proof of Fact (D) and the inequality in (15.4.7).

Now, to prove Fact (E), recall by Fact(4) that $f = h_{j+1,1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} h_{j+1,1}^i$ where for each fixed i , and any nonzero monomial $\Pi_{t=1}^{j+2} f_{t-2}^{\gamma_t}$ in $T_{j+2,i}$ $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $t = 2, 3, \dots, j+2$.

Then for fixed j , and $0 \leq i \leq d_{j+2} - 1$,

$f - f_j^{d_{j+1}} = h_{j+1,1}^{d_{j+2}} - f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} h_{j+1,1}^i = \sum_1 + \sum_2 + \sum_3$ can be written as follows:

$$(15.4.9) \quad \begin{cases} \sum_1 &= h_{j+1,1}^{d_{j+2}} - f_j^{d_{j+1}} = \{f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(1)} f_j^i\}^{d_{j+2}} - f_j^{d_{j+1}}, \\ \sum_2 &= \sum_{i=0}^{d_{j+1}-2} T_{j+2,i} h_{j+1,1}^i, \\ \sum_3 &= T_{j+2,d_{j+2}-1} h_{j+1,1}^{d_{j+2}-1}. \end{cases}$$

Note that \sum_1 can be rewritten as follows: With nonzero constants b_s ,

$$\sum_1 = \sum_{s=1}^{d_{j+2}} b_s (f_j^{n_{j+1}})^{d_{j+2}-s} (\sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(1)} f_j^i)^s.$$

First, by the assumption, observe by (15.4.6) that there is a nonzero monomial

$$(15.4.10) \quad (\Pi_{k=1}^{j+2} f_{k-2}^{\gamma'_k}) (f_j^{n_{j+1}})^{d_{j+2}-1} = (\Pi_{k=1}^{j+1} f_{k-2}^{\gamma'_k}) f_j^{d_{j+1}-1},$$

in \sum_3 because $\gamma'_{j+2} + n_{j+1}(d_{j+2} - 1) = n_{j+1} - 1 + d_{j+1} - n_{j+1} = d_{j+1} - 1$.

Next, we claim that $(\Pi_{k=1}^{j+1} f_{k-2}^{\gamma'_k}) f_j^{d_{j+1}-1} \in \sum_3$ does not belong to both \sum_1 and \sum_2 .

If the claim is proved, then the above monomial of (15.4.10) in \sum_3 would belong to $f(y, z, f_1, \dots, f_j) - f_j^{d_{j+1}}$. But it is impossible because such an monomial does not belong to $f - f_j^{d_{j+1}}$ by induction assumption on $f = f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i$.

Now, to prove the claim, it is enough to consider the following two cases :

Case (1): Whenever $\Pi_{k=1}^{j+2} f_{k-2}^{\tau_k}$ is in \sum_1 with $\tau_1 > 0$ and $\tau_k < n_{k-1}$ for $2 \leq k \leq j+1$, then $\tau_{j+2} \leq n_{j+1}(d_{j+2} - 1) + n_{j+1} - 2 = d_{j+1} - 2$.

Case (2): Whenever $\Pi_{k=1}^{j+2} f_{k-2}^{\tau_k}$ is in \sum_2 with $\tau_1 > 0$ and $\tau_k < n_{k-1}$ for $2 \leq k \leq j+1$, then $\tau_{j+2} < n_{j+1}(d_{j+2} - 2) + n_{j+1} = d_{j+1} - n_{j+1} \leq d_{j+1} - 2$.

So, we proved the claim. Therefore, we can find a contradiction, what we wanted. Thus, we proved Fact(E). \square

For the proof of Theorem 15.4, we prove Sublemma 15.5 and Sublemma 15.6, respectively.

Sublemma 15.5. Assumptions *By the same way as we have seen in (15.4.3) of the conclusion of Sublemma 15.4.α, we may assume that $(h_{j+1,1}, f)$ can be constructed as follows:*

$$(15.5.1) \quad \begin{cases} h_{j+1,1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(1)} f_j^i, \\ f &= h_{j+1,1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} h_{j+1,1}^i, \end{cases}$$

satisfying the facts, denoted by Fact(A), Fact(B), Fact(C), Fact(D) and Fact(E). For brevity of notation, let $h_1 = h_{j+1,1}$, $R_i^{(1)} = R_{j+1,i}^{(1)}$ for $0 \leq i \leq n_{j+1} - 2$ and $T_i^{(1)} = T_{j+2,i}^{(1)} = T_{j+2,i}$ for $0 \leq i \leq d_{j+2} - 1$, respectively.

Conclusions *Then, (f_{j+1}, f) for f can be constructed as follows:*

Case [I]: *If $T_{d_{j+2}-1}^{(1)}$ in (h_1, f) is zero, let $f_{j+1} = h_1$, $R_{j+1,i} = R_i^{(1)}$ for $0 \leq i \leq n_{j+1} - 2$ and $S_{j+1,i} = T_i^{(1)}$ for $0 \leq i \leq d_{j+2} - 2$, respectively. Then, the construction of (f_{j+1}, f) has been already finished.*

Case [II]: If $T_{d_{j+2}-1}^{(1)}$ is not zero, for finding such a construction of (f_{j+1}, f) , it suffices to follow two steps, Step(1) and Step(2).

Step(1) for Case[II] Then, there is a sequence of pairs, $H = \{(h_p, f) : p = 1, 2, \dots\}$, each pair of which can be constructed with five properties, called Property(1), Property(2), Property(3), Property(4) and Property(5), as follows:

$$(15.5.2.1) \quad \begin{cases} h_1 &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(1)} f_j^i \quad \text{with } R_i^{(1)} = R_{j+1,i}^{(1)} \text{ in (15.5.1),} \\ f &= h_1^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(1)} h_1^i \quad \text{with } T_i^{(1)} = T_{j+2,i}^{(1)} \text{ in (15.5.1),} \end{cases}$$

$$(15.5.2.2) \quad \begin{cases} h_2 &= h_1 + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(1)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(2)} f_j^i, \\ f &= h_2^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(2)} h_2^i, \end{cases}$$

$$(15.5.2.3) \quad \begin{cases} h_3 &= h_2 + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(2)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(3)} f_j^i, \\ f &= h_3^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(3)} h_3^i, \end{cases}$$

...

satisfying the following properties and notations:

Property(1) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$. Then $R_i^{(p+1)} = R_i^{(p+1)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \prod_{t=1}^j n_t$ in z and has a multiplicity $\geq (n_{j+1} - i) \prod_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

Property(2) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$. Then $T_i^{(p+1)} = T_i^{(p+1)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \prod_{t=1}^{j+1} n_t$ in z and has a multiplicity $\geq (d_{j+2} - i) \prod_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$.

Consider y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} .

Property(3) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$. Then for any nonzero monomial $\prod_{t=1}^{j+1} f_{t-2}^{\delta_t}$ in $R_i^{(p+1)} = R_i^{(p+1)}(y, z, f_1, \dots, f_{j-1}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$.

Property(4) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$. Then for any nonzero monomial $\prod_{t=1}^{j+2} f_{t-2}^{\delta_t}$ in $T_i^{(p+1)} = T_i^{(p+1)}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, j+2$.

Property(5) In particular, if $i = d_{j+2} - 1$ for $T_i^{(p+1)}$ of Property(4), then $\delta_{j+2} \leq n_{j+1} - 2$.

Step(2) for Case[II] By Step(1), there is a pair $(h_{\nu+1}, f) \in H$ which satisfies the following property:

Property(6) There is an integer $\nu \leq \frac{n_{j+1}+1}{2}$ such that $T_{d_{j+2}-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu$ and $T_{d_{j+2}-1}^{(\nu+1)} = T_{d_{j+2}-1}^{(\nu+2)} = \dots = 0$. That is, $(h_\nu, f) \neq (f_{j+1}, f)$ and $(h_{\nu+1}, f) = (f_{j+1}, f)$ for an integer $\nu \leq \frac{n_{j+1}+1}{2}$.

Remark 15.5.1. (a) It is clear by Sublemma 15.4.α that Property(1), Property(2), Property(3), Property(4) and Property(5) for $(h_1, f) = (h_{j+1,1}, f)$ in (15.5.2.1) are equivalent to Fact(A), Fact(C), Fact(B), Fact(D) and Fact(E) for $(h_{j+1,1}, f)$ in (15.4.3), respectively.

(b) By (a), it is clear by Sublemma 15.4.α that (h_1, f) was already constructed with five properties. \square

Proof of Sublemma 15.5. We prove two cases, respectively.

Case[I]: By Sublemma 15.4.α, there is nothing to prove for this sublemma.

Case[II]: Assume that $T_{d_{j+2}-1}^{(1)}$ is not zero. For the proof of this case, it suffices to follow two steps, Step(1) and Step(2) in order.

Step(1) for Case[II] We will prove this case by induction on the integer $p \geq 1$ that each pair of H satisfies Property(1), Property(2), \dots , Property(5) where $H = \{(h_p, f) : p = 1, 2, \dots\}$ is called a sequence of pairs, defined by (15.5.2.1), (15.5.2.2), \dots .

Step(2) for Case[II] We will prove by Step(1) that there is some integer $\nu \leq \frac{n_{j+1}+1}{2}$ which satisfies Property(6), being equivalent to the fact that $(h_\nu, f) \neq (f_{j+1}, f)$ and $(h_{\nu+1}, f) = (f_{j+1}, f)$.

The Proof of Step(1) for Case[II] As we have seen in Remark 15.4.1, (h_1, f) with $T_{d_{j+2}-1}^{(1)} \neq 0$ has already constructed by (15.5.1) and Sublemma 15.4. α . The general case is proved by induction. Suppose we have shown that for $p = 1, 2, \dots, w$, each pair (h_p, f) satisfies the same kind of properties corresponding to the properties Property(1), Property(2), \dots , Property(5), which the pair (h_1, f) does.

In order to construct such a pair (h_{w+1}, f) , first define h_{w+1} by $h_{w+1} = h_w + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(w)}$ where $T_{d_{j+2}-1}^{(w)}$ was already defined from (h_w, f) . By induction assumption on (h_w, f) , note that $T_{d_{j+2}-1}^{(w)}$ satisfies the corresponding properties, denoted by Property(2), Property(4) and Property(5) of Case [II], if $p = w$. So, h_{w+1} is rewritten in the form

$$(15.5.3) \quad \begin{aligned} h_{w+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(w)} f_j^i + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(w)} \\ &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(w+1)} f_j^i, \end{aligned}$$

and then $R_i^{(w+1)}$ satisfies the same kind of properties as $R_i^{(w)}$ does in Property(1) and Property(3) in Case [II] of the lemma, which can be proved by the following way:

Since for any nonzero monomial $\Pi_{t=1}^{j+2} f_t^{\gamma_t}$ in $T_{d_{j+2}-1}^{(w)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$, $\gamma_1 > 0$, $\gamma_t < n_{t-1}$ for $t = 2, \dots, j+1$ and $\gamma_{j+2} \leq n_{j+1} - 2$, then we can put $T_{d_{j+2}-1}^{(w)} = \sum_{i=0}^{n_{j+1}-2} U_i^{(w)} f_j^i$ where for any nonzero monomial $\Pi_{t=1}^{j+1} f_t^{\delta_t}$ in $U_i^{(w)} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$, and for each i , $U_i^{(w)}$ has a multiplicity $\geq \Pi_{t=1}^{j+1} n_t - i \Pi_{t=1}^j n_t = (n_{j+1} - i) \Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$ because $T_{d_{j+2}-1}^{(w)}$ has a multiplicity $\geq \Pi_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$ by Property(2).

Now, apply the WDT to f with a divisor h_{w+1} , and then it can be proved by Theorem 15.2 that f is written in the form

$$(15.5.4) \quad f = h_{w+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(w+1)} h_{w+1}^i,$$

and then $T_i^{(w+1)}$ satisfies the same kind of properties as $T_i^{(w)}$ does in Property(2), Property(4) and Property(5) in Case [II] of the lemma. Thus, the proof of Step(1) is done.

The Proof of Step(2) for Case[II] Let $T_{d_{j+2}-1}^{(1)}$ in (h_1, f) be not zero. For the proof of Step(2), it is enough to show by Step(1) that $(f_{j+1}, f) = (h_{\nu+1}, f)$ with the following property:

$$(15.5.5) \quad \begin{aligned} T_{d_{j+2}-1}^{(\nu+1)} &= 0 \quad \text{for some integer } \nu \leq \frac{n_{j+1}+1}{2}, \\ T_{d_{j+2}-1}^{(p)} &\neq 0 \quad \text{for } p = 1, 2, \dots, \nu. \end{aligned}$$

First of all, we have already proved by Step(1) that for given (h_p, f) , (h_{p+1}, f) can be written in the form

$$(15.5.6) \quad \begin{cases} h_{p+1} &= h_p + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(p)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(p+1)} f_j^i, \\ f &= h_{p+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(p+1)} h_{p+1}^i, \end{cases}$$

satisfying the properties Property(1), Property(2), \dots , Property(5) in this sublemma except possibly for the above equation (15.5.5).

In preparation for the proof of the equation in (15.5.5), consider y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} as we have seen in the proof of Step(1), if necessary. For brevity of notation, if $T = T(y, z, f_1, \dots, f_j)$ is not zero in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$ for $j \geq 0$, then define $\deg_j T$ by the degree of T as a polynomial in f_j , when T is in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ as a polynomial in f_j with coefficients in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$. For such notations, write $\tau_p = \deg_j(T_{d_{j+2}-1}^{(p)})$ for all $p \geq 1$, and so $0 \leq \tau_p \leq n_{j+1} - 2$ by Property(5).

First, in preparation for the proof of an equality in (15.5.5), it is very interesting to show that if $T_{d_{j+2}-1}^{(w)}$ and $T_{d_{j+2}-1}^{(w+1)}$ is not zero for any integer $w > 0$, then the following inequality is true:

$$(15.5.7) \quad n_{j+1} - 2 \geq \tau_w > \tau_{w+1} \geq 0 \quad \text{with} \quad \tau_w \geq \frac{n_{j+1} - 1}{2},$$

because if $T_{d_{j+2}-1}^{(p)}$ is not zero for any $p \geq 1$ and an inequality in (15.5.7) is true, then $\{\tau_p : p = 1, 2, \dots\}$ is an infinite sequence which is strictly decreasing and bounded with $n_{j+1} - 2 \geq \tau_p > \tau_{p+1} \geq 0$. This would be impossible, which is equivalent to the fact that there is an integer $\nu < n_{j+1}$ such that $T_{d_{j+2}-1}^{(\nu+1)} = 0$.

Now, assuming that $T_{d_{j+2}-1}^{(w)}$ and $T_{d_{j+2}-1}^{(w+1)}$ is not zero for any integer $w > 0$, then we are going to prove that an inequality in (15.5.7) is true.

Since $h_{w+1} = h_w + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(w)}$, then a sequence in (15.5.6) implies that

$$(15.5.8) \quad \begin{aligned} f &= h_w^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(w)} h_w^i \\ &= h_{w+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(w+1)} h_{w+1}^i \\ (15.5.9) \quad &= \left(h_w + \frac{1}{d_{j+2}} T_{d_{j+2}-1}^{(w)} \right)^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_i^{(w+1)} h_{w+1}^i. \end{aligned}$$

Therefore, $f - h_w^{d_{j+2}} - T_{d_{j+2}-1}^{(w)} h_w^{d_{j+2}-1} = \sum_0^{(w)} = \sum_{1,1}^{(w)} + \sum_{1,2}^{(w)}$ such that

$$(15.5.10) \quad \sum_0^{(w)} = \sum_{i=0}^{d_{j+1}-2} T_i^{(w)} h_w^i \quad \text{by (15.5.8),}$$

$$(15.5.11) \quad \begin{cases} \sum_{1,1}^{(w)} = \sum_{q=0}^{d_{j+2}-2} c_q (T_{d_{j+2}-1}^{(w)})^{d_{j+2}-q} h_w^q & \text{by (15.5.9),} \\ \sum_{1,2}^{(w)} = \sum_{i=0}^{d_{j+2}-1} T_i^{(w+1)} h_{w+1}^i, \end{cases}$$

with some nonzero constant c_q by (15.5.9).

Then, to prove that $\tau_w > \tau_{w+1} \geq 0$ and $\tau_w \geq \frac{n_{j+1}-1}{2}$, note that given any nonzero monomial $\Pi_{t=1}^{j+2} f_{t-2}^{\delta_t}$ in either $T_i^{(w)}$ or $T_i^{(w+1)}$ for all i , $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $2 \leq t \leq j+2$, and that $h_{w+1} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ is a W -poly of degree n_{j+1} in f_j .

Also, note by (15.4.7) that for each fixed $j \geq 1$ and for any nonzero monomial $y^\alpha z^\beta$ in $\Pi_{t=1}^{j+1} f_{t-2}^{\gamma_t}$, where $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $2 \leq t \leq j+1$,

$$(15.5.12) \quad \beta < \Pi_{t=1}^j n_t = \text{the multiplicity of } f_j \text{ at } 0 \in \mathbb{C}^2.$$

Now, to finish the proof of (15.5.7), let μ_1, μ_2 and μ_3 be defined by $\max\{\gamma_{j+2}\}$ among the degree of f_j for all nonzero monomials $\Pi_{t=1}^{j+2} f_{t-2}^{\gamma_t}$ in $\sum_0^{(w)}$, $\sum_{1,1}^{(w)}$ and $\sum_{1,2}^{(w)}$, respectively, where $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $2 \leq t \leq j+1$.

Then, we need only to consider three subcases (i), (ii) and (iii), respectively.

(i) In $\sum_0^{(w)}$, $\mu_1 < n_{j+1} + n_{j+1}(d_{j+2} - 2) = d_{j+1} - n_{j+1}$ because $d_{j+1} = n_{j+1}d_{j+2}$.

(ii) In $\sum_{1,1}^{(w)}$, we have two possibilities by (15.5.12), noting by Property(5) that $\tau_w = \deg_j(T_{d_{j+2}-1}^{(w)}) \leq n_{j+1} - 2$.

(iia) If $\beta < \frac{1}{2}\Pi_{t=1}^j n_t$ for all β in (15.5.12), then $\mu_2 = 2\delta_{j+2} + n_{j+1}(d_{j+2} - 2) = 2\tau_w + d_{j+1} - 2n_{j+1} \leq 2(n_{j+1} - 2) + d_{j+1} - 2n_{j+1} = d_{j+1} - 4$ for some nonzero monomials $\Pi_{t=1}^{j+2} f_{t-2}^{\delta_t}$ in $T_{d_{j+2}-1}^{(w)}$.

(iib) If $\beta \geq \frac{1}{2}\Pi_{t=1}^j n_t$ for some β in (15.5.12), then $\mu_2 = 2\delta_{j+2} + 1 + n_{j+1}(d_{j+2} - 2) = 2\tau_w + 1 + d_{j+1} - 2n_{j+1} \leq 2(n_{j+1} - 2) + 1 + d_{j+1} - 2n_{j+1} = d_{j+1} - 3$ for some nonzero monomials $\Pi_{t=1}^{j+2} f_{t-2}^{\delta_t}$ in $T_{d_{j+2}-1}^{(w)}$.

(iii) In $\sum_{1,2}^{(w)}$, $\mu_3 = \delta'_{j+2} + n_{j+1}(d_{j+2} - 1) = \tau_{w+1} + d_{j+1} - n_{j+1} \geq d_{j+1} - n_{j+1}$ for some nonzero monomials $\Pi_{t=1}^{j+2} f_{t-2}^{\delta'_t}$ in $T_{d_{j+2}-1}^{(w+1)}$ where $\tau_{w+1} = \deg_j(T_{d_{j+2}-1}^{(w+1)})$.

By (i) and (iii), it is clear that $\mu_3 > \mu_1$.

Then, we claim that $\mu_2 = \mu_3$. To prove the claim, assume the contrary.

Then, it suffices to consider two possibilities (A) and (B), respectively:

(A) If $\mu_3 > \mu_2$, then there is a nonzero monomial $(\Pi_{t=1}^{j+2} f_{t-2}^{\delta'_t}) f_j^{n_{j+1}(d_{j+2}-1)}$ which belongs to $\sum_{1,2}^{(w)}$, but does not belong to $\sum_{1,1}^{(w)}$ and $\sum_0^{(w)}$ because $\mu_3 > \mu_1$. It would be a contradiction to $\sum_0^{(w)} = \sum_{1,1}^{(w)} + \sum_{1,2}^{(w)}$.

(B) If $\mu_2 > \mu_3$, then there is a nonzero monomial $(\Pi_{t=1}^{j+1} f_{t-2}^{\delta_t}) f_{j+2}^{\mu_2}$ which belongs to $\sum_{1,1}^{(w)}$, but does not belong to $\sum_0^{(w)}$ and $\sum_{1,2}^{(w)}$ because $\mu_2 > \mu_1$. It would be a contradiction to $\sum_0^{(w)} = \sum_{1,1}^{(w)} + \sum_{1,2}^{(w)}$.

Thus, we proved the claim.

First, in preparation for the proof of an inequality in (15.5.7), using (ii) and (iii) with $\mu_2 = \mu_3$, then it suffices to consider two possibilities (a) and (b), as follows:

(a) It is clear by (iia) and (iii) that $\mu_2 = \mu_3$ implies $2\tau_w = \tau_{w+1} + n_{j+1}$.

(b) It is clear by (iib) and (iii) that $\mu_2 = \mu_3$ implies $2\tau_w + 1 = \tau_{w+1} + n_{j+1}$.

Recall by Property(5) of Case(II) in this sublemma that τ_p was defined by $\tau_p = \deg_j(T_{d_{j+2}-1}^{(p)})$ for all $p \geq 1$, and so $0 \leq \tau_p \leq n_{j+1} - 2$, if exists.

(a) First, if $2\tau_w = \tau_{w+1} + n_{j+1}$, then it is trivial that $\tau_w - \tau_{w+1} = \frac{\tau_{w+1} + n_{j+1}}{2} - \tau_{w+1} = \frac{n_{j+1} - \tau_{w+1}}{2} > 0$ because $\tau_{w+1} \leq n_{j+1} - 2$.

(b) Next, if $2\tau_w + 1 = \tau_{w+1} + n_{j+1}$, then it is trivial that $\tau_w - \tau_{w+1} = \frac{\tau_{w+1} + n_{j+1} - 1}{2} - \tau_{w+1} = \frac{n_{j+1} - \tau_{w+1} - 1}{2} > 0$ because $\tau_{w+1} \leq n_{j+1} - 2$.

Moreover, it is clear by (a) and (b) that $\tau_w \geq \frac{n_{j+1}-1}{2}$, which is an inequality in (15.5.7).

Therefore, we proved that $\tau_w > \tau_{w+1} \geq 0$ with $\tau_w \geq \frac{n_{j+1}-1}{2}$ for (15.5.7).

So, assuming that $T_{d_{j+2}-1}^{(p)}$ is not zero for any $p \geq 1$, since an inequality in (15.5.7) is true, then $\{\tau_p : p = 1, 2, \dots\}$ is an infinite sequence which is strictly decreasing and bounded with $n_{j+1} - 2 \geq \tau_p > \tau_{p+1} \geq 0$. This would be impossible, and therefore there is an integer $\nu < n_{j+1}$ such that $T_{d_{j+2}-1}^{(\nu+1)} = 0$.

Finally, to prove that $\nu \leq \frac{n_{j+1}-1}{2}$, define a function $\psi : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ by $\psi(p) = \tau_p$ where $\tau_p = \deg_j(T_{d_{j+2}-1}^{(p)})$ for all $p \geq 1$.

Then, $\nu - 2 \leq (\psi(1) - \psi(2)) + (\psi(2) - \psi(3)) + \dots + (\psi(\nu - 2) - \psi(\nu - 1)) = \psi(1) - \psi(\nu - 1) = \tau_1 - \tau_{\nu-1} \leq n_{j+1} - 2 - \frac{n_{j+1}-1}{2} = \frac{n_{j+1}-3}{2}$, because $\psi(\nu - 1) \neq 0$. So, $\nu \leq \frac{n_{j+1}+1}{2}$. Thus, we proved Property(6), and so the proof of Step(2) is done.

Thus, we completed the proof of Sublemma 15.5. \square

Sublemma 15.6. *For a given f , the construction of a sequence $\{f_0 = z, f_1, \dots, f_j\}$ in the sense of Sublemma 15.5 or Theorem 15.4 must be unique.*

Proof of Sublemma 15.6. The proof will be by induction on the integer j with $0 \leq j \leq \ell - 1$. If $j = 0$, it is trivial. So, for $0 \leq j \leq \ell - 2$, suppose we have shown by induction assumption on j that such an existence of a sequence $\{f_0, f_1, \dots, f_j\}$ is unique for a given f . For the proof of the uniqueness on the integer $(j + 1)$, assume that there are two representation for f as follows:

Let $\phi_{-1} = \psi_{-1} = y$, $\phi_0 = \psi_0 = z$, and for each $k = 1, 2, \dots, j + 1$, let

$$(15.6.1) \quad \begin{cases} \phi_k &= \phi_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R'_{k,i} \phi_{k-1}^i, \\ f &= \phi_{j+1}^{d_{j+2}} + \sum_{p=0}^{d_{j+2}-2} S'_{j+2,p} \phi_{j+1}^p, \end{cases}$$

and

$$(15.6.2) \quad \begin{cases} \psi_k &= \psi_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R''_{k,i} \psi_{k-1}^i, \\ f &= \psi_{j+1}^{d_{j+2}} + \sum_{p=0}^{d_{j+2}-2} S''_{j+2,p} \psi_{j+1}^p, \end{cases}$$

where each of $f = f(y, z, \phi_2, \dots, \phi_{j+1})$ and $f = (y, z, \psi_2, \dots, \psi_{j+1})$ satisfies the same kind of properties and notations as we have done in the conclusion of Theorem 15.4 except possibly for the uniqueness.

In preparation for the proof of the uniqueness on the integer $(j + 1)$, by the same method as in (15.4.7), note that for any nonzero monomial $y^s z^t$ in $\Pi_{k=1}^{j+1} \phi_{k-2}^{\delta'_k}$ of $R'_{j+1,i} = R'_{j+1,i}(y, z, \phi_1, \dots, \phi_{j-1})$,

$$(15.6.3) \quad t < \Pi_{k=1}^j n_k = \text{the multiplicity of } \phi_j \text{ at } 0 \in \mathbb{C}^2 \quad \text{and} \quad s > 0,$$

and for any nonzero monomial $y^\mu z^\nu$ in $\Pi_{k=1}^{j+2} \phi_{k-2}^{\gamma'_k}$ of $S'_{j+2,p} = S'_{j+2,p}(y, z, \phi_1, \dots, \phi_j)$,

$$(15.6.4) \quad \nu < \Pi_{k=1}^{j+1} n_k = \text{the multiplicity of } \phi_{j+1} \text{ at } 0 \in \mathbb{C}^2 \quad \text{and} \quad \mu > 0.$$

Similarly, for any nonzero monomial $y^s z^t$ in $\Pi_{k=1}^{j+1} \psi_{k-2}^{\delta''_k}$ of $R''_{j+1,i} = R''_{j+1,i}(y, z, \psi_1, \dots, \psi_{j-1})$,

$$(15.6.5) \quad t < \Pi_{k=1}^j n_k = \text{the multiplicity of } \psi_j \text{ at } 0 \in \mathbb{C}^2 \quad \text{and} \quad s > 0,$$

and for any nonzero monomial $y^\mu z^\nu$ in $\Pi_{k=1}^{j+2} \psi_{k-2}^{\gamma''_k}$ of $S''_{j+2,p} = S''_{j+2,p}(y, z, \psi_1, \dots, \psi_j)$,

$$(15.6.6) \quad \nu < \Pi_{k=1}^{j+1} n_k = \text{the multiplicity of } \psi_{j+1} \text{ at } 0 \in \mathbb{C}^2 \quad \text{and} \quad \mu > 0.$$

To prove the uniqueness on the integer $(j + 1)$, it suffices to follow two steps, Step(1) and Step(2). More rigorously, first we will construct the detailed statement for Step(1) and the detailed statement for Step(2), and next prove two statements, respectively.

Step(1) We prove by induction assumption on j that

$$(15.6.7) \quad f_k = \phi_k = \psi_k \quad \text{for} \quad 1 \leq k \leq j,$$

and so $f(y, z, \phi_1, \dots, \phi_j) = f(y, z, \psi_1, \dots, \psi_j)$.

Step(2) We prove by Step(1), (15.6.1) and (15.6.2) that $\phi_{j+1} = \psi_{j+1}$, and by the uniqueness of The WDT or Theorem 15.2 that there is nothing to prove for the uniqueness.

In more detail, we write each step with proof, respectively.

Step(1) To construct the statement for Step(1), first it suffices to prove two claims, i.e., Claim(I) and Claim(II), and after then there is nothing to prove for Step(1) with (15.6.7).

Claim(I) $f(y, z, \phi_1, \dots, \phi_j)$ of (15.6.1) can be represented in the form

$$(15.6.8) \quad \begin{cases} \phi_j &= \phi_{j-1}^{n_j} + \sum_{i=0}^{n_j-2} R'_{j,i} \phi_{j-1}^i, \\ f &= \phi_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S'_{j+1,i} \phi_j^i, \end{cases}$$

where f of (15.6.8) is viewed as $f(y, z, \phi_1, \dots, \phi_j)$, satisfying the following property:

(1) Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. For each $j \geq 1$, $S'_{j+1,i} = S'_{j+1,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{t=1}^j n_t$ in z and $S'_{j+1,i}(0, z) = 0$.

(2) Consider $y, z, \phi_1, \dots, \phi_{j-1}$ as independent complex $(j+1)$ -variables at the origin in \mathbb{C}^{j+1} . Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. For any nonzero monomial $\Pi_{t=1}^{j+1} \phi_{t-2}^{\delta'_t}$ in $S'_{j+1,i} = S'_{j+1,i}(y, z, \phi_1, \dots, \phi_{j-1}) \in \mathbb{C}\{y\}[z, \phi_1, \dots, \phi_{j-1}]$, $\delta'_1 > 0$ and $\delta'_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$.

(3) As in (2) of Remark 15.4.1, assuming that f is a W -poly of degree $n \geq 2$ in z with the multiplicity n at $0 \in \mathbb{C}^2$, then $S'_{j+1,i} \in \mathbb{C}\{y\}[z]$ has a multiplicity $\geq (d_{j+1} - i)\Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

Claim(II) $f(y, z, \psi_1, \dots, \psi_j)$ of (15.6.2) can be represented in the form

$$(15.6.9) \quad \begin{cases} \psi_j &= \psi_{j-1}^{n_j} + \sum_{i=0}^{n_j-2} R''_{j,i} \psi_{j-1}^i, \\ f &= \psi_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S''_{j+1,i} \psi_j^i, \end{cases}$$

where f of (15.6.9) is viewed as $f(y, z, \psi_1, \dots, \psi_j)$, satisfying the following property:

(1) Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. For each $j \geq 1$, $S''_{j+1,i} = S''_{j+1,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{t=1}^j n_t$ in z with $S''_{j+1,i}(0, z) = 0$.

(2) Consider $y, z, \psi_1, \dots, \psi_{j-1}$ as independent complex $(j+1)$ -variables at the origin in \mathbb{C}^{j+1} . Let i be fixed with $0 \leq i \leq d_{j+1} - 2$. For any nonzero monomial $\Pi_{t=1}^{j+1} \psi_{t-2}^{\delta''_t}$ in $S''_{j+1,i} = S''_{j+1,i}(y, z, \psi_1, \dots, \psi_{j-1}) \in \mathbb{C}\{y\}[z, \psi_1, \dots, \psi_{j-1}]$, $\delta''_1 > 0$ and $\delta''_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$.

(3) As in (iii) of Remark 15.4.1, assuming that f is a W -poly of degree $n \geq 2$ in z with the multiplicity n at $0 \in \mathbb{C}^2$, then $S''_{j+1,i} \in \mathbb{C}\{y\}[z]$ of (15.4.1) has a multiplicity $\geq (d_{j+1} - i)\Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

If the proofs of Claim(I) and Claim(II) are done, then it can be proved by (15.6.8) and (15.6.9) and by induction assumption on j that $f_k = \phi_k = \psi_k$ for $1 \leq k \leq j$ in (15.6.7), and so $f(y, z, \phi_1, \dots, \phi_j) = f(y, z, \psi_1, \dots, \psi_j)$.

The Proof of Step(1) First, in order to prove Claim(I), using the equations for $k = j$ and $k = j+1$ in (15.6.1), then we get the following:

$$(15.6.10) \quad f = \phi_{j+1}^{d_{j+2}} + \sum_{p=0}^{d_{j+2}-2} S'_{j+2,p} \phi_{j+1}^p$$

$$(15.6.11) \quad = \sum_1 + \sum_2,$$

$$\begin{aligned} \text{where } \sum_1 &= \{\phi_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} \phi_j^i\}^{d_{j+2}} \quad \text{and} \\ \sum_2 &= \sum_{p=0}^{d_{j+2}-2} S'_{j+2,p} \{\phi_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} \phi_j^i\}^p. \end{aligned}$$

Now, consider $f - \phi_j^{d_{j+1}}$ as

$$(15.6.12) \quad f - \phi_j^{d_{j+1}} = \left(\sum_1 - \phi_j^{d_{j+1}} \right) + \sum_2.$$

Recall that $\phi_j(y, z) \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^j n_t$ in z with the multiplicity $\Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$ by the induction assumption on j , and that $R'_{j+1,i}(y, z)$ is a polynomial of degree $< \Pi_{t=1}^j n_t$ in z and $R'_{j+1,i}(0, z) = 0$ by (15.6.3) and that $S'_{j+2,i}(y, z)$ is a polynomial of degree $< \Pi_{t=1}^{j+1} n_t$ in z and $S'_{j+2,i}(0, z) = 0$ by (15.6.4).

In preparation for the proof of Claim(I), first of all, we will prove the following:

- (i) If we write $\sum_1 -\phi_j^{d_{j+1}} = \sum b_{p,q} y^p z^q$ with some nonzero constant $b_{p,q}$, then $p > 0$ and $q < (d_{j+1} - 1)\Pi_{t=1}^j n_t$.
- (ii) If we write $\sum_2 = \sum c_{r,s} y^r z^s$ with some nonzero constant $c_{r,s}$, then $r > 0$ and $s < (d_{j+1} - n_{j+1})\Pi_{t=1}^j n_t$.

For any nonzero monomial $y^p z^q \in \sum_1 -\phi_j^{d_{j+1}}$, it is clear that $p > 0$ and by (15.6.3) that $q < (n_{j+1}(d_{j+2} - 1) + n_{j+1} - 2)\Pi_{t=1}^j n_t + \Pi_{t=1}^j n_t = (d_{j+1} - 1)\Pi_{t=1}^j n_t$. Thus, the proof of (i) is done.

For any nonzero monomial $y^r z^s \in \sum_2$, it is clear that $r > 0$ and by (15.6.3) and (15.6.4) that $s < (d_{j+2} - 2)n_{j+1}\Pi_{t=1}^j n_t + \Pi_{t=1}^{j+1} n_t = (d_{j+1} - 2n_{j+1} + n_{j+1})\Pi_{t=1}^j n_t = (d_{j+1} - n_{j+1})\Pi_{t=1}^j n_t$. Thus, the proof of (ii) is done.

Therefore, we proved that whenever any nonzero monomial $y^\alpha z^\beta \in f - \phi_j^{d_{j+1}}$ then $\alpha > 0$ and $\beta < (d_{j+1} - 1)\Pi_{t=1}^j n_t$, noting that $\phi_j(y, z) \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^j n_t$ in z with the multiplicity $\Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$ by the induction assumption on j .

Now, in order to prove Claim(I), apply the WDT with a divisor ϕ_j to f as an element of $\mathbb{C}\{y\}[z]$. Since for any nonzero monomial $y^\alpha z^\beta \in f - \phi_j^{d_{j+1}}$ $\alpha > 0$ and $\beta < (d_{j+1} - 1)\Pi_{t=1}^j n_t$ by (i) and (ii), then it is clear by (i) and (ii) of Theorem 15.2 that f of (15.6.11) can be rewritten as follows:

$$(15.6.13) \quad f = \phi_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-1} T_i^{(j+1)} \phi_j^i \quad \text{with } T_{d_{j+1}-1}^{(j+1)} = 0,$$

where for each $i = 0, 1, \dots, d_{j+1} - 2$, $T_i^{(j+1)} = T_i^{(j+1)}(y, z) = \sum a_{p,q} y^p z^q$ with a nonzero constant $a_{p,q}$ such that $p > 0$ and $q < \Pi_{t=1}^j n_t$ and that $T_i^{(j+1)}(0, z) = 0$ and $T_i^{(j+1)}$ has a multiplicity $\geq (d_{j+1} - i)\Pi_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$.

Now, to finish the proof of Claim(I), since $\phi_{j-1} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^{j-1} n_t$ in z with the multiplicity $\Pi_{t=1}^{j-1} n_t$ at $0 \in \mathbb{C}^2$, apply the WDT with a divisor ϕ_{j-1} to $T_i^{(j+1)}$ for each $i = 0, 1, \dots, d_{j+2} - 2$. Then by Theorem 15.2, each $T_i^{(j+1)}$ can be written as

$$(15.6.14) \quad T_i^{(j+1)} = \sum_{k_1=0}^{n_j-1} Q_{k_1} \phi_{j-1}^{k_1},$$

where if exists, each $Q_{k_1} \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{k=1}^{j-1} n_t$ in z with $Q_{k_1}(0, z) = 0$. Using the similar technique as we have seen in the proof of Fact(D) in (15.4.3), since $\phi_{j-2} \in \mathbb{C}\{y\}[z]$ is a W -poly of degree $\Pi_{t=1}^{j-2} n_t$ in z with the multiplicity $\Pi_{t=1}^{j-2} n_t$ at $0 \in \mathbb{C}^2$, then for each fixed k_1 apply the WDT with a divisor ϕ_{j-2} to Q_{k_1} for each $k_1 = 0, 1, \dots, n_j - 1$, again. Then by Theorem 15.2, each Q_{k_1} may be written in the form

$$(15.6.15) \quad Q_{k_1} = \sum_{k_2=0}^{n_{j-1}-1} Q_{k_1, k_2} \phi_{j-2}^{k_2},$$

where if exists, each $Q_{k_1, k_2} \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $< \Pi_{t=1}^{j-2} n_t$ in z with $Q_{k_1, k_2}(0, z)$ zero.

Thus, continue the same process as above, and consider $y, z, \phi_1, \dots, \phi_{j-1}$ as independent complex $(j+1)$ -variables at the origin in \mathbb{C}^{j+1} . Let i be fixed with $0 \leq i \leq d_{j+1} - 2$.

Then, we proved that for each fixed $i = 0, 1, \dots, d_{j+1} - 2$, and for any nonzero monomial $\Pi_{t=1}^{j+1} \phi_{t-2}^{\delta_t}$ in $T_{j+1,i} = T_{j+1,i}(y, z, \phi_1, \dots, \phi_{j-1}) \in \mathbb{C}\{y\}[z, \phi_1, \dots, \phi_{j-1}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$. If we write $S'_{j+1,i} = T_i^{(j+1)}$ for $0 \leq i \leq d_{j+1} - 2$, then the proof of Claim(I) with (15.6.8) is done.

By the same method as we have used in the proof of Claim(I), the proof of Claim(II) with (15.6.9) can be proved, too.

Therefore, by induction assumption on j and (15.6.8) and (15.6.9), we proved that

$$(15.6.16) \quad f_k = \phi_k = \psi_k \quad \text{for } 1 \leq k \leq j,$$

and then $f(y, z, \phi_1, \dots, \phi_j) = f(y, z, \psi_1, \dots, \psi_j)$. Thus, the proof of Step(1) is done.

Step(2) For the proof, it suffices to show that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i \leq n_{j+1} - 2$, because it is clear by the uniqueness of The WDT or Theorem 15.2 that $\phi_{j+1} = \psi_{j+1}$ if and only if $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i \leq n_{j+1} - 2$ where $\phi_{j+1} = f_j^{n_j} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} f_j^i$ by (15.6.1) and $\psi_{j+1} = f_j^{n_j} + \sum_{i=0}^{n_{j+1}-2} R''_{j+1,i} f_j^i$ by (15.6.2), noting by Step(1) that $f_j = \phi_j = \psi_j$.

The Proof of Step(2) In preparation for the proof of this step, using the equation in either (15.6.7) of Step(1) or (15.6.16), then apply (15.6.16) to (15.6.1) with $k = j+1$, and also apply (15.6.16) to (15.6.2) with $k = j+1$. Then, it is clear that f can be rewritten in the form

$$(15.6.17) \quad \begin{aligned} f &= \sum_1 + \sum_2 \quad \text{for (15.6.1)} \\ &= \sum_3 + \sum_4 \quad \text{for (15.6.2),} \end{aligned}$$

where

$$(15.6.18) \quad \begin{cases} \sum_1 &= (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} f_j^i)^{d_{j+2}}, \\ \sum_2 &= \sum_{p=0}^{d_{j+2}-2} S'_{j+2,p} (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} f_j^i)^p, \end{cases}$$

$$(15.6.19) \quad \begin{cases} \sum_3 &= (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R''_{j+1,i} f_j^i)^{d_{j+2}}, \\ \sum_4 &= \sum_{p=0}^{d_{j+2}-2} S''_{j+2,p} (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R''_{j+1,i} f_j^i)^p. \end{cases}$$

Now, consider $f - f_j^{d_{j+1}}$ as

$$(15.6.20) \quad \begin{aligned} f - f_j^{d_{j+1}} &= (\sum_1 - f_j^{d_{j+1}}) + \sum_2 \\ &= (\sum_3 - f_j^{d_{j+1}}) + \sum_4. \end{aligned}$$

For the proof of this step, consider y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} . Using the same method as we have seen in the proof of Fact(D) in (15.4.3) of Sublemma 15.4. α , then \sum_2 , $\sum_1 - f_j^{d_{j+1}}$, \sum_4 and $\sum_3 - f_j^{d_{j+1}}$ which are polynomials in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, can be written by (A), (B), (C) and (D) in order, as follows:

(A) Firstly, for any nonzero monomial $y^r z^s \in \sum_2$ it is clear by (15.6.3) and (15.6.4) that $r > 0$ and $s < (d_{j+2} - 2)n_{j+1} \Pi_{t=1}^j n_t + \Pi_{t=1}^{j+1} n_t = (d_{j+1} - 2n_{j+1} + n_{j+1}) \Pi_{t=1}^j n_t = (d_{j+1} - n_{j+1}) \Pi_{t=1}^j n_t$.

So, considering \sum_2 as a polynomial in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, by the same method as we have seen in the proof of Fact(D) in (15.4.3), \sum_2 can be rewritten as follows:

$$(15.6.21) \quad \text{Whenever } \Pi_{k=1}^{j+2} f_k^{\delta_k} \text{ is in } \sum_2, \text{ then } \delta_1 > 0,$$

$$\delta_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta_{j+2} < d_{j+1} - n_{j+1}.$$

(B) Secondly, assuming that $R'_{j+1,i}$ is nonzero for some i , for any nonzero monomial $y^r z^s \in \sum_1 -f_j^{d_{j+1}}$ it is clear by (15.6.3) that $r > 0$ and $s < \{(d_{j+2} - 1)n_{j+1} + n_{j+1} - 2\}\Pi_{t=1}^j n_t + \Pi_{t=1}^j n_t = (d_{j+1} - 1)\Pi_{t=1}^j n_t$.

If $R'_{j+1,i}$ is nonzero for some i , there is a nonzero monomial $y^\alpha z^\beta \in R'_{j+1,i}$, and so there is a nonzero monomial $y^{r'} z^{s'} \in \sum_1 -f_j^{d_{j+1}}$ such that $r' > 0$ and $s' \geq (d_{j+2} - 1)n_{j+1}\Pi_{t=1}^j n_t = (d_{j+1} - n_{j+1})\Pi_{t=1}^j n_t$, which implies that $y^{r'} z^{s'} \notin \sum_2$.

So, considering $\sum_1 -f_j^{d_{j+1}}$ as a polynomial in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, by the same method as we have seen in the proof of Fact(D) in (15.4.3), if $R'_{j+1,i}$ is nonzero for some i then $\sum_1 -f_j^{d_{j+1}}$ can be rewritten as follows:

$$(15.6.22) \quad \text{Whenever } \Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \text{ is in } \sum_1 -f_j^{d_{j+1}}, \text{ then } \delta_1 > 0,$$

$$\delta_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta_{j+2} < d_{j+1} - 1.$$

$$\text{Also, there is a monomial } \Pi_{k=1}^{j+2} f_{k-2}^{\delta'_k} \in \sum_1 -f_j^{d_{j+1}} \text{ with } \Pi_{k=1}^{j+2} f_{k-2}^{\delta'_k} \notin \sum_2$$

$$\text{such that } \delta'_1 > 0, \delta'_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta'_{j+2} \geq d_{j+1} - n_{j+1}.$$

(C) Thirdly, for any nonzero monomial $y^{r'} z^{s'} \in \sum_4$ it is clear by (15.6.5) and (15.6.6) that $r > 0$ and $s < (d_{j+2} - 2)n_{j+1}\Pi_{t=1}^j n_t + \Pi_{t=1}^{j+1} n_t = (d_{j+1} - 2n_{j+1} + n_{j+1})\Pi_{t=1}^j n_t = (d_{j+1} - n_{j+1})\Pi_{t=1}^j n_t$.

So, considering \sum_4 as a polynomial in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, by the same method as we have seen in the proof of Fact(D) in (15.4.3), \sum_4 can be rewritten as follows:

$$(15.6.23) \quad \text{Whenever } \Pi_{k=1}^{j+2} f_{k-2}^{\delta'_k} \text{ is in } \sum_4, \text{ then } \delta'_1 > 0,$$

$$\delta'_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta'_{j+2} < d_{j+1} - n_{j+1}.$$

(D) Fourthly, assuming that $R''_{j+1,i}$ is nonzero for some i , for any nonzero monomial $y^r z^s \in \sum_3 -f_j^{d_{j+1}}$ it is clear by (15.6.5) that $r > 0$ and $s < \{(d_{j+2} - 1)n_{j+1} + n_{j+1} - 2\}\Pi_{t=1}^j n_t + \Pi_{t=1}^j n_t = (d_{j+1} - 1)\Pi_{t=1}^j n_t$.

If $R''_{j+1,i}$ is nonzero for some i , there is a nonzero monomial $y^\alpha z^\beta \in R''_{j+1,i}$, and so there is a nonzero monomial $y^{r''} z^{s''} \in \sum_3 -f_j^{d_{j+1}}$ such that $r'' > 0$ and $s'' \geq (d_{j+2} - 1)n_{j+1}\Pi_{t=1}^j n_t = (d_{j+1} - n_{j+1})\Pi_{t=1}^j n_t$, which implies that $y^{r''} z^{s''} \notin \sum_2$.

So, considering $\sum_3 -f_j^{d_{j+1}}$ as a polynomial in $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, by the same method as we have seen in the proof of Fact(D) in (15.4.3), if $R''_{j+1,i}$ is nonzero for some i then $\sum_3 -f_j^{d_{j+1}}$ can be rewritten as follows:

$$(15.6.24) \quad \text{Whenever } \Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \text{ is in } \sum_3 -f_j^{d_{j+1}}, \text{ then } \delta_1 > 0,$$

$$\delta_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta_{j+2} < d_{j+1} - 1.$$

$$\text{Also, there is a monomial } \Pi_{k=1}^{j+2} f_{k-2}^{\delta''_k} \in \sum_3 -f_j^{d_{j+1}} \text{ with } \Pi_{k=1}^{j+2} f_{k-2}^{\delta''_k} \notin \sum_4$$

$$\text{such that } \delta''_1 > 0, \delta''_t < n_t \text{ for } 2 \leq t \leq j+1 \text{ and } \delta''_{j+2} \geq d_{j+1} - n_{j+1}.$$

Recall by (15.6.1), (15.6.2) and Step(1) that

$$(15.6.25) \quad \begin{aligned} \phi_{j+1} &= f_j^{n_j} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} f_j^i \quad \text{and} \\ \psi_{j+1} &= f_j^{n_j} + \sum_{i=0}^{n_{j+1}-2} R''_{j+1,i} f_j^i. \end{aligned}$$

In order to prove that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i \leq n_{j+1} - 2$, first let

$$(15.6.26) \quad \begin{aligned} m &= \max\{i : R'_{j+1,i} \text{ is nonzero in } \phi_{j+1} \text{ of (15.6.25)}\}, \quad \text{and} \\ r &= \max\{i : R''_{j+1,i} \text{ is nonzero in } \psi_{j+1} \text{ of (15.6.25)}\}. \end{aligned}$$

In order to prove that $m = r$ and $R'_{j+1,m} = R''_{j+1,m}$, it suffices to consider two subcases:

Subcases(1). If there does not exist an integer m satisfying (15.6.26), then $R'_{j+1,i}$ is identically zero for all $i = 0, 1, \dots, n_{j+1} - 2$, and so $R''_{j+1,i}$ is identically zero for all $i = 0, 1, \dots, n_{j+1} - 2$ by (15.6.20), (15.6.21), (15.6.22), (15.6.23) and (15.6.24). Therefore, there is nothing to prove that $\phi_{j+1} = \psi_{j+1}$.

Subcase(2). If there exist such integers $m \geq 0$ and $r \geq 0$, $\sum_1 - f_j^{d_{j+1}}$ and $\sum_3 - f_j^{d_{j+1}}$ can be written as follows:

(a)(a1) $\sum_1 - f_j^{d_{j+1}} = \sum a_{\gamma_1 \gamma_2 \dots \gamma_{j+2}} \prod_{k=1}^{j+2} f_k^{\gamma_k}$ where each $a_{\gamma_1 \gamma_2 \dots \gamma_{j+2}}$ is a nonzero constant and $\gamma_1 > 0$, $\gamma_k < n_{k-1}$ for $2 \leq k \leq j+1$, and $\gamma_{j+2} \geq 0$.

(a2) Then, let τ be defined by $\max\{\gamma_{j+2}\}$ for all nonzero monomials $\prod_{k=1}^{j+2} f_k^{\gamma_k}$ in $\sum_1 - f_j^{d_{j+1}}$ where $\gamma_1 > 0$, $\gamma_k < n_{k-1}$ for $2 \leq k \leq j+1$, and $\gamma_{j+2} \geq 0$. Then, $\tau = n_{j+1}(d_{j+2} - 1) + m$ by (15.6.22), (15.6.3) and (15.6.4).

(b)(b1) $\sum_3 - f_j^{d_{j+1}} = \sum b_{\delta_1 \delta_2 \dots \delta_{j+2}} \prod_{k=1}^{j+2} f_k^{\delta_k}$ where each $b_{\delta_1 \delta_2 \dots \delta_{j+2}}$ is a nonzero constant and $\delta_1 > 0$, $\delta_k < n_{k-1}$ for $2 \leq k \leq j+1$, and $\delta_{j+2} \geq 0$.

(b2) Then, let ω be defined by $\max\{\delta_{j+2}\}$ for all nonzero monomials $\prod_{k=1}^{j+2} f_k^{\delta_k}$ in $\sum_3 - f_j^{d_{j+1}}$ where $\delta_1 > 0$, $\delta_k < n_{k-1}$ for $2 \leq k \leq j+1$, and $\delta_{j+2} \geq 0$. Then, $\omega = n_{j+1}(d_{j+2} - 1) + r$ by (15.6.24), (15.6.5) and (15.6.5).

(c) Either $\tau = \omega$ or $m = r$ by (a2) and (b2).

(d) To prove that $R'_{j+1,m} = R''_{j+1,m}$, first note that for any nonzero monomial $\prod_{k=1}^{j+2} f_k^{\gamma_k} \in f_j^{n_{j+1}(d_{j+2}-1)+m} R'_{j+1,m}$, $\gamma_{j+2} = \tau$ by (15.6.3), and that $\gamma'_{j+2} < \tau$ for any nonzero monomial $\prod_{k=1}^{j+2} f_k^{\gamma'_k} \in f_j^{n_{j+1}(d_{j+2}-1)+i} R'_{j+1,i}$ with $i < m$. By the similar result as in $\sum_3 - f_j^{d_{j+1}}$, we can get that

$$(15.6.27) \quad f_j^{n_{j+1}(d_{j+2}-1)+m} R'_{j+1,m} = f_j^{n_{j+1}(d_{j+2}-1)+m} R''_{j+1,m},$$

and so $R'_{j+1,m} = R''_{j+1,m}$.

If $m = r = 0$, there is nothing to prove. To prove that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i < m$, under the condition that $m > 0$, then it suffices to show that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i < m$.

From (15.6.25) and (15.6.26) again, let m_1 and r_1 be defined by

$$(15.6.28) \quad \begin{aligned} m_1 &= \max\{i : R'_{j+1,i} \text{ is nonzero with } i < m\}, \\ r_1 &= \max\{i : R''_{j+1,i} \text{ is nonzero with } i < r = m\}. \end{aligned}$$

For brevity of notation, let $R_{j+1,m} = R'_{j+1,m} = R''_{j+1,m}$. Noting that \sum_1, \dots, \sum_4 were already defined by (15.6.18) and (15.6.19), then $\sum_{1,1}$ and $\sum_{1,3}$ can be defined by

$$(15.6.29) \quad \begin{aligned} \sum_{1,1} &= \sum_1 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}} \quad \text{and} \\ \sum_{1,3} &= \sum_3 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}. \end{aligned}$$

By (15.6.17) and (15.6.29), $f - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}$ can be rewritten as follows:

$$(15.6.30) \quad f - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}} = \sum_{1,1} + \sum_2 = \sum_{3,1} + \sum_4.$$

In preparation for the proof of $m_1 = r_1$ and $R'_{j+1,m_1} = R''_{j+1,r_1}$, for convenience of notation, $\sum_{1,1}$ and $\sum_{3,1}$ can be rewritten as follows:

$$(15.6.31) \quad \sum_{1,1} = \sum_1 -(f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}} = \phi_{j+1}^{d_{j+2}} - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}$$

$$= \left\{ \sum_{i=0}^{m-1} R'_{j+1,i} f_j^i \right\} \cdot \left\{ \sum_{i=1}^{d_{j+2}} \phi_{j+1}^{d_{j+2}-i} (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{i-1} \right\}$$

$$(15.6.32) \quad \text{where } \sum_1 = \phi_{j+1}^{d_{j+2}} \quad \text{with } \phi_{j+1} = (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R'_{j+1,i} f_j^i),$$

$$\sum_{3,1} = \sum_3 -(f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}} = \psi_{j+1}^{d_{j+2}} - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}$$

$$= \left\{ \sum_{i=0}^{m-1} R''_{j+1,i} f_j^i \right\} \cdot \left\{ \sum_{i=1}^{d_{j+2}} \psi_{j+1}^{d_{j+2}-i} (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{i-1} \right\}$$

$$\text{where } \sum_3 = \psi_{j+1}^{d_{j+2}} \quad \text{with } \psi_{j+1} = (f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R''_{j+1,i} f_j^i).$$

For the proof, it suffices to consider two subcases:

Subcases(1). If there does not exist an integer m_1 satisfying (15.6.28), then $R'_{j+1,i}$ is identically zero for all $i = 0, 1, \dots, m-1$, and so $R''_{j+1,i}$ is identically zero for all $i = 0, 1, \dots, m-1$, because otherwise $\sum_{3,1}$ satisfies the same kind of properties as $\sum_3 - f_j^{d_{j+1}}$ does in by (15.6.20), (15.6.22) and (15.6.24). Therefore, there is nothing to prove that $\phi_{j+1} = \psi_{j+1}$.

Subcase(2). If there exist such integers $m_1 \geq 0$ and $r_1 \geq 0$, $\sum_{1,1} = \sum_1 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}$ and $\sum_{3,1} = \sum_3 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m)^{d_{j+2}}$ can be written as follows:

(a) Let τ_1 be defined by $\text{Max}\{\gamma'_{j+2}\}$ for all nonzero monomials $\Pi_{k=1}^{j+2} f_k^{\gamma_{k-2}}$ in $\sum_{1,1}$, where $\gamma'_1 > 0$, $\gamma'_k < n_{k-1}$ for $2 \leq k \leq j+1$ and $\gamma'_{j+2} \geq 0$. Then, $\tau_1 = n_{j+1}(d_{j+2}-1) + m_1 < \tau$ by (15.6.3) because $\sum_{1,1}$ satisfies the same kind of properties as $\sum_1 - f_j^{d_{j+1}}$ does in (15.6.22).

(b) Let ω_1 be defined by $\text{Max}\{\gamma''_{j+2}\}$ for all nonzero monomials $\Pi_{k=1}^{j+2} f_k^{\gamma''_{k-2}}$ in $\sum_{3,1}$ where $\gamma''_1 > 0$, $\gamma''_k < n_{k-1}$ for $2 \leq k \leq j+1$ and $\gamma''_{j+2} \geq 0$. Then, $\omega_1 = n_{j+1}(d_{j+2}-1) + r_1 < \omega$ by (15.6.5) because $\sum_{3,1}$ satisfies the same kind of properties as $\sum_3 - f_j^{d_{j+1}}$ does in (15.6.24).

(c) Either $\tau_1 = \omega_1$ or $m_1 = r_1$ by (a) and (b).

(d) To prove that $R'_{j+1,m_1} = R''_{j+1,r_1}$, first note that for any nonzero monomial $\Pi_{k=1}^{j+2} f_k^{\gamma_k} \in f_j^{n_{j+1}(d_{j+2}-1)+m_1} R'_{j+1,m_1}$, $\gamma_{j+2} = \tau_1$ by (15.6.3), and if $i < m_1$ then for any nonzero monomial $\Pi_{k=1}^{j+2} f_k^{\gamma_k} \in f_j^{n_{j+1}(d_{j+2}-1)+i} R'_{j+1,i}$, $\gamma'_{j+2} < \tau_1$.

Note that whenever $\Pi_{k=1}^{j+2} f_k^{\gamma_k} \in f_j^{n_{j+1}(d_{j+2}-1)+m_1} R'_{j+1,m_1}$ then $\Pi_{k=1}^{j+2} f_k^{\gamma_k} \in \sum_{1,1}$.

By the similar result as in $\sum_{3,1}$, we can get that

$$(15.6.33) \quad f_j^{n_{j+1}(d_{j+2}-1)+m_1} R'_{j+1,m_1} = f_j^{n_{j+1}(d_{j+2}-1)+m_1} R''_{j+1,m_1},$$

and so $R'_{j+1,m_1} = R''_{j+1,r_1}$ with $m_1 = r_1$.

If $m_1 = r_1 = 0$, there is nothing to prove. To prove that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i < m_1$, under the condition that $m_1 > 0$, then it suffices to show that $R'_{j+1,i} = R''_{j+1,i}$ for $0 \leq i < m_1$.

From (15.6.25) and (15.6.26) again, let m_2 and r_2 be defined by

$$(15.6.34) \quad m_2 = \text{Max}\{i : R'_{j+1,i} \text{ is nonzero with } i < m_1\},$$

$$r_2 = \text{Max}\{i : R''_{j+1,i} \text{ is nonzero with } i < r_1 = m_1\}.$$

For brevity of notation, let $R_{j+1,m_1} = R'_{j+1,m_1} = R''_{j+1,r_1}$. Noting that \sum_1, \dots, \sum_4 were already defined by (15.6.18) and (15.6.19), then $\sum_{1,2}$ and $\sum_{3,2}$ can be defined by

$$(15.6.35) \quad \begin{aligned} \sum_{1,2} &= \sum_1 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m + R_{j+1,m_1} f_j^{m_1})^{d_{j+2}} \quad \text{and} \\ \sum_{3,2} &= \sum_3 - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m + R_{j+1,m_1} f_j^{m_1})^{d_{j+2}}. \end{aligned}$$

By (15.6.17) and (15.6.35), $f - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m + R_{j+1,m_1} f_j^{m_1})^{d_{j+2}}$ can be rewritten as follows:

$$(15.6.36) \quad f - (f_j^{n_{j+1}} + R_{j+1,m} f_j^m + R_{j+1,m_1} f_j^{m_1})^{d_{j+2}} = \sum_{1,2} + \sum_2 = \sum_{3,2} + \sum_4.$$

By the same method as we have used in (15.6.31) and (15.6.32), it can be easily shown that $R'_{j+1,m_2} = R''_{j+1,r_2}$ with $m_2 = r_2$. Thus, repeating the above process finitely many times, it can be proved that $R'_{j+1,i} = R''_{j+1,i}$ for each $i = 0, 1, \dots, n_{j+1} - 2$, if exists. So, the proof of Step(2) is done, and then we finished the proof of Sublemma 15.6. \square

By Sublemma 15.5 and Sublemma 15.6, we completed the proof of Theorem 15.4. \square

Chapter X: Irreducibility criterion of W-polys of two complex variables

§16. In preparation for irreducibility criterion of W-polys of two complex variables

§16.1. Known preliminaries on irreducibility criterion of germs of analytic functions of two complex variables

Lemma 16.0. Assumptions Let $f(y, z) = z^n + a_{n-2}y^{\alpha_{n-2}}z^{n-2} + \cdots + a_1y^{\alpha_1}z + a_0y^{\alpha_0}$ be irreducible in ${}_2\mathcal{O}_0$ with multiplicity $n \geq 2$ at $(0, 0) \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y, z)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Assume that $d = \gcd(n, \alpha_0) > 1$. Then, we can write $n = dn_1$ and $\alpha_0 = d\alpha_{1,0,1}$ with $\gcd(n_1, \alpha_{1,0,1}) = 1$ and $2 \leq n_1 < \alpha_{1,0,1}$.

In particular, if $a_i(y, z) = a_i(y)$ for all i , then $f(y, z)$ is called a W-poly in z .

Conclusions Then, f can be written uniquely in the form

$$(16.0.1) \quad f = (z^{n_1} + \xi y^{\alpha_{1,0,1}})^d + \sum_{p,q \geq 0} c_{p,q} y^p z^q \quad \text{with} \quad n_1 p + \alpha_{1,0,1} q > n_1 \alpha_{1,0,1} d,$$

where the $c_{p,q}$ are nonzero complex numbers for some nonnegative integers p and q such that $n_1 p + \alpha_{1,0,1} q > n_1 \alpha_{1,0,1} d$, satisfying the following properties:

(i) ξ is a unique nonzero number such that

$$(16.0.2) \quad {}_d C_i \xi^i = a_{n-in_1}(0, 0) \quad \text{for} \quad 1 \leq i \leq d.$$

(ii) $\frac{\alpha_0}{n} = \frac{\alpha_{1,0,1}}{n_1} = \frac{\alpha_{n-in_1}}{in_1}$, i.e., $\alpha_{n-in_1} = i\alpha_{1,0,1}$ for $1 \leq i \leq d$, and

$$(16.0.3) \quad \frac{\alpha_{n-j}}{j} > \frac{\alpha_{1,0,1}}{n_1} \quad \text{for any } j \neq n_1, 2n_1, \dots, (d-1)n_1 \quad \text{where } 1 \leq j \leq n.$$

(iii) If $q < n$, then $n_1 p + \alpha_{1,0,1} q > n_1 \alpha_{1,0,1} d$ if and only if $\frac{p}{n-q} > \frac{\alpha_0}{n} = \frac{\alpha_{1,0,1}}{n_1}$.

Moreover, if $f(y, z)$ is a W-poly in z , note that $p > 0$ and $q \leq n-2$.

Proof of Lemma 16.0. It just follows from Hensel's lemma or Theorem 3.2.

Remark 16.0.1. Note by (i) that $\xi = \frac{1}{d} a_{n-n_1}(0, 0)$ and the $a_{n-in_1}(0, 0)$ are nonzero for all $i = 1, \dots, d$.

Theorem 16.1 (By Theorem 3.6 and Theorem 3.7).

Assumptions

(a) Let $V(f) = \{(y, z) : f(y, z) = 0\}$, and $V(F) = \{(y, z) : F(y, z) = 0\}$ be analytic varieties at $(0, 0)$ in \mathbb{C}^2 , each of which is written respectively in the form,

$$(16.1.1) \quad f = (z^{n_1} + \varepsilon y^{k_1})^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} y^\alpha z^\beta \quad \text{with} \quad n_1 \alpha + k_1 \beta > n_1 k_1 d,$$

$$F = y^{\delta_1} z^{\delta_2} f,$$

satisfying the properties (i), (ii), \dots , (vi):

- (i) $\gcd(n_1, k_1) = 1$ with $1 \leq n_1 < k_1$ and d is a positive integer.
- (ii) ε is a unit in $\mathbb{C}\{y, z\}$.
- (iii) The $c_{\alpha, \beta}$ are nonzero complex numbers for some nonnegative integers α and β such that $n_1 \alpha + k_1 \beta > n_1 k_1 d$.
- (iv) δ_1 and δ_2 are nonnegative integers.
- (v) If $n_1 = 1$, assume that δ_2 is a positive integer.
- (vi) If $d \geq 2$ and $n_1 = 1$, then assume in addition that $V(f)$ has an isolated singular point at the origin as a reduced variety.

(b) Let $V(G) = \{(y, z) : G(y, z) = 0\}$ be another analytic variety with isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(16.1.2) \quad \begin{aligned} g &= z^{n_1} + y^{k_1} \quad \text{with } \gcd(n_1, k_1) = 1 \text{ and } 1 \leq n_1 < k_1, \\ G &= z^\gamma g, \end{aligned}$$

satisfying the properties (i) and (ii):

- (i) If $n_1 = 1$, then $\gamma = 1$.
- (ii) If $n_1 \geq 2$, then $\gamma = 0$, and so $G(y, z) = g(y, z)$.

Conclusions Let $\tau_m = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : M^{(m)} \rightarrow \mathbb{C}^2$ be the compositions of a finite number m of successive blow-ups π_i which is needed to get the standard resolution of the singular point of $V(G)$. If $V(g)$ has the singular point at the origin, then as compared with the above τ_m , exactly the same τ_m can be also used for the standard resolution of the singular point of $V(g)$ as far as the blow-ups process is concerned.

(a1) We can use just one coordinate patch of the local coordinates for each blow-up π_i of τ_m with $1 \leq i \leq m$ in the sense of Lemma 2.12.

(a2) Just as above, we can use the same τ_m for the composition of the first finite number m of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of $V(F)$.

(a3) Also, we can use just the common one coordinate patch of the given local coordinates for each blow-up π_i of the above τ_m in (a1), in order to study any of $V^{(i)}(F)$ for all $i = 1, 2, \dots, m$ in the sense of Lemma 2.14.

(b) For simplicity of notations, let (v, u) be the common one of the local coordinates for the m -th blow-up $\pi_m : M^{(m)} \rightarrow M^{(m-1)}$ at $(0, 0)$ which is the quasisingular point of $V^{(m-1)}(G)$ in the sense of Definition 2.6. Being viewed as an analytic mapping, $\tau_m : M^{(m)} \rightarrow \mathbb{C}^2$ can be written in the form

$$(16.1.3) \quad \tau_m(v, u) = (y, z) = (v^{n_1} u^a, v^{k_1} u^b),$$

where

- (b1) a and $b > 0$ are nonnegative integers such that $bn_1 - ak_1 = 1$,
- (b2) $E_m = \{v = 0\}$ is defined by the m -th exceptional curve of the first kind.

(c) By (b), along $v = 0$, $(F \circ \tau_m)_{total}$ can be written in the following form:

$$(16.1.4) \quad \begin{aligned} (F \circ \tau_m)_{total} &= v^{e_m} u^\varepsilon (f \circ \tau_m)_{proper} \quad \text{with} \\ (f \circ \tau_m)_{proper} &= (u + \varepsilon')^d + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} v^{n_1 \alpha + k_1 \beta - n_1 k_1 d} u^{\varepsilon_{\alpha, \beta}}, \\ (G \circ \tau_m)_{total} &= v^{k_1 \gamma + n_1 k_1} u^{b \gamma + a k_1} (u + 1), \end{aligned}$$

where

- (i) $e_m = n_1 \delta_1 + k_1 \delta_2 + n_1 k_1 d$, $\varepsilon = a \delta_1 + b \delta_2 + a k_1 d$ and $\varepsilon_{\alpha, \beta} = a \alpha + b \beta - a k_1 d \geq 0$,
- (ii) by assumption of $V(G)$, if $n_1 = 1$ then $\gamma = 1$, and if $n_1 \geq 2$ then $\gamma = 0$,
- (iii) ε' is a unit in $\mathbb{C}\{u + 1, v\}$.

(d) By (c), we have the following:

- (d1) Whether $n \geq 2$ or not, $G \in \text{the type}[1]$ under τ_m .
- (d2) If $n = 1$ then $f \in \text{the type}[0]$ and $F \in \text{the type}[1]$ under τ_m , but if $n \geq 2$, then $f \in \text{the type}[1]$ and $F \in \text{the type}[1]$ under τ_m in the sense of Definition 2.8.

Observe by the notation that $(f \circ \tau_m)_{proper}$ is the local defining equation for the proper transform $V^{(m)}(f)$ and that $(F \circ \tau_m)_{total}$ is the local defining equation for the total transform of $V(F)$ under τ_m .

§16.2. An algorithm for computing a complete irreducibility criterion of any W-poly in $\mathbb{C}\{y, z\}$ which has the same multiplicity sequence as the standard Puiseux expansion ($y = t^n$ and $z = t^\alpha + t^\beta$) does

In §16.2, the first aim is to find an algorithm for computing a complete irreducibility criterion of any W-poly in $\mathbb{C}\{y, z\}$ which has the same multiplicity sequence as the standard Puiseux expansion ($y = t^n$ and $z = t^\alpha + t^\beta$) does by Theorem 16.5 and Theorem 16.4 with Proposition 16.2 and Proposition 16.3, whose proofs can be represented by §16.3.

In §17, the main aim is to finish a complete irreducibility criterion of any W-poly in $\mathbb{C}\{y, z\}$ having the same multiplicity sequence as the standard Puiseux expansion does, which can be represented by Theorem 16.6 with Proposition 16.7 and Proposition 16.8.

Proposition 16.2. Assumptions Let $f(y, z) = z^n + a_{n-2}y^{\alpha_{n-2}}z^{n-2} + \dots + a_1y^{\alpha_1}z + a_0y^{\alpha_0}$ be an irreducible W -poly in z with multiplicity $n \geq 2$ at $(0, 0) \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Let $d_2 = \gcd(n, \alpha_0) > 1$ with $n = d_2n_1$ and $\alpha_0 = d_2\alpha_{1,0,1}$. Note that $2 \leq n_1 < \alpha_{1,0,1}$.

Conclusions Then, (g_1, f) can be written in the form

$$(16.2.1) \quad \begin{cases} g_1 &= z^{n_1} + \xi_1 y^{\sigma_1} & \text{with } \sigma_1 = \alpha_{1,0,1}, f_{-1} = y \text{ and } f_0 = z, \\ f &= g_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} g_1^i, \end{cases}$$

where, considering f_{-1}, f_0, g_1 as independent complex 3-variables at $0 \in \mathbb{C}^3$,

(i) $n = d_2n_1$ with $d_2 \geq 2$ and $n_1 \geq 2$, and $n = d_1$ if necessary;

(ii) $\sigma_1 = \alpha_{n-n_1} = \alpha_{1,0,1}$ and $\xi_1 = \frac{1}{d_2} a_{n-n_1}(0)$;

(iii) $T_{2,i} = T_{2,i}(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}, f_0\}$ of f in (16.2.1) for $i = 0, 1, \dots, d_2 - 1$;

(iv) $g_1 = g_1(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}\}[f_0]$;

(v) $f = f(f_{-1}, f_0, g_1) \in \mathbb{C}\{f_{-1}, f_0\}[g_1]$ of f in (16.2.1),

satisfying two conditions, denoted by The Necessary and Sufficient Condition[A] for $g_1(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, each of which is represented respectively, as follows:

[1] The Necessary and Sufficient Condition[A] for $g_1(y, z) \in \text{the type}[1]$:

$g_1 \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W -poly of degree n_1 in f_0 with a coefficient of $f_0^{n_1-1}$ zero in $\mathbb{C}\{f_{-1}\}$, and $g_1(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5

To find The Necessary and Sufficient Condition[A] for $g_1(y, z) \in \text{the type}[1]$, it is clear that $g_1(y, z) \in \mathbb{C}\{y\}[z]$ of (16.2.1) itself is an irreducible W -poly of degree n_1 in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$, and $g_1(y, z) \in \text{the type}[1]$.

[2] The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$:

$f \in \mathbb{C}\{f_{-1}, f_0\}[g_1]$ is an irreducible W -poly of degree d_2 in g_1 with a coefficient of $g_1^{d_2-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5

To find the Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, it suffices to show that $f = g_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} g_1^i \in \mathbb{C}\{f_{-1}, f_0\}[g_1]$ of (16.2.1) satisfies two properties (1) and (2): Note that either $\ell = 2$ or $\ell \geq 3$ and that T_{2,d_2-1} may not be zero.

(1) Each $T_{2,i} \in \mathbb{C}\{f_{-1}, f_0\}$ of f in (16.2.1) satisfies the properties (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_2 - 1$.

(1a)(1a-1) For any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\delta_k}$ in $T_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(1a-2) In particular, for any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\delta_k}$ in T_{2,d_2-1} , $\delta_1 > 0$ and $\delta_2 \leq n_1 - 2$.

(1b) Let \mathbb{N}_0 be the set of nonnegative integers and \mathbb{N}_0^2 be its two dimensional copy.

(1b-1) Let $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ be integer-valued functions, each of which is defined respectively, as follows:

$\theta_1(t) = t$ for each $t \in \mathbb{N}_0$, and

$\bar{\theta}_2(t_1, t_2) = t_2\theta_1(\alpha_{1,0,1}) + n_1\theta_1(t_1) = t_2\alpha_{1,0,1} + n_1t_1$ for each $(t_1, t_2) \in \mathbb{N}_0^2$.

(1b-2) For any two nonzero monomials $\prod_{k=1}^2 f_{k-2}^{\beta_k}$ and $\prod_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}$ with i fixed,

$$(16.2.2) \quad \bar{\theta}_2(\beta_k)_{k=1}^2 = \bar{\theta}_2(\gamma_k)_{k=1}^2 \text{ if and only if } \beta_k = \gamma_k \text{ for } k = 1, 2.$$

So, there is a unique nonzero monomial $C_{2,i} \prod_{k=1}^2 f_{k-2}^{\beta_{2,i,k}}$ in $T_{2,i}$

with a constant $C_{2,i}$ such that $\bar{\theta}_2(\beta_{2,i,k})_{k=1}^2 = \min\{\bar{\theta}_2(\gamma_k)_{k=1}^2\}$

for any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}$.

(1c) For each $i = 0, 1, \dots, d_2 - 1$,

$$(16.2.3) \quad \bar{\theta}_2(\beta_{2,i,1}, \beta_{2,i,2}) > (d_2 - i)n_1\theta_1(\alpha_{1,0,1}).$$

(1d) For all $i = 0, 1, \dots, d_2 - 1$, the following hold:

$$(16.2.4) \quad \gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) \geq 1 \quad \text{and} \\ \frac{\bar{\theta}_2(\beta_{2,i,1}, \beta_{2,i,2})}{d_2 - i} \geq \frac{\bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})}{d_2}.$$

Then, either $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k})_{k=1}^2) = 1$ or $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k})_{k=1}^2) \leq d_2$.

(1d-1) Suppose $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k})_{k=1}^2) = 1$. Then f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type [2] in the sense of Definition 2.5 if and only if the inequality in (16.2.4) holds and g_1 is irreducible in ${}_2\mathcal{O}_0$ with $g_1 \in$ the type [1] in the sense of Definition 2.5.

(1d-2) Suppose $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k})_{k=1}^2) \leq d_2$ in (16.2.4). To find an irreducible criterion, it remains to study two subcases respectively:

Subcase(i) of (1d-2) Let $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) = d_2$ in (16.2.4). Then, f is either irreducible or not in ${}_2\mathcal{O}_0$. If f is irreducible in ${}_2\mathcal{O}_0$ then $f \in$ the type $[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5.

Subcase(ii) of (1d-2) Let $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) < d_2$ in (16.2.4). Then, f is either irreducible or not in ${}_2\mathcal{O}_0$. If f is irreducible in ${}_2\mathcal{O}_0$ then $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_2n_1$ at $0 \in \mathbb{C}^2$. Also, either $f \in$ the type [2] or $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2b) $f = f(f_{-1}, f_0, g_1) \in \mathbb{C}\{f_{-1}, f_0\}[g_1]$ of (16.2.1) is an irreducible W -poly of degree d_2 in g_1 with coefficients in $\mathbb{C}\{f_{-1}, f_0\}$ and with multiplicity d_2 at $0 \in \mathbb{C}^3$ where f_{-1}, f_0, g_1 are viewed as independent complex three variables at the origin in \mathbb{C}^3 .

Remark 16.2.1.1 (1) It is clear that $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is a W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_2n_1$ at $0 \in \mathbb{C}^2$.

(2) It is clear that g_1 of (16.2.1) satisfies The Necessary and Sufficient Condition[A] for $g_1(y, z) \in$ the type[1].

(3) Whether $T_{2,d_2-1} = 0$ or not, then it is said by Proposition 16.2 that f of (16.2.1) satisfies The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$.

(4) If $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k})_{k=1}^2) = 1$, it is said by (1d) of Proposition 16.2 that f satisfies The Necessary and Sufficient Condition[A] for $f(y, z) \in$ the type[2].

(5) Let $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) < d_2$ in (16.2.4). If f is irreducible in ${}_2\mathcal{O}_0$ with T_{2,d_2-1} either zero or not, it is proved by this proposition that $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5. But, if f is irreducible in ${}_2\mathcal{O}_0$ and $T_{2,d_2-1} \neq 0$, $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2}))$ may be equal to d_2 .

(6) Let $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) = d_2$ in Subcase(i) of (1d-2). If f is irreducible in ${}_2\mathcal{O}_0$ then $T_{2,d_2-1} \neq 0$, otherwise if $T_{2,d_2-1} = 0$, $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,1}, \beta_{2,0,2})) < d_2$ by Hensel's lemma(Theorem 3.2).

(7) Let $f = g_1^5 + \binom{5}{1}y^6zg_1^4 + \binom{5}{2}y^{12}z^2g_1^3 + \binom{5}{3}y^{18}z^3g_1^2 + \{\binom{5}{4}y^{24}z^4 + y^{30}\}g_1 + y^{35}z^2 - y^{37}$ where $g_1 = z^5 + y^7$. Note that $g_1 + y^6z = 0$ and $g_1 = 0$ have the same multiplicity sequence. Then, it is easy to compute that the above pair (g_1, f) satisfies a pair of (16.2.1), which belongs to Subcase(i) of (1d-2) of this proposition with $f(y, z) \in$ the type[2].

Remark 16.2.1.2. Assuming that f is irreducible in ${}_2\mathcal{O}_0$, let $h_2 = h_1 + \frac{1}{d_2}T_{2,d_2-1}$ where $h_1 = g_1$ and T_{2,d_2-1} were defined by (16.2.1). Then, h_2 is an irreducible W -poly of degree n_1 in z , and also two curves defined by $h_2 = 0$ and $g_1 = 0$ have the same multiplicity sequence by Theorem 12.1 because for any nonzero monomial $y^{\delta_1}z^{\delta_2} \in T_{2,d_2-1}$, $\theta_2(\delta_1, \delta_2) > n_1\alpha_{1,0,1}$ by (1c) of Proposition 16.2. So, it is said by Definition of 2.4 that $V(h_2)$ and $V(h_1)$ have the same multiplicity sequence under two standard resolutions, denoted by $h_2 \stackrel{\text{multiseq}}{\sim} h_1$ under two standard resolutions.

Definition 16.2.2. If $h(y, z) \in \mathbb{C}\{y\}[z]$ and $f(y, z) \in \mathbb{C}\{y\}[z]$ are W -polys in z with coefficients in $\mathbb{C}\{y\}$, then (h, f) is called a pair of W -polys in z . Also, assuming that $\ell(y, z) \in \mathbb{C}\{y\}[z]$ and $g(y, z) \in \mathbb{C}\{y\}[z]$ are W -polys in z with coefficients in $\mathbb{C}\{y\}$, it is said for brevity of notation that $(h, f) = (\ell, g)$ if and only if $h(y, z) = \ell(y, z)$ and $f(y, z) = g(y, z)$, for the proof of Proposition 16.3 and its applications.

Proposition 16.3. Assumptions *Let $f(y, z) = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be an irreducible W -poly in z with multiplicity $n \geq 2$ at $(0, 0) \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Note that a_{n-1} is identically zero. Assume that $d_2 = \gcd(n, \alpha_0) > 1$. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $\gcd(n_1, \alpha_{1,0,1}) = 1$. Note that $2 \leq n_1 < \alpha_{1,0,1}$. Without any need of proof, we may assume by Proposition 16.2 and Definition 16.2.2 that there exists a pair of W -polys in z with coefficients in $\mathbb{C}\{y\}$, denoted by (g_1, f) , which can be written in the following form:*

$$(16.3.0) \quad \begin{cases} g_1 &= z^{n_1} + \xi_1 y^{\sigma_1} \text{ with } \xi_1 = \frac{1}{d_2} a_{n-n_1}(0) \neq 0, \\ f &= g_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} g_1^i, \end{cases}$$

$\sigma_1 = \alpha_{n-n_1} = \alpha_{1,0,1}$, satisfying two conditions, denoted by *The Necessary and Sufficient Condition[A]* for $g_1(y, z) \in \text{the type}[1]$ and *The Necessary Condition[B]* for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, as we have seen in Proposition 16.2.

Conclusions *The main aim is to construct a unique pair (f_1, f) such that (f_1, f) can be written in the form*

$$(16.3.1) \quad \begin{cases} f_1 &= z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i} z^i \text{ with } f_{-1} = y \text{ and } f_0 = z, \\ f &= f_1^{d_2} + \sum_{i=0}^{d_2-2} S_{2,i} f_1^i, \end{cases}$$

where f_{-1}, f_0, f_1 are considered as independent complex three variables at the origin in \mathbb{C}^3 if necessary, satisfying the following properties:

(i) *The first problem is how to construct $f_1 = f_1(y, z)$ satisfying the condition in $\widehat{[1]}$ such that $f_1(y, z) \stackrel{\text{multiseq}}{\sim} g_1(y, z)$ under the standard resolutions.*

$\widehat{[1]}$ The Necessary and Sufficient Condition[A] for $f_1(y, z) \in \text{the type}[1]$:
 $f_1 \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W -poly of degree n_1 in f_0 with a coefficient of $f_0^{n_1-1}$ zero in $\mathbb{C}\{f_{-1}\}$, and $f_1(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5

(ii) *The second problem is to prove that $f = f(f_{-1}, f_0, f_1)$ satisfies the condition in $\widehat{[2]}$ which is defined by the same kind of property as $f(f_{-1}, f_0, g_1)$ have done in The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$.*

$\widehat{[2]}$ The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[f_1]$ is an irreducible W -poly of degree d_2 in f_1 with a coefficient of $f_1^{d_2-1}$ zero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5

In order to find the construction of a pair (f_1, f) in (16.3.1), first of all, it suffices to consider the following two cases, depending on the fact that either T_{2,d_2-1} of (16.3.0) is zero or not. For brevity of notations, let $h_1 = g_1$ and $T_{2,i}^{(1)} = T_{2,i}$.

Case(1) *Let $T_{2,d_2-1}^{(1)} = 0$. Put $f_1 = g_1$, $R_{1,0} = \xi_1 y^{\sigma_1}$, and $S_{2,i} = T_{2,i}^{(1)}$ for $0 \leq i \leq d_2 - 2$. Then, it was already proved by Proposition 16.2 that (f_1, f) of the main aim and (g_1, f) are the same pairs in the sense of Definition 16.2.2.*

Case(2) *Let $T_{2,d_2-1}^{(1)} \neq 0$. Then, there is a sequence of pairs of W -polys in z , $\{(h_p, f) : p = 1, 2, \dots\}$ with $h_1 = g_1$ such that $(h_\nu, f) \neq (h_{\nu+1}, f) = (h_{\nu+2}, f) = \dots$ for some integer $\nu \leq \frac{n_1+1}{2}$, each pair of which can be written in the form*

$$(16.3.2) \quad \begin{cases} h_1 &= g_1 = z^{n_1} + \xi_1 y^{\alpha_{1,0,1}}, \\ f &= h_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(1)} h_1^i, \end{cases}$$

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and for each $p = 2, 3, \dots$

$$(16.3.3) \quad \begin{cases} h_p &= h_{p-1} + \frac{1}{d_2} T_{2,d_2-1}^{(p-1)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p)} z^i, \\ f &= h_p^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_p^i, \end{cases}$$

with $T_{2,d_2-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{2,d_2-1}^{(\nu+1)} = T_{2,d_2-1}^{(\nu+2)} = \dots = 0$,

where, considering f_{-1}, f_0, h_p as independent complex 3-variables at $0 \in \mathbb{C}^2$,

(i) $n = d_2 n_1$ with $d_2 \geq 2$ and $n_1 \geq 2$;

(ii) $R_{1,i}^{(p)} = R_{1,i}^{(p)}(f_{-1}) \in \mathbb{C}\{f_{-1}\}$ for $p \geq 1$ and $0 \leq i \leq n_1 - 2$;

(iii) $T_{2,i}^{(p)} = T_{2,i}^{(p)}(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}, f_0\}$ for $p \geq 1$ and $0 \leq i \leq d_2 - 1$;

(iv) $h_p = h_p(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}\}[f_0]$ for $p \geq 1$;

(v) $f = f(f_{-1}, f_0, h_p) \in \mathbb{C}\{f_{-1}, f_0\}[h_p]$ for each fixed (h_p, f) of either (16.3.2) or (16.3.3), satisfying two conditions, denoted by The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, each of which is represented respectively, as follows:

In more detail, for any fixed (h_p, f) of (16.3.3), h_p of (h_p, f) satisfies The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$ and $f(y, z)$ of (h_p, f) satisfies The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$. In particular, if p_1 is a given integer with $p_1 \geq \frac{n_1+1}{2}$, define $(f_1, f) = (h_{p_1}, f)$ by Definition 16.2.2. Then, f_1 of (f_1, f) of (16.3.1)

satisfies The Necessary and Sufficient Condition[A] for $f_1(y, z) \in \text{the type}[1]$ in $\widehat{[1]}$, and f of (f_1, f) of (16.3.1) satisfies The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ in $\widehat{[2]}$ by the same way as (h_p, f) of (16.3.3) does up to the change of notations: Recall that $f_{-1} = y$ and $f_0 = z$.

[1] The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$:

$h_p \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W-poly of degree n_1 in f_0 with a coefficient of $f_0^{n_1-1}$ zero in $\mathbb{C}\{f_{-1}\}$, and $h_p(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5

To find The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$, it suffices to show that $h_p = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p)} z^i$ of (16.3.3) satisfies two properties (1) and (2):

(1) Let p be fixed with $p \geq 1$. Each $R_{1,i}^{(p)} \neq 0$ satisfies the properties (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, n_1 - 2$. Also, for each $p \geq 1$, $h_p \stackrel{\text{multiseq}}{\sim} h_1$ and $h_p \in \mathbb{C}\{y\}[z]$ is an irreducible W-poly in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

(1a) We write $R_i^{(p)} = b_i^{(p)} y^{\alpha_{1,i,1}^{(p)}}$ with a unit $b_i^{(p)}$ in $\mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}^{(p)}$, if exists. For all $p \geq 1$, $\alpha_{1,0,1}^{(p)} = \alpha_{1,0,1}$ and $\xi_1 = b_0^{(p)}(0)$ where ξ_1 was found to be $\frac{1}{d_2} a_{n-n_1}(0)$ as in $(h_1, f) = (g_1, f)$ of (16.3.2).

(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers, by the same way as in Proposition 16.2.

(1c) For all $i = 0, 1, \dots, n_{j+1} - 2$,

$$(16.3.4) \quad \theta_1(\alpha_{1,i,1}^{(p)}) > n_1 - i.$$

(1d) For all $i = 1, \dots, n_{j+1} - 2$,

$$(16.3.5) \quad \begin{aligned} \gcd(n_1, \theta_1(\alpha_{1,0,1}^{(p)})) &= 1 \quad \text{with} \quad \sigma_1 = \alpha_{1,0,1}^{(p)}, \\ \frac{\theta_1(\alpha_{1,i,1}^{(p)})}{n_1 - i} &> \frac{\theta_1(\alpha_{1,0,1}^{(p)})}{n_1} \quad \text{or} \quad n_1 \alpha_{1,i,1}^{(p)} + \alpha_{1,0,1}^{(p)} i > n_1 \alpha_{1,0,1}^{(p)}. \end{aligned}$$

(2)(2a) For each $p \geq 1$, $h_p = h_p(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W-poly in z with coefficients in $\mathbb{C}\{y\}$ and with $h_p \stackrel{\text{multiseq}}{\sim} h_1 = g_1$, and $h_p \in \text{the type}[1]$ in the sense of Definition 2.5.

(2b) $h_p = h_p(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}\}[f_0]$ of (16.3.3) is an irreducible W -poly in f_0 with coefficients in $\mathbb{C}\{f_{-1}\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

[2] The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[h_p]$ is an irreducible W -poly of degree d_2 in h_p with a coefficient of $h_p^{d_2-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5

To find the Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, it suffices to show that for each $p \geq 1$, $f = h_p^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p)} h_p^i$, of (16.3.3) satisfies two properties (1) and (2): Note that either $\ell = 2$ or $\ell > 2$.

(1) Each $T_{2,i}^{(p)}(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}, f_0\}$ of f in (16.3.3) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_2 - 1$.

(1a) For any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p)}$,

$$(16.3.6) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 < n_1.$$

In particular, if $i = d_2 - 1$ for $T_{2,i}^{(p)}$ then $\gamma_1 > 0$ and $\gamma_2 \leq n_1 - 2$.

(1b) Define $\bar{\theta}_2(t_k)_{k=1}^2 = t_2\sigma_1 + n_1 t_1$ for any $(t_k)_{k=1}^2 \in N_0^2$ by the same way as we have seen in Proposition 16.2, (1b) of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$.

For any two nonzero monomials $\Pi_{k=1}^2 f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p)}$,

$$(16.3.7) \quad \bar{\theta}_2(\beta_k)_{k=1}^2 = \bar{\theta}_2(\gamma_k)_{k=1}^2 \text{ if and only if } \beta_k = \gamma_k \text{ for } k = 1, 2.$$

So, there is a unique nonzero monomial $C_{2,i}^{(p)} \Pi_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(p)}}$ in $T_{2,i}^{(p)}$

with a constant $C_{2,i}^{(p)}$ such that $\bar{\theta}_2(\beta_{2,i,k}^{(p)})_{k=1}^2 = \min\{\bar{\theta}_2(\gamma_k)_{k=1}^2\}$

for any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p)}$.

(1c) For all $i = 0, 1, \dots, d_2 - 1$,

$$(16.3.8) \quad \bar{\theta}_2(\beta_{2,i,k}^{(p)})_{k=1}^2 > (d_2 - i)n_1\theta_1(\sigma_1).$$

(1d) For all $i = 0, 1, \dots, d_2 - 1$,

$$(16.3.9) \quad \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) \geq 1, \\ \frac{\bar{\theta}_2(\beta_{2,i,k}^{(p)})_{k=1}^2}{d_2 - i} \geq \frac{\bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2}{d_2}.$$

Then, either $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) = 1$ or $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) \leq d_2$.

(1d-1) Let $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if the inequality in (16.3.9) holds. In this case, $f \in \text{the type } [2]$ in the sense of Definition 2.5, but note that $T_{2,d_2-1}^{(p)}$ may not be zero where

$$(16.3.10) \quad h_p = h_{p-1} + \frac{1}{d_2} T_{2,d_2-1}^{(p-1)} \quad \text{and} \quad f = h_p^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p)} h_p^i.$$

(1d-2) Let $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) \leq d_2$. There is a positive integer ν with $\nu \leq \frac{n_1+1}{2}$ such that $T_{2,d_2-1}^{(\nu+1)} = T_{2,d_2-1}^{(\nu+2)} = \dots = 0$ and $T_{2,d_2-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$. In this case, $f \in \text{the type } [\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5 and note that

$$(16.3.11) \quad 1 \leq \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(\nu+1)})_{k=1}^2) < d_2.$$

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_2 n_1$ at $0 \in \mathbb{C}^2$. Also, either $f \in$ the type $[2]$ or $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2b) $f = f(f_{-1}, f_0, h_{\nu+1}) \in \mathbb{C}\{f_{-1}, f_0\}[h_{\nu+1}]$ of (16.3.3) is an irreducible W -poly of degree d_2 in $h_{\nu+1}$ with coefficients in $\mathbb{C}\{f_{-1}, f_0\}$ and with multiplicity d_2 at $0 \in \mathbb{C}^3$ where $y, z, h_{\nu+1}$ are viewed as independent complex three variables at the origin in \mathbb{C}^3 .

Remark 16.3.0. (a) As a corollary of Proposition 16.3, following the same properties and notations in **Case(2)** of a conclusion of Proposition 16.3, let $f_1 = h_r$ for any $r \geq \nu + 1$ where $\nu + 1$ is the smallest positive integer such that $T_{2, d_2-1}^{(\nu+1)}$ is equal to zero.

For notation, if $p = \nu + 1$, it is clear that $f = f(f_{-1}, f_0, h_p) \in \mathbb{C}\{f_{-1}, f_0\}[h_p]$ satisfies the following up to the change of notations:

(i) h_p of (h_p, f) in (16.3.3) satisfies the same kind of The Necessary and Sufficient Condition[A] for $h_p(y, z) \in$ the type[1] as f_1 of (f_1, f) in (16.3.1) does in $\widehat{[1]}$ up to the change of notations.

(ii) f of (h_p, f) in (16.3.3) satisfies the same kind of The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$ as f of (f_1, f) in (16.3.1) does in $\widehat{[2]}$ up to the change of notations: Recall that $f_{-1} = y$ and $f_0 = z$ up to the change of notations.

(b) Let (h_p, f) of (16.3.3) be defined as above, as we have seen in **Case(2)**. In addition, assuming that $p = \nu + 1$ and $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p)})_{k=1}^2) = 1$, then the statement of $[2]$ in (16.3.3)

can be replaced by the statement of $\widehat{[2]}$ in (16.3.1), as follows:

$[2]$ The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[h_p]$ is an irreducible W -poly of degree d_2 in h_p with a coefficient of $h_p^{d_2-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5

$\widehat{[2]}$ The Necessary and Sufficient Condition for $f(y, z) \in$ the type $[2]$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[h_p]$ is an irreducible W -poly of degree d_2 in h_p with a coefficient of $h_p^{d_2-1}$ zero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in$ the type $[2]$ in the sense of Definition 2.5

Theorem 16.4(A complete algorithm for finding an irreducibility criterion of arbitrary W -polys in $\mathbb{C}\{Y, Z\}$ which have the same multiplicity sequences as the standard Puiseux expansion($y = t^n$ and $z = t^\alpha + t^\beta$) does).

Assumptions Let $f(y, z) = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be an irreducible W -poly in z with multiplicity $n \geq 2$ at $0 \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists, and the α_i are positive integers. Note that a_{n-1} is identically zero. Assume that $d_2 = \gcd(n, \alpha_0) > 1$. Write $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ with $\gcd(n_1, \alpha_{1,0,1}) = 1$. Note that $2 \leq n_1 < \alpha_{1,0,1}$. Without any need of proof, we may assume by Proposition 16.2 and Definition 16.2.2 that (g_1, f) can be written in the form

$$(16.4.1) \quad \begin{cases} g_1 &= z^{n_1} + \xi_1 y^{\alpha_{1,0,1}} \text{ with } \xi_1 = \frac{1}{d_2} a_{n-n_1}(0), \\ f &= g_1^{d_2} + \sum_{i=1}^{d_2-1} T_{2,i} g_1^i, \end{cases}$$

satisfying the same properties and notations as in the conclusion of Proposition 16.2.

Conclusions Then, (f_1, f) can be uniquely written in the form

$$(16.4.2) \quad \begin{cases} f_1 &= z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i} z^i \text{ with } f_{-1} = y \text{ and } f_0 = z, \\ f &= f_1^{d_2} + \sum_{i=0}^{d_2-2} S_{2,i} f_1^i, \end{cases}$$

where, considering f_{-1}, f_0, f_1 as independent complex 3-variables at the origin in \mathbb{C}^3 ,

- (i) $n = d_2 n_1$ with $d_2 \geq 2$ and $n_1 \geq 2$, and $n = d_1$ if necessary;
- (ii) $R_{1,i} = R_{1,i}(y) \in \mathbb{C}\{y\}$ for each $i = 0, 1, \dots, n_1 - 2$, and
- (iii) $S_{2,i} = S_{2,i}(y, z) \in \mathbb{C}\{y\}[z]$ for each $i = 0, 1, \dots, d_2 - 2$, and

- (iv) $f_1 = f_1(y, z) \in \mathbb{C}\{y\}[z]$;
(v) $f = f(y, z, f_1) \in \mathbb{C}\{y\}[z, f_1] \subseteq \mathbb{C}\{y, z\}[f_1]$ in (16.4.2),

satisfying two algorithms, an algorithm for finding an irreducibility criterion for $f_1(y, z) \in$ the type[1] in the sense of Definition 2.5 and an algorithm for finding an irreducibility criterion for $f(y, z) \in$ the type[2] in the sense of Definition 2.5, each of which is represented as follows:
Write $f_{-1} = y$ and $f_0 = z$, if necessary.

[1] An algorithm for finding an irreducibility criterion for $f_1(y, z) \in$ the type[1]:
 $f_1 \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W-poly of degree n_1 in f_0 with a coefficient
of $f_0^{n_1-1}$ zero, and $f_1(y, z) \in$ the type[1] in the sense of Definition 2.5

Note that $f \in$ the type[ℓ] with $\ell \geq 2$.

Let $d_2 = \gcd(n, \alpha_0)$ with $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ for some integers n_1 and $\alpha_{1,0,1}$.

Then, $f_1 = f_0^{n_1} + \sum_{i=0}^{n_2} R_{1,i} f_0^i \in \mathbb{C}\{y\}[f_0]$ of (16.4.2) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying the following (1) and (2):

- (1) Each $R_{1,i} \in \mathbb{C}\{y\}$ in f_1 of (16.4.2) satisfies (1a), (1b), (1c) and (1d) for $i=0, 1, \dots, n_1-2$.
(1a) For $0 \leq i \leq n_1 - 2$, each $R_{1,i} = b_i y^{\alpha_{1,i,1}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.
(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers.
(1c) $\theta_1(\alpha_{1,i,1}) > (n_1 - i)$ for all $i = 0, 1, \dots, n_1 - 2$.
(1d) For all $i = 0, 1, \dots, n_1 - 2$,

$$(16.4.3) \quad \gcd(n_1, \alpha_{1,0,1}) = 1 \quad \text{and} \quad \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} > \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}.$$

(2)(2a) $f_1 = f_1(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W-poly in z with coefficients in $\mathbb{C}\{y\}$ and with $f_1 \stackrel{\text{multiseq}}{\sim} h_1 = g_1$, and $f_1 \in$ the type [1] in the sense of Definition 2.5.

(2b) $f_1 = f_0^{n_1} + \sum_{i=0}^{n_2} R_{1,i} f_0^i \in \mathbb{C}\{f_{-1}\}[f_0]$ of (16.4.2) is an irreducible W-poly in f_0 with coefficients in $\mathbb{C}\{f_{-1}\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

[2] An algorithm for finding an irreducibility criterion for $f(y, z) \in$ the type[2]:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[f_1]$ is an irreducible W-poly of degree d_2 in f_1 with a coefficient
of $f_1^{d_2-1}$ zero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f = f(y, z) \in$ the type[ℓ] with $\ell \geq 2$
in the sense of Definition 2.5

Note that $2 \leq \ell$ where j was already given by (16.4.1) and that $n = \prod_{i=1}^r q_i$.

Then, $f = f_1^{d_2} + \sum_{i=0}^{d_2-2} S_{2,i} f_1^i$ of (16.4.2) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying the following (1) and (2):

- (1) Each $S_{2,i} \in \mathbb{C}\{y, z\}$ of f in (16.4.2) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_2 - 2$.
(1a) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $S_{2,i}$,

$$(16.4.4) \quad \delta_1 > 0 \quad \text{and} \quad \delta_2 < n_1.$$

(1b) Define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by $\theta_2(t_k)_{k=1}^2 = t_2 \theta_1(\alpha_{1,0,1}) + n_1 \theta_1(t_1) = t_2 \alpha_{1,0,1} + n_1 t_1$ for each $(t_1, t_2) \in \mathbb{N}_0^2$, by the same way as in Proposition 16.4.2.

Then, for any two nonzero monomials $\Pi_{k=1}^{j+1} f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $S_{2,i}$ with i fixed,

$$(16.4.5) \quad \theta_2(\beta_k)_{k=1}^2 = \theta_2(\delta_k)_{k=1}^2 \text{ if and only if } \beta_k = \delta_k \text{ for } k = 1, 2.$$

So, there exists a unique nonzero monomial $B_{2,i} \Pi_{k=1}^2 f_{k-2}^{\beta_{2,i,k}}$ in $S_{2,i}$

with a nonzero constant $B_{2,i}$ such that $\theta_2(\beta_{2,i,k})_{k=1}^2 = \min\{\theta_2(\delta_k)_{k=1}^2\}$

for any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\delta_k}$ in $S_{2,i}$ with i fixed.

(1c) For all $i = 0, 1, \dots, d_2 - 2$,

$$(16.4.6-1) \quad \theta_2(\beta_{2,i,k})_{k=1}^2 > (d_2 - i)n_2\theta_2(\beta_{2,0,k})_{k=1}^2 \quad \text{for all } i = 0, 1, \dots, d_2 - 2.$$

(1d) For all $i = 0, 1, \dots, d_2 - 2$,

$$(16.4.6-2) \quad \begin{aligned} \gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) &\geq 1 \quad \text{and} \\ \frac{\theta_2(\beta_{2,i,k})_{k=1}^2}{d_2 - i} &\geq \frac{\theta_2(\beta_{2,0,k})_{k=1}^2}{d_2}. \end{aligned}$$

Then, either $\gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) = 1$ or $1 < \gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) < d_2$.

(1d-1) Let $\gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if the inequality in (16.4.6.2) holds and $f \in$ the type [2] in the sense of Definition 2.5.

(1d-2) Let $1 < \gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) < d_2$. If f is irreducible in ${}_2\mathcal{O}_0$, then $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2)(2a) By (1d) we can find a complete algorithm for finding an irreducibility criterion of Weierstrass polynomials in $\mathbb{C}\{y, z\}$ which have the same multiplicity sequences as the standard Puiseux expansion ($y = t^n$ and $z = t^\alpha + t^\beta$) does.

(2b) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n in z with a coefficient of z^{n-1} zero in $\mathbb{C}\{y\}$ and with multiplicity $n = d_2 n_1$ at $0 \in \mathbb{C}^2$. Also, either $f \in$ the type [2] or $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2c) $f = f(y, z, f_1) \in \mathbb{C}\{y, z\}[f_1]$ of (16.4.2) is an irreducible W -poly of degree d_2 in f_1 with a coefficient of $f_1^{d_2-1}$ zero in $\mathbb{C}\{y, z\}$ and with multiplicity d_2 at $0 \in \mathbb{C}^3$.

§16.3. The proofs of Theorem 16.4 with Proposition 16.2 and Proposition 16.3

Proof of Proposition 16.2. We prove by (16.2.1) that g_1 of (g_1, f) satisfies The Necessary and Sufficient Condition[A] for $g_1(y, z) \in$ the type[1] and that f of (g_1, f) satisfies The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$, respectively.

The proof of The Necessary and Sufficient Condition[A] for $g_1(y, z) \in$ the type[1]: It is clear.

The proof of The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$: First we show that (1a), (1b), (1c) and (1d) of (1) are true, and next that (2a) and (2b) of (2) are true.

(1) We show that (1a), (1b), (1c) and (1d) are true, respectively.

(1a) Apply the WDT with a divisor $g_1 = z^{n_1} + \xi_1 y^{\alpha_{1,0,1}}$ to f where $\xi_1 = \frac{1}{d_2} a_{n-n_1}(0)$.

(1a-1) By (1) of Theorem 15.2, f may be written uniquely in the form

$$(16.2.5) \quad f = g_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i} g_1^i,$$

where each $T_{2,i} = T_{2,i}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree $< n_1$ such that $T_{2,i}(0, z) = 0$. Thus, the proof of (1a-1) is done.

(1a-2) Assume the contrary. Then, we get that $\delta_2 = n_1 - 1$ for a nonzero monomial $y^{\delta_1} z^{\delta_2} \in T_{2,d_2-1}$, and so it would be a nonzero monomial $y^{\delta_1} z^{\delta_2} (z^{n_1})^{d_2-1} \in T_{2,d_2-1} g_1^{d_2-1}$ as in $\mathbb{C}\{y\}[z]$. But, note that $y^{\delta_1} z^{\delta_2} (z^{n_1})^{d_2-1} = y^{\delta_1} z^{n-1}$ because $\delta_2 + n_1(d_2 - 1) = n_1 - 1 + n_1(d_2 - 1) = n - 1$. Now, we claim that there is a nonzero monomial $cy^{\delta_1} z^{n-1} \in f(y, z)$ for some nonzero number c . If the claim is proved, then it would be a contradiction to the assumption that a_{n-1} is zero, and so there is nothing to prove for (1a-2).

For the proof of the claim, first it is needed to compute any nonzero monomial $y^{\gamma_1} z^{\gamma_2} \in T_i g_1^i$ for $1 \leq i \leq d_2 - 2$, and also any nonzero monomial $y^{\gamma'_1} z^{\gamma'_2} \in g_1^{d_2}$, as in $\mathbb{C}\{y\}[z]$.

For any nonzero monomial $y^{\gamma_1} z^{\gamma_2} \in T_i g_1^i$ with $1 \leq i \leq d_2 - 2$,

$$(16.2.6) \quad \gamma_2 \leq n_1 - 1 + n_1 i = n_1(i + 1) - 1 \leq n_1(d_2 - 1) - 1 < n - n_1 \leq n - 2.$$

If a nonzero monomial $y^{\gamma'_1} z^{\gamma'_2}$ with $\gamma'_1 > 0$ is in $g_1^{d_2} = (z^{n_1} + \xi y^{\alpha_{1,0,1}})^{d_2}$, it is clear that $\gamma'_2 \leq n - 2$ because $2 \leq n_1$. Thus, the claim is proved, and so the proof of (1a-2) is done.

(1b) It suffices to show that if $\theta_2(\gamma_1, \gamma_2) = \theta_2(\delta_1, \delta_2)$ where $0 < \gamma_1$, $0 < \delta_1$, $0 \leq \gamma_2 < n_1$ and $0 \leq \delta_2 < n_1$, then $\gamma_i = \delta_i$ for $i = 1, 2$. By definition, $n_1 \gamma_1 + \alpha_{1,0,1} \gamma_2 = n_1 \delta_1 + \alpha_{1,0,1} \delta_2$, i.e., $n_1(\delta_1 - \gamma_1) = \alpha_{1,0,1}(\gamma_2 - \delta_2)$. Since $\gcd(n_1, \alpha_{1,0,1}) = 1$ and $|\gamma_2 - \delta_2| < n_1$, it is clear that $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$. Thus, the proof of (1b) is done.

(1c) By (1b), let $C_{2,i} y^{\gamma_{2,i,1}} z^{\gamma_{2,i,2}}$ be a unique monomial in $T_{2,i}$ with a constant $C_{2,i}$ such that $\theta_2(\gamma_{2,i,1}, \gamma_{2,i,2}) = \min\{\theta_2(\delta_1, \delta_2)\}$ for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $T_{2,i}$ with i fixed.

To prove an inequality in (16.2.3), it suffices to prove that the following are true:

(1c-1) For each $i = 0, 1, \dots, d_2 - 1$, $y^{\gamma_{2,i,1}} z^{\gamma_{2,i,2}} z^{in_1}$ belongs to $T_{2,i} g_1^i$.

(1c-2) $\theta_2(\gamma_{2,i,1}, \gamma_{2,i,2}) + in_1 \alpha_{1,0,1} > d_2 n_1 \alpha_{1,0,1}$ for all $i = 0, 1, \dots, d_2 - 1$.

To prove (1c-1), note that for any nonzero monomial $y^\gamma z^\delta \in T_i g_1^i$, $n_1 \gamma + \alpha_{1,0,1} \delta \geq n_1 \gamma_{2,i,1} + \alpha_{1,0,1} \gamma_{2,i,2} + in_1 \alpha_{1,0,1} = \theta_2(\gamma_{2,i,1}, \gamma_{2,i,2}) + in_1 \alpha_{1,0,1}$, because $y^\gamma z^\delta = y^{\delta_1} z^{\delta_2} y^{q\alpha_{1,0,1}} z^{pn_1}$ for a nonzero monomial $y^{\delta_1} z^{\delta_2} \in T_{2,i}$ where $g_1 = z^{n_1} + \xi y^{\alpha_{1,0,1}}$ with $p+q = i$. Since $T_i g_1^i \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree $(in_1 + \alpha)$ where $0 \leq \alpha < n_1$, it is clear that $y^{\gamma_{2,i,1}} z^{\gamma_{2,i,2}} z^{in_1}$ belongs to $T_i g_1^i$. Thus, the proof of (1c-1) can be done. Also, it is trivial by (1c-1) that (1c-2) is true because $y^{\gamma_{2,i,1}} z^{\gamma_{2,i,2}} z^{in_1} \in \sum_{i=0}^{d_2-1} T_{2,i} g_1^i$ by Lemma 16.0 and Theorem 16.1. Thus, we proved an inequality in (16.2.3). Then, the proof of (1c) is done.

(1d) In preparation for the proof of an inequality in (16.2.4), whenever a nonzero monomial $y^\gamma z^\delta \in T_{2,i} g_1^i$ is chosen arbitrary for each $i = 0, 1, \dots, d_2 - 1$, then it was already proved by (16.2.3) that $n_1 \gamma + \alpha_{1,0,1} \delta > n_1 \alpha_{1,0,1} d_2$.

In order to apply Theorem 16.1 to the proof of an inequality in (16.2.4), recall by (16.2.5) that $V(g_1) = \{(y, z) : g_1(y, z) = 0\}$ is an analytic variety with isolated singularity at the origin in \mathbb{C}^2 defined by the form

$$(16.2.7) \quad g_1 = z^{n_1} + \xi_1 y^{\alpha_{1,0,1}} \quad \text{with } \gcd(n_1, k_1) = 1 \text{ and } 2 \leq n_1 < k_1 = \alpha_{1,0,1}.$$

Let τ_{λ_1} be the composition of a finite number λ_1 of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(g_1)$. For each $t = 1, 2, \dots, \lambda_1$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\{\pi_i : M^{(i)} \rightarrow M^{(i-1)} \text{ is a blow-up of } M^{(i-1)} \text{ at some point for } 1 \leq i \leq t\}$ with $M^{(0)} = \mathbb{C}^2$. For brevity of notation, let $V^{(t)}(g_1)$ be the proper transform under τ_t for $1 \leq t \leq \lambda_1$.

Let $E^{(\lambda_1)} = \tau_{\lambda_1}^{-1}(0, 0)$, and let $E^{(\lambda_1)} = \cup E_i$, $1 \leq i \leq \lambda_1$, be the decomposition of $E^{(\lambda_1)}$ into irreducible components where each E_i is called an exceptional curve of the first kind. By Theorem 16.1, we can use the following consequences.

Consequence(1). In order to study $V^{(t)}(g_1)$ under τ_t , we can find just one coordinate patch of the local coordinates for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$, where $1 \leq t \leq \lambda_1$ and $M^{(0)} = \mathbb{C}^2$.

Consequence(2). By Consequence(1), we can use the same τ_{λ_1} for the composition of the first finite number λ_1 of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of $V(f)$.

Consequence(3). In order to study each proper transform of both $V(g_1)$ and $V(f)$ under τ_t , without using a nonsingular change of coordinates, we can use the common one coordinate patch of the same local coordinates simultaneously, as it has been already used for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$ in Consequence(1), where $1 \leq t \leq \lambda_1$.

After λ_1 iterations of blow-ups, let $(v_{\lambda_1}, u_{\lambda_1})$ and $(v'_{\lambda_1}, u'_{\lambda_1})$ be the local coordinates for $M^{(\lambda_1)}$ where $\pi_{\lambda_1} : M^{(\lambda_1)} \rightarrow M^{(\lambda_1-1)}$ was defined to be the λ_1 -th blow-up at some point of $M^{(\lambda_1-1)}$ with $u'_{\lambda_1} = 1/u_{\lambda_1}$ and $v'_{\lambda_1} = v_{\lambda_1} u_{\lambda_1}$. Note that $E_{\lambda_1} = \{v_{\lambda_1} = 0\} \cup \{v'_{\lambda_1} = 0\}$. For brevity of notation, write $(v, u) = (v_{\lambda_1}, u_{\lambda_1})$ and $(v', u') = (v'_{\lambda_1}, u'_{\lambda_1})$.

Being viewed as an analytic mapping, $\tau_{\lambda_1} : M^{(\lambda_1)} \rightarrow \mathbb{C}^2$ can be written in the form

$$(16.2.8) \quad \tau_{\lambda_1}(v, u) = (y, z) = (v^{n_1} u^a, v^{k_1} u^b) \quad \text{with } k_1 = \alpha_{1,0,1}.$$

where

- (i) a and $b > 0$ are nonnegative integers such that $bn_1 - ak_1 = 1$,
- (ii) $E_{\lambda_1} = \{v = 0\}$ is defined by the $\lambda_1 - th$ exceptional curve of the first kind.

By Theorem 16.1, along $v = 0$, $(f \circ \tau_{\lambda_1})_{total}$ can be written in the following form:

$$\begin{aligned}
 (16.2.9) \quad & (g_1 \circ \tau_{\lambda_1})_{total} = v^{n_1 k_1} u^{a k_1} (u + \xi_1), \\
 & (f \circ \tau_{\lambda_1})_{total} = v^{e_{\lambda_1}} u^\varepsilon (f \circ \tau_{\lambda_1})_{proper}, \\
 & (f \circ \tau_{\lambda_1})_{proper} = (u + \xi_1)^{d_2} + \sum_{i=0}^{d_2-1} T'_i (u + \xi_1)^i, \quad \text{with} \\
 & T'_i = T'_i(u, v) = b_i v^{M'_i}, \\
 & M'_i = \theta_2(\gamma_{2,i,1}, \gamma_{2,i,2}) + in_1 k_1 - d_2 n_1 k_1 > 0,
 \end{aligned}$$

where

- (i) $e_{\lambda_1} = n_1 k_1 d_2$ and $\varepsilon = a k_1 d_2$,
- (ii) ξ_1 is a nonzero constant and each b_i is a unit in $\mathbb{C}\{u + \xi_1, v\}$.

Since $(f \circ \tau_{\lambda_1})_{proper}$ is irreducible in $\mathbb{C}\{u + \xi, v\}$, by Theorem 3.2 and (16.2.9) we have an inequality for each $i = 1, 2, \dots, d_2 - 1$:

$$(16.2.10) \quad \frac{M'_i}{d_2 - i} \geq \frac{M'_0}{d_2}, \quad \text{that is,} \quad \frac{\theta_2(\gamma_{2,i,1}, \gamma_{2,i,2})}{d_2 - i} \geq \frac{\theta_2(\gamma_{2,0,1}, \gamma_{2,0,2})}{d_2}.$$

Thus, we proved the inequality in (16.2.4).

(1d-1) Suppose $\gcd(d_2, \theta_2(\gamma_{2,0,1}, \gamma_{2,0,2})) = 1$. It is clear by Corollary 3.3 and Theorem 3.7.

(1d-2) Suppose $1 < \gcd(d_2, \theta_2(\gamma_{2,0,1}, \gamma_{2,0,2})) \leq d_2$. In order to prove that if T_{2,d_2-1} is zero then $\gcd(d_2, \theta_2(\gamma_{2,0,1}, \gamma_{2,0,2})) < d_2$, it suffices to consider the following three subcases (i), (ii) and (iii) of (1d-2): Note that $\gcd(d_2, \theta_2(\gamma_{2,0,1}, \gamma_{2,0,2})) = \gcd(d_2, M'_0)$ where $M'_0 = \theta_2(\gamma_{2,0,1}, \gamma_{2,0,2}) - d_2 n_1 k_1 > 0$ by (16.2.9).

(i) of (1d-2). Let $d_2 < M'_0$. Apply Theorem 3.2 to $(f \circ \tau_{\lambda_1})_{proper}$ in (16.2.10). Since T_{2,d_2-1} is zero, it is clear that $\gcd(d_2, M'_0) < d_2$.

(ii) of (1d-2). Let $d_2 = M'_0$. By Theorem 3.2, T_{d_2-1} cannot be zero.

(iii) of (1d-2). Let $d_2 > M'_0$. It is clear.

Also, note by Theorem 3.2 that $\gcd(d, M'_0)$ may be equal to M'_0 whether T_{d-1} is zero or not. Thus, the proof of (1d) is done.

(2)(2a) The proof is trivial by (1d) of (1).

(2b) Note that $T_{2,i} g_1^i \in \mathbb{C}\{y\}[z, g_1] \subseteq \mathbb{C}\{y, z\}[g_1]$ for $0 \leq i \leq d_2 - 1$ and $f \in \mathbb{C}\{y, z\}[g_1]$, considering y, z, g_1 as independent three complex variables. Let $y^{\delta_1} z^{\delta_2} g_1^i \in T_{2,i} g_1^i$ be arbitrary for $0 \leq i \leq d_2 - 1$ where $\delta_1 > 0$ and $\delta_2 < n_1$. Then, $n_1 \alpha_{1,0,1}(\delta_1 + \delta_2 + i) > n_1 \delta_1 + \alpha_{1,0,1} \delta_2 + in_1 \alpha_{1,0,1} \geq \theta_2(\gamma_{2,i,1}, \gamma_{2,i,2}) + in_1 \alpha_{1,0,1} > (d_2 - i) n_1 \alpha_{1,0,1} + in_1 \alpha_{1,0,1} = d_2 n_1 \alpha_{1,0,1}$ by (b) and (1c). Thus, $\delta_1 + \delta_2 + i > d_2$. So, $f(y, z, g_1)$ is an irreducible W -poly of degree d_2 in g_1 with coefficients in $\mathbb{C}\{y, z\}$ and with multiplicity d at the origin in \mathbb{C}^3 because ${}_3\mathcal{O}_0$ is a unique factorization domain and $f \in \mathbb{C}\{y\}[z]$ is irreducible in $\mathbb{C}\{y, z\}$. Thus, the proof of (2b) is done. Therefore, the proof of this proposition is done. \square

Proof of Proposition 16.3. For the construction of a pair (f_1, f) in (16.3.1), it suffices to consider the following two cases, depending on the fact that T_{2,d_2-1} of (16.3.0) is either zero or not. For brevity of notations, let $h_1 = g_1$ and $T_{2,i}^{(1)} = T_{2,i}$ for $0 \leq i \leq d_2 - 1$.

Case(1): Let $T_{2,d_2-1}^{(1)}$ be zero. It is clear.

Case(2): Let $T_{2,d_2-1}^{(1)}$ be nonzero. It has been already shown by Sublemma 15.5 and Sublemma 15.6 in the proof of Theorem 15.4 that the following assertion is true:

There is a sequence of W -polys in z of pairs, $\{(h_p, f) : p = 1, 2, \dots\}$ such that

$$(16.3.12) \quad (h_{\nu+1}, f) = (h_{\nu+2}, f) = \dots \quad \text{for some integer } \nu \leq \frac{n_1 + 1}{2},$$

each pair of which can be written in the form

$$(16.3.13) \quad \begin{cases} h_1 &= g_1 = z^{n_1} + \xi_1 y^{\alpha_{1,0,1}}, \\ f &= h_1^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(1)} h_1^i, \end{cases}$$

and for each $p = 2, 3, \dots$

$$(16.3.14) \quad \begin{cases} h_p &= h_{p-1} + \frac{1}{d_2} T_{2,d_2-1}^{(p-1)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p)} z^i, \\ f &= h_p^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p)} h_p^i, \end{cases}$$

with $T_{2,d_2-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{2,d_2-1}^{(\nu+1)} = T_{2,d_2-1}^{(\nu+2)} = \dots = 0$ where $T_{2,i}^{(p)} = T_{2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ for $p \geq 1$ and $0 \leq i \leq d_2-1$, and $R_{2,i}^{(p)} = R_{2,i}^{(p)}(y) \in \mathbb{C}\{y\}$ for $p \geq 1$ and $0 \leq i \leq n_1-2$, if exist, satisfying the same kind of the properties and notations as we have seen in Sublemma 15.5 of Theorem 15.4, as follows:

(16.3.15-1) Property(1) and Property(3) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_1-2$. Then, $R_{1,i}^{(p)} = R_{1,i}^{(p)}(y) \in \mathbb{C}\{y\}$ with $R_{2,i}^{(p)}(0) = 0$ and has a multiplicity $\geq (n_1 - i)$ at $0 \in \mathbb{C}^2$.

(16.3.15-2) Property(2) and Property(4) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_2-1$. Then, $T_{2,i}^{(p)} = T_{2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial of degree $\delta < n_1$ in z and has a multiplicity $\geq (d_2 - i)n_1$ at $0 \in \mathbb{C}^2$.

Also, for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $T_{2,i}^{(p)} = T_{2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(16.3.15-3) Property(5) In particular, if $i = d_2 - 1$ for $T_{2,i}^{(p)}$ of Property(4), then $\delta_2 \leq n_1 - 2$.

(16.3.15-4) Property(6) There is an integer $\nu < n_1$ such that $T_{2,d_2-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu$ and $T_{2,d_2-1}^{(\nu+1)} = T_{2,d_2-1}^{(\nu+2)} = \dots = 0$. Also, $\nu \leq \frac{n_1+1}{2}$.

Remark. Without any need of proof, Property(1), Property(2), \dots , Property(6), which are mentioned just above, follow clearly from Sublemma 15.5 of Theorem 15.4, which belongs to Case[II] with $j = 0$ in the conclusion of Sublemma 15.5 of Theorem 15.4. In Sublemma 15.5, note that $f_{-1} = y$ and $f_0 = z$.

For the proof of this proposition in Case(2), it suffices to show that two properties, denoted by, The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ can be satisfied, respectively. Then, the proof will be by induction on the integer $p \geq 1$.

Now, it is enough to consider the following two subcases for Case(2), respectively:

Subcase(A) $p = 1$, and Subcase(B) $p \geq 1$.

Subcase(A) of Case(2): Let $p = 1$. Then it suffices to show that (h_1, f) given by (16.3.2), satisfies The Necessary and Sufficient Condition[A] for $g_1(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, which was already proved by Proposition 16.2.

Subcase(B) of Case(2): Let $p \geq 1$. For the proof of this subcase, it suffices to show by Subcase(A) of Case(2) that the following sublemma is true:

Sublemma 16.3.1 for Subcase(B) of Case(2).

Assumptions For the induction proof, suppose we have shown on the integer $p \geq 1$ that The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$ and

The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ are true for (h_p, f) , following the same notations and properties as we have seen in (16.3.3), (16.3.4), \dots , (16.3.11).

Conclusions Then, (h_{p+1}, f) can be written by

$$(16.3.16) \quad \begin{cases} h_{p+1} &= h_p + \frac{1}{d_2} T_{2,d_2-1}^{(p)} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p+1)} z^i, \\ f &= h_{p+1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i, \end{cases}$$

with $T_{2,d_2-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{2,d_2-1}^{(\nu+1)} = T_{2,d_2-1}^{(\nu+2)} = \dots = 0$ where $T_{2,i}^{(p)} = T_{2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ for $p \geq 1$ and $0 \leq i \leq d_2 - 1$, and $R_{1,i}^{(p)} = R_{1,i}^{(p)}(y) \in \mathbb{C}\{y\}$ for $p \geq 1$ and $0 \leq i \leq n_1 - 2$, if exist, satisfying the following properties, denoted by The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ inductively as we have seen in the conclusion of this proposition:

**[1] The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$:
 $h_{p+1} \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W-poly of degree n_1 in f_0 with a coefficient of $f_0^{n_1-1}$ zero in $\mathbb{C}\{f_{-1}\}$, and $h_{p+1}(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5**

To find The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$, as an element in $\mathbb{C}\{y, z\}$ if necessary, it suffices to show that $h_{p+1} = z^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i}^{(p+1)} z^i$ of (16.3.16) satisfies the following (1) and (2):

(1) Let p be fixed with $p \geq 1$. Each $R_{1,i}^{(p+1)} \neq 0$ satisfies the properties (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, n_1 - 2$. Also, for each $p \geq 1$, $h_{p+1} \stackrel{\text{multiseq}}{\sim} h_1$ and $h_{p+1} \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

(1a) We write $R_{1,i}^{(p+1)} = b_i^{(p+1)} y^{\alpha_{1,i,1}^{(p+1)}}$ with a unit $b_i^{(p+1)}$ in $\mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}^{(p+1)}$, if exists. For all $p \geq 1$, $\alpha_{1,0,1}^{(p+1)} = \alpha_{1,0,1}$ and $\xi_1 = b_0^{(p)}(0)$ where ξ_1 was found to be $\frac{1}{d_2} a_{n-n_1}(0)$ as in $(h_1, f) = (g_1, f)$ of (16.3.2).

(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers, by the same way as in Proposition 16.2.

(1c) For all $i = 0, 1, \dots, n_{j+1} - 2$,

$$(16.3.17) \quad \theta_1(\alpha_{1,i,1}^{(p+1)}) > n_1 - i.$$

(1d) For all $i = 0, 1, \dots, n_1 - 2$,

$$(16.3.18) \quad \gcd(n_1, \theta_1(\alpha_{1,0,1}^{(p+1)})) = 1 \quad \text{with} \quad \sigma_1 = \alpha_{1,0,1}^{(p+1)},$$

$$\frac{\theta_1(\alpha_{1,i,1}^{(p+1)})}{n_1 - i} \geq \frac{\theta_1(\alpha_{1,0,1}^{(p+1)})}{n_1}.$$

Note that $\alpha_{1,i,1}^{(p+1)} n_1 + i \alpha_{1,0,1}^{(p+1)} > n_1 \alpha_{1,0,1}^{(p+1)}$ for all $i = 1, \dots, n_1 - 2$.

(2)(2a) For each $p \geq 1$, $h_{p+1} = h_{p+1}(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z with coefficients in $\mathbb{C}\{y\}$ and with $h_{p+1} \stackrel{\text{multiseq}}{\sim} h_1 = g_1$, and $h_{p+1} \in \text{the type}[1]$ in the sense of Definition 2.5.

(2b) $h_{p+1} = h_{p+1}(f_{-1}, f_0) \in \mathbb{C}\{f_{-1}\}[f_0]$ of (16.3.16) is an irreducible W -poly in f_0 with coefficients in $\mathbb{C}\{f_{-1}\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

**[2] The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[h_{p+1}]$ is an irreducible W-poly of degree d_2 in h_{p+1} with a coefficient of $h_{p+1}^{d_2-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ in the sense of Definition 2.5**

To find The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, as an element in $\mathbb{C}\{y, z\}$ if necessary, it is enough to show that for each $p \geq 1$, $f = h_{p+1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i$, of (16.3.16) satisfies the following (1) and (2): Note that either $\ell = 2$ or $\ell > 2$.

(1) Each $T_{2,i}^{(p+1)} \in \mathbb{C}\{f_{-1}, f_0\}$ of f in (16.3.3) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_2 - 1$.

(1a) For any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p+1)}$,

$$(16.3.19) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 < n_1.$$

In particular, if $i = d_2 - 1$ for $T_{2,i}^{(p+1)}$ then $\gamma_1 > 0$ and $\gamma_2 \leq n_1 - 2$.

(1b) Define $\bar{\theta}_2(t_k)_{k=1}^2 = t_2\sigma_1 + n_1t_1$ for any $(t_k)_{k=1}^2 \in N_0^2$ by the same way as we have seen in the definition of the integer valued-function $\bar{\theta}_2$ in the conclusion of Proposition 16.2.

For any two nonzero monomials $\Pi_{k=1}^2 f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p+1)}$,

$$(16.3.20) \quad \bar{\theta}_2(\beta_k)_{k=1}^2 = \bar{\theta}_2(\gamma_k)_{k=1}^2 \text{ if and only if } \beta_k = \gamma_k \text{ for } k = 1, 2.$$

So, there is a unique nonzero monomial $C_{2,i}^{(p+1)} \Pi_{k=1}^2 f_{k-2}^{\beta_{2,i,k}^{(p+1)}}$ in $T_{2,i}^{(p+1)}$

with a constant $C_{2,i}^{(p+1)}$ such that $\bar{\theta}_2(\beta_{2,i,k}^{(p+1)})_{k=1}^2 = \min\{\bar{\theta}_2(\gamma_k)_{k=1}^2\}$

for any nonzero monomial $\Pi_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p+1)}$.

(1c) For all $i = 0, 1, \dots, d_2 - 1$,

$$(16.3.21) \quad \bar{\theta}_2(\beta_{2,i,k}^{(p+1)})_{k=1}^2 > (d_2 - i)n_1\theta_1(\sigma_1).$$

(1d) For all $i = 0, 1, \dots, d_2 - 1$,

$$(16.3.22) \quad \begin{aligned} \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) &\geq 1, \\ \frac{\bar{\theta}_2(\beta_{2,i,k}^{(p+1)})_{k=1}^2}{d_2 - i} &\geq \frac{\bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2}{d_2}. \end{aligned}$$

Then, either $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) = 1$ or $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) \leq d_2$.

(1d-1) Let $\gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if the inequality in (16.3.22) holds. In this case, $f \in$ the type [2] in the sense of Definition 2.5, but note that $T_{2,d_2-1}^{(p+1)}$ may not be zero where

$$(16.3.23) \quad h_{p+1} = h_p + \frac{1}{d_2} T_{2,d_2-1}^{(p)} \quad \text{and} \quad f = h_{p+1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i.$$

(1d-2) Let $1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(p+1)})_{k=1}^2) \leq d_2$. There is a positive integer ν with $\nu \leq \frac{n_1+1}{2}$ such that $T_{2,d_2-1}^{(\nu+1)} = 0$ and $T_{2,d_2-1}^{(p+1)} \neq 0$ for $p+1 = 1, 2, \dots, \nu$. In this case, $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5 and note that

$$(16.3.24) \quad 1 < \gcd(d_2, \bar{\theta}_2(\beta_{2,0,k}^{(\nu+1)})_{k=1}^2) < d_2.$$

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_2 n_1$ at $0 \in \mathbb{C}^2$. Also, either $f \in$ the type [2] or $f \in$ the type $[\ell]$ with $\ell \geq 3$ in the sense of Definition 2.5.

(2b) $f = f(f_{-1}, f_0, h_{p+1}) \in \mathbb{C}\{f_{-1}, f_0\}[h_{p+1}]$ of (16.3.16) is an irreducible W -poly of degree d_2 in h_{p+1} with coefficients in $\mathbb{C}\{f_{-1}, f_0\}$ and with multiplicity d_2 at $0 \in \mathbb{C}^3$.

By The Necessary Condition[A] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$, let $f_1 = h_r$ for any $r \geq \nu + 1$ where $\nu + 1$ is the smallest positive integer such that $T_{2,d_2-1}^{(\nu+1)}$ is equal to zero.

**[3] The Necessary Condition[A] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0\}[f_1]$ is an irreducible W -poly of degree d_2 in f_1 with a coefficient**

of $f_1^{d_2-1}$ zero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f = f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$
in the sense of Definition 2.5 \square

The proof of Sublemma 16.3.1 for Subcase(B) of Case(2) Let $p \geq 1$ with $T_{2,d_2-1}^{(p)} \neq 0$. Suppose we have shown that The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ are true for (h_p, f) .

We will prove that h_{p+1} of (h_{p+1}, f) satisfies The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$ and that f of (h_{p+1}, f) satisfies The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, respectively.

[1] The proof of The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$

(1) (1a) It is clear by either (16.3.15-1) or Sublemma 15.5.5 of Theorem 15.4.

(1b) There is nothing to prove.

(1c) Since $h_{p+1} = h_p + \frac{1}{d_2} T_{2,d_2-1}^{(p)}$, it suffices to show by induction on p that for any nonzero monomial $y^\alpha z^\beta \in T_{2,d_2-1}^{(p)}$, $\alpha + \beta > n_1$. Since $T_{2,d_2-1}^{(p)}$ of (h_p, f) satisfies (16.3.7) and an inequality in (16.3.8) by induction on p , $n_1 < \alpha_{1,0,1}$ implies that

$$(16.3.25) \quad \alpha_{1,0,1}(\alpha + \beta) > n_1\alpha + \alpha_{1,0,1}\beta = \theta_2(\alpha, \beta) \geq \theta_2(\beta_{2,d_2-1,1}^{(p)}, \beta_{2,d_2-1,2}^{(p)}) > n_1\alpha_{1,0,1},$$

for any nonzero monomials $y^\alpha z^\beta$ in $T_{2,d_2-1}^{(p+1)}$. Thus, it is clear that $\alpha + \beta > n_1$.

(1d) By induction on p , it is clear by (16.3.25) that for any nonzero monomial $y^\alpha z^\beta \in T_{2,d_2-1}^{(p+1)}$ with $\beta = 0$, $\alpha > \alpha_{1,0,1}$. So, $\alpha_{1,0,1}^{(p+1)} = \alpha_{1,0,1}$.

For any nonzero monomial $y^\alpha z^\beta \in T_{2,d_2-1}^{(p+1)}$, write $i = \beta$. Then the proof of (1d) can be done because it is clear by (16.3.25) that for all $i = 1, \dots, n_1 - 2$,

$$(16.3.26) \quad \gcd(n_1, \theta_1(\alpha_{1,0,1})) = 1 \quad \text{with} \quad \alpha_{1,0,1}^{(p+1)} = \alpha_{1,0,1} = \sigma_1, \\ \frac{\alpha}{n_1 - \beta} > \frac{\alpha_{1,0,1}}{n_1} \quad \text{or} \quad n_1\alpha + \alpha_{1,0,1}\beta > n_1\alpha_{1,0,1}.$$

(2) The proofs of (2a) and (2b) are clear by (1). Thus, the proof of The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[1]$ is done.

[2] The proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$

(1) (1a) It is clear by either (16.3.15-2) and (16.3.15-3), or Sublemma 15.5.5 of Theorem 15.4.

(1b) Using the same method as we have used in the proof of (16.2.2) in (1b) of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$, it is clear that (16.3.7) in (1b) of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ is true. So, it is clear that there exists a unique nonzero monomial $C_{2,i}^{(p+1)} y^{\beta_{2,i,1}^{(p+1)}} z^{\beta_{2,i,2}^{(p+1)}} \in T_{2,i}^{(p+1)}$ with a constant $C_{2,i}^{(p+1)}$ such that $\theta_2(\beta_{2,i,1}^{(p+1)}, \beta_{2,i,2}^{(p+1)}) = \min\{\theta_2(\gamma_1, \gamma_2)\}$ for any nonzero monomial $y^{\gamma_1} z^{\gamma_2}$ in $T_{2,i}^{(p+1)}$.

(1c) In preparation for the proof of (16.3.21), note by (16.0.1) and (16.3.26) that for any nonzero monomial $y^\alpha z^\beta \in h_{p+1} - h_1$ and for any nonzero monomial $y^{\alpha'} z^{\beta'} \in h_{p+1}^i$ with $p \geq 1$,

$$(16.3.27) \quad \theta_2(\alpha, \beta) > n_1\alpha_{1,0,1} \quad \text{and} \quad \theta_2(\alpha', \beta') \geq in_1\alpha_{1,0,1}.$$

First note that $f(y, z)$ can be rewritten as follows:

$$(16.3.28) \quad f = h_{p+1}^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i = (h_{p+1} - h_1 + h_1)^{d_2} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i, \\ = (h_{p+1} - h_1)^{d_2} + h_1^{d_2} + \sum_{i=1}^{d_2-1} d_2 C_i (h_{p+1} - h_1)^i h_1^{d_2-i} + \sum_{i=0}^{d_2-1} T_{2,i}^{(p+1)} h_{p+1}^i$$

Since $\bar{\theta}_2(\beta_{2,i,k}^{(p+1)})_{k=1}^2 = \min\{\bar{\theta}_2(\gamma_k)_{k=1}^2\}$ for any nonzero monomial $\prod_{k=1}^2 f_{k-2}^{\gamma_k}$ in $T_{2,i}^{(p+1)}$ by (16.3.20), note by (16.3.27) that for any nonzero monomial $y^\gamma z^\delta \in T_{2,i}^{(p+1)} h_{p+1}^i$,

$$(16.3.29) \quad n_1 \gamma + \alpha_{1,0,1} \delta \geq n_1 \beta_{2,i,1} + \alpha_{1,0,1} \beta_{2,i,2} + i n_1 \alpha_{1,0,1} = \theta_2(\beta_{2,i,1}, \beta_{2,i,2}) + i n_1 \alpha_{1,0,1},$$

because $y^\gamma z^\delta = y^{\delta_1} z^{\delta_2} y^{t\alpha_{1,0,1}} z^{sn_1}$ for a nonzero monomial $y^{\delta_1} z^{\delta_2} \in T_{2,i}^{(p+1)}$ where $s + t = i$.

By (16.0.1) of Lemma 16.0, (16.3.27), (16.3.28) and (16.3.29), to prove an inequality in (16.3.21), we show by the induction on the integer q that the following is true:

$$(16.3.30) \quad \theta_2(\beta_{2,d_2-q,k})_{k=1}^2 + q n_1 \alpha_{1,0,1} > d_2 n_1 \alpha_{1,0,1}$$

Let $q = 1$ for the proof of (16.3.30). It is clear by (16.3.19) that $y^{\beta_{2,d_2-1,1}^{(p+1)}} z^{\beta_{2,d_2-1,2}^{(p+1)}} z^{n_1} \in T_{2,d_2-1}^{(p+1)} h_{p+1}^{d_2-1}$ belongs to $f - h_{p+1}^{d_2}$. So, it is clear that an inequality in (16.3.21) is true. Let $q = 2$ for the proof of (16.3.30). To prove an inequality in (16.3.30), if $y^{\beta_{2,d_2-2,1}^{(p+1)}} z^{\beta_{2,d_2-2,2}^{(p+1)}} z^{2n_1} \in T_{2,d_2-2}^{(p+1)} h_{p+1}^2$ belongs to $f - h_{p+1}^{d_2}$, there is nothing to prove by (16.0.1) of Lemma 16.0. Otherwise, it is clear by (16.3.19) that $y^{\beta_{2,d_2-2,1}^{(p+1)}} z^{\beta_{2,d_2-2,2}^{(p+1)}} z^{2n_1}$ would belong to $T_{2,d_2-1}^{(p+1)} h_{p+1}$, which implies by (16.3.29) that $\theta_2(\beta_{2,d_2-2,k})_{k=1}^2 + 2n_1 \alpha_{1,0,1} \geq \theta_2(\beta_{2,d_2-1,k})_{k=1}^2 + n_1 \alpha_{1,0,1} > d_2 n_1 \alpha_{1,0,1}$. Thus, the proof of inequality for $q = 2$ is done. If $q \geq 3$, by the same method as we have used for $q = 1$ and $q = 2$ and by induction on the integer q , the proof of (16.3.21) is easily done.

(1d) For the proof of an inequality in (16.3.22), by the same method as we have used in the proof of Proposition 16.2, there is nothing to prove.

(2) The proof is trivial by (1d) of (1). Thus, the proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq 2$ is done, and then the proof of Sublemma 16.3.1 for Subcase(B) of Case(2) is done. So, the proof for Case(2) is finished and therefore, the proof of the proposition can be completed. \square

Proof of Theorem 16.4. The remaining proof of the theorem just follows from Proposition 16.2 and Proposition 16.3 with Remark 16.3.0. \square

§ 17. Irreducibility criterion of W-polys of two complex variables(A generalized representation of irreducible W-polys of two complex variables)

Observe by Theorem 1.13(Irreducibility criterion of W-polys of two complex variables) of § 1.7 in Part[A] that the necessary and sufficient condition for $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{the type}[\ell]$ in the sense of Definition 2.5 can be represented without proof. In this section, the first aim is to prove that the sufficiency of the condition in Theorem 1.13 can be proved by Theorem 16.5. The second aim is to prove that the necessity of the condition in Theorem 1.13(the converse of Theorem 16.5) can be represented by Theorem 16.6 with proof. Then, we can find The 2nd Algorithm for computing irreducibility criterion of all the W-polys of two complex variables in the process of the proof of Theorem 16 with Proposition 16.7 and Proposition 16.8 completely and rigorously, using Theorem 15.4.

Theorem 16.5(A sufficient condition for any W-poly $f(y, z)$ to be irreducible in $\mathbb{C}\{y, z\}$ in terms of The Division Algorithm for W-polys(Irreducibility criterion of W-polys of two complex variables)).

Assumptions Let $f(y, z) = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be a W-poly in z with multiplicity $n \geq 2$ at $0 \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists, and the α_i are positive integers. Note that a_{n-1} is identically zero. Note that a_{n-1} is identically zero for convenience. Write $n = \prod_{k=1}^{j+1} n_k$ with positive integers $n_k \geq 2$ for all k where the n_k may not be the factorization of prime numbers.

In addition, the following inequality holds:

$$(16.5.0) \quad 2 \leq n \leq \alpha_0.$$

Suppose by Theorem 15.4(The Division Algorithm for W-polys) that for each fixed positive integer j and for each $k = 1, 2, \dots, j$, f_k and f can be written in the form

$$(16.5.1) \quad \begin{cases} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \\ f &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i} f_j^i \end{cases}$$

where, considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} with $f_{-1} = y$ and $f_0 = z$,

- (i) $n = \prod_{k=1}^{j+1} n_k$ with $n_k \geq 2$ for $1 \leq k \leq j+1$;
 - (ii) for each fixed k and for each i with $0 \leq i \leq n_k - 2$, $R_{k,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}$;
 - (iii) for each $k = 1, 2, \dots, j$, $f_k = f_k(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}]$ and
 - (iv) $f = f(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ with $f = f_{j+1}$,
- satisfying a finite number of conditions, each of which is represented respectively, as follows:

(1) Condition[A] for $f_1(y, z) \in \text{the type}[1]$ in the sense of Definition 2.5:

$R_{1,i} \in \mathbb{C}\{y\}$ satisfies the properties (1a), (1b) and (1c) for each $i = 0, 1, \dots, n_1 - 2$, and then $f_1 = f_1(y, z) \in \mathbb{C}\{y, z\}$ satisfies the properties (1d).

(1a) For $0 \leq i \leq n_1 - 2$, each $R_{1,i} = b_i y^{\alpha_{1,i,1}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.

(1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers.

(1c) $\theta_1(\alpha_{1,i,1}) > (n_1 - i)$ for all $i = 0, 1, \dots, n_1 - 2$.

(1d) For all $i = 0, 1, \dots, n_1 - 2$,

$$(16.5.2) \quad \begin{aligned} \gcd(n_1, \alpha_{1,0,1}) &= 1 \quad \text{and} \\ \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} &= \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}. \end{aligned}$$

(2) Condition[A] for $f_2(y, z) \in \text{the type}[2]$ in the sense of Definition 2.5:

$R_{2,i} \in \mathbb{C}\{y\}[z]$ satisfies the properties (2a), (2b) and (2c) for each $i = 0, 1, \dots, n_2 - 2$, and then $f_2 = f_2(y, z, f_1) \in \mathbb{C}\{y, z\}[f_1]$ satisfies the properties (2d).

(2a) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(2b) Let \mathbb{N}_0^2 be two-dimensional cartesian product of \mathbb{N}_0 . For given integers $n_1, \alpha_{1,0,1}$ and a function θ_1 in $\text{Cond}[A]$ of the 1st type, define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by

$$(16.5.3) \quad \theta_2(t_1, t_2) = t_2 \theta_1(\alpha_{1,0,1}) + n_1 \theta_1(t_1) = t_2 \alpha_{1,0,1} + n_1 t_1 \quad \text{for each } (t_1, t_2) \in \mathbb{N}_0^2.$$

Then, for any two nonzero monomials $y^{\alpha_1} z^{\alpha_2}$ and $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed,

$$(16.5.3-1) \quad \begin{aligned} \theta_2(\alpha_1, \alpha_2) &= \theta_2(\delta_1, \delta_2) \text{ if and only if } \alpha_1 = \delta_1 \text{ and } \alpha_2 = \delta_2. \\ \text{So, there exists a unique nonzero monomial } A_{2,i} y^{\alpha_{2,i,1}} z^{\alpha_{2,i,2}} \text{ in } R_{2,i} \\ \text{with a nonzero constant } A_{2,i} \text{ such that } \theta_2(\alpha_{2,i,1}, \alpha_{2,i,2}) &= \min\{\theta_2(\delta_1, \delta_2)\} \\ \text{for any nonzero monomial } y^{\delta_1} z^{\delta_2} \text{ in } R_{2,i} \text{ with } i \text{ fixed.} \end{aligned}$$

(2c) For all $i = 0, 1, \dots, n_2 - 2$,

$$(16.5.4) \quad \theta_2(\alpha_{2,i,k})_{k=1}^2 > (n_2 - i) n_1 \alpha_{1,0,1}.$$

(2d) For all $i = 0, 1, \dots, n_2 - 2$,

$$(16.5.5) \quad \begin{aligned} \gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) &= 1 \quad \text{and} \\ \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} &\geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2}. \end{aligned}$$

(3) Condition[A] for $f_m(y, z) \in \text{the type}[m]$ in the sense of Definition 2.5:

For each fixed m with $3 \leq m \leq j+1$, $R_{m,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}$ satisfies the properties (3a), (3b) and (3c) for each $i = 0, 1, \dots, n_m - 2$, and then $f_m = f_m(y, z, f_1, \dots, f_{m-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}[f_{m-1}]$ satisfies the properties (3d).

(3a) For any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with $f_{-1} = y$ and $f_0 = z$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $k = 2, 3, \dots, m$.

(3b) By induction assumption on the integer $(m-1) \leq j$, there exists a sequence $\{f_3, f_4, \dots, f_{m-1}\}$, each of which satisfies the same kind of properties and notations as we have seen in Condition[A] for $f_2(y, z) \in \text{the type}[2]$ in the sense of Definition 2.5. Then inductively, define $\theta_m : \mathbb{N}_0^m \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^m is its m -dimensional cartesian product by

$$(16.5.6) \quad \theta_m(t_k)_{k=1}^m = t_m \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1} + n_{m-1} \theta_{m-1}(t_k)_{k=1}^{m-1} \quad \text{for each } (t_k)_{k=1}^m \in \mathbb{N}_0^m,$$

where recall by induction assumption that for a fixed i , $A_{m-1,i} \Pi_{k=1}^{m-1} f_{k-2}^{\alpha_{m-1,i,k}}$ is a unique nonzero monomial in $R_{m-1,i}$ with a constant $A_{m-1,i}$ such that

$$(16.5.7) \quad \theta_{m-1}(\alpha_{m-1,i,k})_{k=1}^{m-1} = \min\{\theta_{m-1}(\delta_k)_{k=1}^{m-1}\},$$

for any nonzero monomial $\Pi_{k=1}^{m-1} f_{k-2}^{\delta_k}$ in $R_{m-1,i}$.

Then, for any two nonzero monomials $\Pi_{k=1}^m f_{k-2}^{\alpha_k}$ and $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with i fixed,

$$(16.5.7-1) \quad \begin{aligned} \theta_m(\alpha_k)_{k=1}^m &= \theta_m(\delta_k)_{k=1}^m \text{ if and only if } \alpha_k = \delta_k \text{ for } k = 1, 2, \dots, m. \\ \text{So, there exists a unique nonzero-monomial } A_{m,i} \Pi_{k=1}^m f_{k-2}^{\alpha_{m,i,k}} \text{ in } R_{m,i} \\ \text{with a constant } A_{m,i} \text{ such that } \theta_m(\alpha_{m,i,k})_{k=1}^m &= \min\{\theta_m(\delta_k)_{k=1}^m\} \\ \text{for any nonzero monomial } \Pi_{k=1}^m f_{k-2}^{\delta_k} \text{ in } R_{m,i}. \end{aligned}$$

(3c) For all $i = 0, 1, \dots, n_m - 2$,

$$(16.5.8) \quad \theta_m(\alpha_{m,i,k})_{k=1}^m > (n_m - i) n_{m-1} \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1}.$$

(3d) For all $i = 0, 1, \dots, n_m - 2$,

$$(16.5.9) \quad \gcd(n_m, \theta_m(\alpha_{m,0,k})_{k=1}^m) = 1 \quad \text{and} \\ \frac{\theta_m(\alpha_{m,i,k})_{k=1}^m}{n_m - i} \geq \frac{\theta_m(\alpha_{m,0,k})_{k=1}^m}{n_m}.$$

Conclusions Consider the sequence $S = \{f_k : 1 \leq k \leq j+1\}$ with $f_{j+1} = f$ where $f_k = f_k(y, z, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-1}\}$, which have the same properties and notations as we have seen in (16.5.1) of the assumptions of Theorem 16.5.

(1) $f = f_{j+1}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ is an irreducible W -poly of degree n_{j+1} in f_j with a coefficient of $f_j^{n_{j+1}-1}$ zero in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$ and with multiplicity n_{j+1} at the origin in \mathbb{C}^{j+1} .

(2) $f \in \mathbb{C}\{y, z\}$ is an irreducible W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity n at the origin in \mathbb{C}^2 , which can be written as follows:

$f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ is an irreducible W -poly in z with $f_{j+1} \in$ the type[j+1] in the sense of Definition of 2.5 and with multiplicity n at the origin in \mathbb{C}^2 where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists, and the α_i are positive integers, and $n = \Pi_{i=1}^{j+1} n_i$. Note that a_{n-1} is zero.

Proof of Theorem 16.5. The proof of the theorem just follows from Theorem 12.0.

Corollary 16.5.1. Assumptions Under the same assumptions and notations as in Theorem 16.5, note that f_k is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $(0, 0)$ in \mathbb{C}^2 for $r \geq 1$. In particular, for each $k = 1, 2, \dots, j+1$, let $V(H_{j+1}) = \{(y, z) : H_{j+1}(y, z) = 0\}$ be an analytic variety at $(0, 0)$ in \mathbb{C}^2 , each of which is defined as follows:

$$(16.5.1.1) \quad \begin{aligned} \text{(i)} \quad & H_1 = z^{n_1} + y^{\alpha_{1,0,1}} \quad \text{with } n_1 \geq 2 \text{ and } \alpha_{1,0,1} \geq 2. \\ \text{(ii)} \quad & H_2 = H_1^{n_2} + y^{\alpha_{2,0,1}} z^{\alpha_{2,0,2}}. \\ \text{(iii)} \quad & H_3 = H_2^{n_3} + y^{\alpha_{3,0,1}} z^{\alpha_{3,0,2}} H_1^{\alpha_{3,0,3}}. \\ & \dots \dots \dots \\ \text{(j+1)} \quad & H_{j+1} = H_j^{n_{j+1}} + y^{\alpha_{j+1,0,1}} z^{\alpha_{j+1,0,2}} H_1^{\alpha_{j+1,0,3}} \dots H_{j-1}^{\alpha_{j+1,0,j+1}}. \end{aligned}$$

Conclusions Then, $f_{j+1} \in \mathbb{C}\{y, z\}$ and $H_{j+1} \in \mathbb{C}\{y, z\}$ have the same multiplicity sequence at $(0, 0)$ in \mathbb{C}^2 . \square

Remark 16.5.2. (I) Note by Theorem 5.0 that H_{j+1} is irreducible in $\mathbb{C}\{y, z\}$ with $H_{j+1} \in$ the type[j+1] in the sense of Definition of 2.5 $\iff \gcd(n_1, \alpha_{1,0,1}) = 1, \gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) = 1, \dots, \gcd(n_{j+1}, \theta_{j+1}(\alpha_{j+1,0,k})_{k=1}^{j+1}) = 1$.

(II) Note that f_{j+1} is irreducible in $\mathbb{C}\{y, z\}$ with $f_{j+1} \in$ the type[j+1] in the sense of Definition of 2.5 \iff the following hold:

$$\begin{aligned} (1) \quad & \gcd(n_1, \alpha_{1,0,1}) = 1 \text{ and } \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}. \\ (2) \quad & \gcd(n_2, \theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})) = 1 \text{ and } \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} \geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2}. \\ & \dots \dots \dots \\ \text{(j+1)} \quad & \gcd(n_{j+1}, \theta_{j+1}(\alpha_{j+1,0,k})_{k=1}^{j+1}) = 1 \text{ and } \frac{\theta_m(\alpha_{m,i,k})_{k=1}^m}{n_m - i} \geq \frac{\theta_m(\alpha_{m,0,k})_{k=1}^m}{n_m}. \end{aligned}$$

Theorem 16.6(How to find an algorithm for computing irreducibility criterion of all the W-polys defining plane curve singularities at $0 \in \mathbb{C}^2$ (The converse of Theorem 16.5)).

Let N_0 be the set of nonnegative integers and N_0^k be its k -dimensional copy. Let r be an arbitrary positive integer.

Assumptions Let $f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i$ be a W-poly in z with multiplicity $n \geq 2$ at $0 \in \mathbb{C}^2$ where for $0 \leq i \leq n-2$, each $a_i = a_i(y)$ is a unit in ${}_2\mathcal{O}_0$ if exists and the α_i are positive integers. Note that a_{n-1} is zero and that n can be viewed as $n = \prod_{i=1}^r q_i$ where each integer q_i is a prime number.

Conclusions To find a complete algorithm for computing the irreducibility criterion of the Weierstrass polynomials $f(y, z)$ in $\mathbb{C}\{y, z\}$, it suffices to prove the following:

Whenever f is irreducible in $\mathbb{C}\{y, z\}$, we can find a positive integer ℓ with $\ell \leq r$ such that $f \in$ the type $[\ell]$ in the sense of Definition 2.5 with the following property, and conversely:

We may start to assume for brevity of representation that $\ell \geq 2$ (that is, $\gcd(n, \alpha_0) > 1$), because it was already proved by Corollary 3.3 and Proposition 16.2 that f is irreducible in $\mathbb{C}\{y, z\}$ with $f \in$ the type $[1]$ if and only if (16.6.0) holds.

$$(16.6.0) \quad \frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n} \quad \text{for} \quad 0 \leq i \leq n-2 \quad \text{and} \quad \gcd(n, \alpha_0) = 1.$$

For each fixed j with $0 \leq j \leq \ell-1$, there exists a sequence of irreducible W-polys in z , $\{f_k : k = 1, 2, \dots, j\}$ where each $f_k \in \mathbb{C}\{y\}[z]$ is a W-polys in z with coefficients in $\mathbb{C}\{y\}$ and $f_k \in$ the type $[k]$ and $f_j \neq f$, satisfying the following properties and notations: Note that $f_{-1} = y$, $f_0 = z$.

For each fixed j with $1 \leq k \leq j$, f can be uniquely written in the form

$$(16.6.1) \quad \begin{cases} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}] \\ f &= f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j], \end{cases}$$

where, considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at $0 \in \mathbb{C}^{j+2}$,

- (i) $n = d_{j+1} \prod_{k=1}^j n_k$ with $d_{j+1} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j$, and $n = d_1$ if $j = 0$;
- (ii) for each fixed k and for each i with $0 \leq i \leq n_k - 2$, $R_{k,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{k-2}]$;
- (iii) for each i with $0 \leq i \leq d_{j+1} - 2$, $S_{j+1,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$;
- (iv) for each $k = 1, 2, \dots, j$, $f_k = f_k(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{k-1}]$ and
- (v) $f = f(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$,

satisfying a finite number of algorithms, each of which is represented respectively, as follows:

[1] An algorithm for computing an irreducibility criterion for $f_1(y, z) \in$ the type $[1]$: $f_1 \in \mathbb{C}\{f_{-1}\}[f_0]$ is an irreducible W-poly of degree n_1 in f_0 with a coefficient of $f_0^{n_1-1}$ zero, and $f_1(y, z) \in$ the type $[1]$ in the sense of Definition 2.5

Note that $f = f(f_{-1}, f_0, f_1) \in$ the type $[\ell]$ with $\ell \geq 2$, as an element in $\mathbb{C}\{y, z\}$.

Let $d_2 = \gcd(n, \alpha_0)$ with $n = d_2 n_1$ and $\alpha_0 = d_2 \alpha_{1,0,1}$ for some integers n_1 and $\alpha_{1,0,1}$.

Then, $f_1 = f_0^{n_1} + \sum_{i=0}^{n_1-2} R_{1,i} f_0^i \in \mathbb{C}\{y\}[f_0]$ of (16.6.1) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying two properties (1) and (2):

- (1) Each $R_{1,i} \in \mathbb{C}\{y\}$ in f_1 of (16.6.1) satisfies (1a), (1b), (1c), (1d) for $i=0, 1, \dots, n_1-2$.
 - (1a) For $0 \leq i \leq n_1-2$, each $R_{1,i} = b_i y^{\alpha_{1,i,1}}$ with a unit $b_i \in \mathbb{C}\{y\}$ and a positive integer $\alpha_{1,i,1}$ if exists. Denote $A_{1,i}$ by $b_i(0)$ for convenience of notations.
 - (1b) Define a function $\theta_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by $\theta_1(t) = t$ where \mathbb{N}_0 is the set of nonnegative integers.
 - (1c) $\theta_1(\alpha_{1,i,1}) > (n_1 - i)$ for all $i = 0, 1, \dots, n_1-2$.
 - (1d) For all $i = 0, 1, \dots, n_1-2$,

$$(16.6.2) \quad \gcd(n_1, \alpha_{1,0,1}) = 1 \quad \text{and} \quad \frac{\theta_1(\alpha_{1,i,1})}{n_1 - i} = \frac{\alpha_{1,i,1}}{n_1 - i} \geq \frac{\alpha_{1,0,1}}{n_1} = \frac{\theta_1(\alpha_{1,0,1})}{n_1}.$$

(2)(2a) $f_1 = f_1(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n_1 in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$, and $f_1 \in$ the type [1].

(2b) $f_1 = f_0^{n_1} + \sum_{i=0}^{n_2} R_{1,i} f_0^i$ of (16.6.1) is an irreducible W -poly of degree n_1 in f_0 with coefficients in $\mathbb{C}\{f_{-1}\}$ and with multiplicity n_1 at $0 \in \mathbb{C}^2$.

Remark. Note that an inequality in (16.6.2) is the necessary and sufficient condition for f_1 to be irreducible in $\mathbb{C}\{y, z\}$ with $f_1 \in$ the type [1]. So, $f_1 \stackrel{\text{multiseq}}{\sim} H_1$ where $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$.

[2] An algorithm for computing an irreducibility criterion for $f_2(y, z) \in$ the type[2]:
 $f_2 \in \mathbb{C}\{f_{-1}, f_0\}[f_1]$ is an irreducible W -poly of degree n_2 in f_1 with a coefficient of $f_1^{n_2-1}$ zero in $\mathbb{C}\{f_{-1}, f_0\}$, and $f_2 = f_2(y, z) \in$ the type[2] in the sense of Definition 2.5
 Note that $f = f(f_{-1}, f_0, f_1, f_2) \in$ the type $[\ell]$ with $\ell \geq 3$, as an element in $\mathbb{C}\{y, z\}$.

Then, $f_2 = f_1^{n_2} + \sum_{i=0}^{n_2-2} R_{2,i} f_1^i \in \mathbb{C}\{y, z\}[f_1]$ of (16.6.1) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying two properties (1) and (2):

(1) Each $R_{2,i} \in \mathbb{C}\{y\}[z]$ in f_2 of (16.6.1) satisfies (1a), (1b), (1c), and (1d) for $i=0, 1, \dots, n_2-2$.

(1a) For any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$, $\delta_1 > 0$ and $\delta_2 < n_1$.

(1b) Let \mathbb{N}_0^2 be two-dimensional cartesian product of \mathbb{N}_0 . For given integers $n_1, \alpha_{1,0,1}$ and a function θ_1 in $\text{Cond}[A]$ of the 1st type, define a function $\theta_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by

$$(16.6.3) \quad \theta_2(t_1, t_2) = t_2 \theta_1(\alpha_{1,0,1}) + n_1 \theta_1(t_1) = t_2 \alpha_{1,0,1} + n_1 t_1 \quad \text{for each } (t_1, t_2) \in \mathbb{N}_0^2.$$

Then, for any two nonzero monomials $y^{\alpha_1} z^{\alpha_2}$ and $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed,

$$(16.6.3-1) \quad \theta_2(\alpha_1, \alpha_2) = \theta_2(\delta_1, \delta_2) \text{ if and only if } \alpha_1 = \delta_1 \text{ and } \alpha_2 = \delta_2.$$

So, there exists a unique nonzero monomial $A_{2,i} y^{\alpha_{2,i,1}} z^{\alpha_{2,i,2}}$ in $R_{2,i}$

with a nonzero constant $A_{2,i}$ such that $\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2}) = \min\{\theta_2(\delta_1, \delta_2)\}$

for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $R_{2,i}$ with i fixed.

(1c) For all $i = 0, 1, \dots, n_2 - 2$,

$$(16.6.4) \quad \theta_2(\alpha_{2,i,k})_{k=1}^2 > (n_2 - i) n_1 \alpha_{1,0,1}.$$

(1d) For all $i = 0, 1, \dots, n_2 - 2$,

$$(16.6.5) \quad \gcd(n_2, \theta_2(\alpha_{2,0,k})_{k=1}^2) = 1 \quad \text{and} \quad \frac{\theta_2(\alpha_{2,i,1}, \alpha_{2,i,2})}{n_2 - i} \geq \frac{\theta_2(\alpha_{2,0,1}, \alpha_{2,0,2})}{n_2}.$$

Assuming that f_1 is an irreducible W -poly with $f_1 \in$ the type[1], then an inequality of (16.6.5) is the necessary and sufficient condition for f_2 to be irreducible in $\mathbb{C}\{y, z\}$ with $f_2 \in$ the type [2].

(2)(2a) $f_2 = f_2(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree $n_1 n_2$ in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n_1 n_2$ at $0 \in \mathbb{C}^2$, and $f_2 \in$ the type [2].

(2b) $f_2 = f_2(f_{-1}, f_0, f_1) \in \mathbb{C}\{f_{-1}, f_0\}[f_1]$ of (16.6.1) is an irreducible W -poly of degree n_2 in f_1 with coefficients in $\mathbb{C}\{f_{-1}, f_0\}$ and with multiplicity n_2 at $0 \in \mathbb{C}^3$.

Remark. By (1d), $f_2 \stackrel{\text{multiseq}}{\sim} H_2 = H_1^{n_2} + y^{\alpha_{2,0,1}} z^{\alpha_{2,0,2}}$ where $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$, if irreducible.

[3] An algorithm for computing an irreducibility criterion for $f_m(y, z) \in$ the type[m]:
 $f_m \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{m-2}\}[f_{m-1}]$ is an irreducible W -poly of degree n_m in f_{m-1}
 with a coefficient of $f_{m-1}^{n_m-1}$ zero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{m-2}\}$
 and $f_m = f_m(y, z) \in$ the type[m] in the sense of Definition 2.5

Let m be arbitrary with $3 \leq m \leq j$ and with $j \leq \ell - 1 \leq r - 1$. Note that $f = f(f_{-1}, f_0, \dots, f_m) \in$ the type $[\ell]$ with $\ell \geq m + 1$, as an element in $\mathbb{C}\{y, z\}$.

Then, $f_m = f_{m-1}^{n_m} + \sum_{i=0}^{n_m-2} R_{m,i} f_{m-1}^i \in \mathbb{C}\{y, z, f_1, \dots, f_{m-2}\}[f_{m-1}]$ of (16.6.1) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying two properties (1) and (2):

(1) Each $R_{m,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{m-2}]$ of f_m in (16.6.1) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, n_m - 2$.

(1a) For any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with $f_{-1} = y$ and $f_0 = z$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $k = 2, 3, \dots, m$.

(1b) By induction assumption on the integer $(m-1) \leq j$, suppose there exists a sequence $\{f_3, f_4, \dots, f_{m-1}\}$, each of which satisfies the same kind of properties and notations as f_2 does in An algorithm for finding an irreducibility criterion for $f_2(y, z) \in \text{the type}[2]$.

Inductively, define $\theta_m : \mathbb{N}_0^m \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^m is its m -dimensional cartesian product by

$$(16.6.6) \quad \theta_m(t_k)_{k=1}^m = t_m \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1} + n_{m-1} \theta_{m-1}(t_k)_{k=1}^{m-1} \quad \text{for each } (t_k)_{k=1}^m \in \mathbb{N}_0^m,$$

where recall by induction assumption that for a fixed i , $A_{m-1,i} \Pi_{k=1}^{m-1} f_{k-2}^{\alpha_{m-1,i,k}}$ is a unique nonzero monomial in $R_{m-1,i}$ with a nonzero constant $A_{m-1,i}$ such that

$$(16.6.7) \quad \theta_{m-1}(\alpha_{m-1,i,k})_{k=1}^{m-1} = \min\{\theta_{m-1}(\delta_k)_{k=1}^{m-1}\},$$

for any nonzero monomial $\Pi_{k=1}^{m-1} f_{k-2}^{\delta_k}$ in $R_{m-1,i}$.

Then, for any two nonzero monomials $\Pi_{k=1}^m f_{k-2}^{\alpha_k}$ and $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$ with i fixed,

$$(16.6.7-1) \quad \theta_m(\alpha_k)_{k=1}^m = \theta_m(\delta_k)_{k=1}^m \quad \text{if and only if } \alpha_k = \delta_k \text{ for } k = 1, 2, \dots, m.$$

So, there exists a unique nonzero-monomial $A_{m,i} \Pi_{k=1}^m f_{k-2}^{\alpha_{m,i,k}}$ in $R_{m,i}$ with a nonzero constant $A_{m,i}$ such that $\theta_m(\alpha_{m,i,k})_{k=1}^m = \min\{\theta_m(\delta_k)_{k=1}^m\}$

for any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $R_{m,i}$.

(1c) For all $i = 0, 1, \dots, n_m - 2$,

$$(16.6.8) \quad \theta_m(\alpha_{m,i,k})_{k=1}^m > (n_m - i) n_{m-1} \theta_{m-1}(\alpha_{m-1,0,k})_{k=1}^{m-1}.$$

(1d) For all $i = 0, 1, \dots, n_m - 2$,

$$(16.6.9) \quad \gcd(n_m, \theta_m(\alpha_{m,0,k})_{k=1}^m) = 1 \quad \text{and} \\ \frac{\theta_m(\alpha_{m,i,k})_{k=1}^m}{n_m - i} \geq \frac{\theta_m(\alpha_{m,0,k})_{k=1}^m}{n_m}.$$

Assuming that f_k is an irreducible W -poly with $f_k \in \text{the type}[k]$ for $1 \leq k \leq m-1$, then an inequality of (16.6.9) is the necessary and sufficient condition for f_m to be irreducible in $\mathbb{C}\{y, z\}$ with $f_m \in \text{the type}[m]$.

(2)(2a) For each m with $3 \leq m \leq j$, $f_m = f_m(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree $\Pi_{k=1}^m n_k$ in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $\Pi_{k=1}^m n_k$ at $0 \in \mathbb{C}^2$, and $f_m \in \text{the type}[m]$.

(2b) $f_m = f_m(f_{-1}, f_0, \dots, f_{m-1}) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{m-2}\}[f_{m-1}]$ of (16.6.1) is an irreducible W -poly of degree n_m in f_{m-1} with coefficients in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{m-2}\}$ and with multiplicity n_m at $0 \in \mathbb{C}^{m+1}$.

Remark. By (1d), $f_m \stackrel{\text{multiseq}}{\sim} H_m$ where $H_m = H_{m-1}^{n_m} + y^{\alpha_{m,0,1}} z^{\alpha_{m,0,2}} H_1^{\alpha_{m,0,3}} \dots H_{m-2}^{\alpha_{m,0,m}}$.

[4] An algorithm for computing an irreducibility criterion for $f(y, z) \in \text{the type}[j+1]$:
 $f \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}[f_j]$ is an irreducible W -poly of degree d_{j+1} in f_j
with a coefficient of $f_j^{d_{j+1}-1}$ zero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$
and $f = f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+1$ in the sense of Definition 2.5

Note that $j+1 \leq \ell$ where j was already given by (16.6.1) and that $n = \Pi_{i=1}^r q_i$.

Then, $f = f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ of (16.6.1) can be viewed as an element in $\mathbb{C}\{y, z\}$ if necessary, satisfying two properties (1) and (2):

(1) Each $S_{j+1,i} \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$ of f in (16.6.1) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_{j+1} - 2$.

(1a) For any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $S_{j+1,i}$ with $f_{-1} = y$ and $f_0 = z$,

$$(16.6.10) \quad \delta_1 > 0 \text{ and } \delta_k < n_{k-1} \text{ for } k = 2, 3, \dots, j+1.$$

(1b) Define a function $\theta_{j+1} : \mathbb{N}_0^{j+1} \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^{j+1} is its $(j+1)$ -dimensional cartesian product by

$$(16.6.11) \quad \theta_{j+1}(t_k)_{k=1}^{j+1} = t_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j + n_j \theta(t_k)_{k=1}^j \text{ for each } (t_k)_{k=1}^{j+1} \in \mathbb{N}_0^{j+1}.$$

Then, for any two nonzero monomials $\Pi_{k=1}^{j+1} f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $S_{j+1,i}$ with i fixed,

$$(16.6.11-1) \quad \theta_{j+1}(\beta_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1} \text{ if and only if } \beta_k = \delta_k \text{ for } k = 1, 2, \dots, j+1.$$

So, there exists a unique nonzero monomial $B_{j+1,i} \Pi_{k=1}^{j+1} f_{k-2}^{\beta_{j+1,i,k}}$ in $S_{j+1,i}$ with a nonzero constant $B_{j+1,i}$ such that $\theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} = \min\{\theta_{j+1}(\delta_k)_{k=1}^{j+1}\}$ for any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $S_{j+1,i}$ with i fixed.

(1c) For all $i = 0, 1, \dots, d_{j+1} - 2$,

$$(16.6.12) \quad \theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} > (d_{j+1} - i) n_j \theta_j(\alpha_{j,0,k})_{k=1}^j \text{ for all } i = 0, 1, \dots, d_{j+1} - 2.$$

(1d) For all $i = 0, 1, \dots, d_{j+1} - 2$,

$$(16.6.13) \quad \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) < d_{j+1} \text{ and } \frac{\theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1}}{d_{j+1} - i} \geq \frac{\theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}}{d_{j+1}}.$$

Then, either $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = 1$ or $1 < \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1})$.

(1d-1) Let $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = 1$. Assuming that $f_j \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly with $f_j \in \text{the type}[j]$, then an inequality of (16.6.13) is the necessary and sufficient condition for f to be irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{the type}[j+1]$. In this case, $f = f_{j+1}$ with $f_{j+1} \in \text{the type}[j+1]$, after replacing d_{j+1} , $B_{j+1,i}$, $\beta_{j+1,i,k}$ and $S_{j+1,i}$ by n_{j+1} , $A_{j+1,i}$, $\alpha_{j+1,i,k}$ and $R_{j+1,i}$ for $i = 0, 1, \dots, n_{j+1} - 2$ and $k = 1, 2, \dots, j+1$, respectively.

(1d-2) Let $1 < \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) < d_{j+1}$. Assuming that f_j is an irreducible W -poly with $f_j \in \text{the type}[j]$, then f is irreducible in $\mathbb{C}\{y, z\}$ with $f \in \text{the type}[\ell]$ with $\ell \geq j+2$. Note that $\ell \leq r$.

Therefore, if $f \in \text{the type}[\ell]$ with $\ell \geq j+2$, by using the induction method on the positive integer ℓ as we have used in (16.6.1), there exists a sequence of irreducible W -polys in z , $\{f_k \in \mathbb{C}\{y\}[z]: k=1, 2, \dots, j+p\}$ for some integer $p > 1$ such that $f_k \in \text{the type}[k]$ for $k = 1, 2, \dots, j+p$ and $f_{j+p} = f$, satisfying the same kind of the properties and notations in the necessary and sufficient condition for f_m to be an irreducible W -poly with $f_m \in \text{the type}[m]$ for $m = 1, 2, \dots, j+p$, by using the same kind of the statements as we have used in (1d) of [3] with finitely many times because $\ell \leq r$ and $\ell = j+p$.

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly of degree n in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_{j+1} \Pi_{k=1}^j n_k$ at $0 \in \mathbb{C}^2$. Also, either $f \in \text{the type}[j+1]$ or $f \in \text{the type}[\ell]$ with $\ell \geq j+2$.

(2b) $f = f(f_{-1}, f_0, \dots, f_j) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}[f_j]$ of (16.6.1) is an irreducible W -poly of degree d_{j+1} in f_j with a coefficient of $f_j^{d_{j+1}-1}$ zero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$ and with multiplicity d_{j+1} at $0 \in \mathbb{C}^{j+2}$.

§18. The proofs of Theorem 16.6 with Proposition 16.7 and Proposition 16.8

Proof of Theorem 16.6. For the proof of the theorem, let j be chosen arbitrary with $0 \leq j \leq \ell - 1$. Then, it suffices to show that the following are true:

(i) First, for each $m = 1, 2, \dots, j$, **An algorithm for finding an irreducibility criterion for $f_m(y, z) \in \text{the type}[m]$** which satisfies two properties (1) and (2), is true, as we have seen in the conclusion of this theorem.

(ii) Next, for $j + 1 \leq \ell$, **An algorithm for finding an irreducibility criterion for $f(y, z) \in \text{the type}[j + 1]$** which satisfies two properties (1) and (2), is true, as we have seen in the conclusion of this theorem.

Now, the proof of the theorem will be by induction on the integer j with $0 \leq j \leq \ell - 1$.

If $\ell = 1$, there is nothing to prove by Proposition 3.2 and Theorem 3.7 because f is irreducible in $\mathbb{C}\{y, z\}$ if and only if $\gcd(n, \alpha_0) = 1$ and $\frac{\alpha_i}{n-i} \geq \frac{\alpha_0}{n}$ for all $i = 1, 2, \dots, n - 2$.

If $\ell \geq 2$ and $j = 0, 1$, then it was already proved by an equation in (16.3.3) of Proposition 16.3 and Theorem 16.4.

For the induction proof with $\ell \geq 2$, suppose we have shown that the theorem holds on the integer j , with $1 \leq j \leq \ell - 1$. Then, we may assume without any need of proof that f can be uniquely written in the form

$$(16.6.14) \quad \begin{cases} f_k &= f_{k-1}^{n_k} + \sum_{i=0}^{n_k-2} R_{k,i} f_{k-1}^i \quad \text{for } k = 1, 2, \dots, j, \\ f &= f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i, \end{cases}$$

satisfying the same properties and notations as in conclusion of the theorem.

For the remaining proof of the theorem, on the integer $j + 1$, it suffices to show that there exists such a W -poly $f_{j+1} \in \mathbb{C}\{y\}[z]$ of (16.6.15) with $f_{j+1} \in \text{the type}[j + 1]$ satisfying the same kind of two properties as we have seen in **An algorithm for finding an irreducibility criterion for $f_j(y, z) \in \text{the type}[j]$** which satisfies two properties (1) and (2) in the theorem, and that $f = f(y, z, f_1, \dots, f_{j+1})$ of (16.6.15) can be constructed with $f(y, z) \in \text{the type}[j + 2]$, satisfying the same kind of two properties as we have seen in **An algorithm for finding an irreducibility criterion for $f(y, z) \in \text{the type}[j + 1]$** which satisfies two properties (1) and (2) in the conclusion of this theorem for $f = f(y, z, f_1, \dots, f_j)$, according to the following notations, if possible,

$$(16.6.15) \quad \begin{cases} f_{j+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i} f_j^i, \\ f &= f_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} S_{j+2,i} f_{j+1}^i, \end{cases}$$

where $n = d_{j+2} \prod_{k=1}^{j+1} n_k$ with $d_{j+2} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j + 1$.

Now, by the above induction assumption on the integer j and by (16.6.14), in preparation for applying Theorem 14.0 and Proposition 14.1 to the proof of this theorem, we need to substitute a new notation, and so let $\{g_k : k = 1, 2, \dots, j + 1\}$ be in $\mathbb{C}\{y, z\}$ such that $g_k = f_k$ for $1 \leq k \leq j$ and $g_{j+1} = f$ where $f = f(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$, and note that g_{j+1} of Theorem 14.0 may not be irreducible in $\mathbb{C}\{y, z\}$ by construction. Then, $\{g_k : k = 1, 2, \dots, j + 1\}$ of the above substitution satisfies the same kind of properties and notations as in the assumption of Theorem 14.0 and Proposition 14.1. For the proof of this theorem, we can use the same kind of conclusion as in Theorem 14.0 and Proposition 14.1 without any need of proof, which can be represented in the following sublemma.

Sublemma 16.6.1 Assumptions By the above induction assumption on the integer j and by (16.6.1), \dots , (16.6.13) in this theorem, for brevity of notation let $\{g_k : k = 1, 2, \dots, j + 1\}$ be in $\mathbb{C}\{y, z\}$ such that $g_k = f_k$ for $1 \leq k \leq j$ and $g_{j+1} = f$ where $f_k = f(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{k-1}]$ for $k = 1, 2, \dots, j$ and $f = f(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ by construction. Then, it is clear without any need of proof that for all $k = 1, 2, \dots, j + 1$, g_1, g_2, \dots, g_j are irreducible in $\mathbb{C}\{y, z\}$ but g_{j+1} may not be irreducible in $\mathbb{C}\{y, z\}$, which satisfy the same kind of properties and notations as in the assumption of Theorem 14.0 and Proposition 14.1.

Conclusions As a consequence, we can use the same kind of results and notations as in Theorem 14.0 and Proposition 14.1, as follows:

Let τ_{λ_j} be the composition of a finite number λ_j of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(f_j)$. For each $t = 1, 2, \dots, \lambda_j$, write $\tau_t = \pi_1 \circ \pi_2 \circ \dots \circ \pi_t : M^{(t)} \rightarrow \mathbb{C}^2$ where $\pi_i : M^{(i)} \rightarrow M^{(i-1)}$ is a blow-up of $M^{(i-1)}$ at some point of $M^{(i-1)}$ for $1 \leq i \leq t$ with $M^{(0)} = \mathbb{C}^2$. For brevity of notation, let $V^{(t)}(f_j)$ be the proper transform under τ_t for $1 \leq t \leq \lambda_j$.

Let $E^{(\lambda_j)} = \tau_{\lambda_j}^{-1}(0, 0)$, and let $E^{(\lambda_j)} = \cup E_i$, $1 \leq i \leq \lambda_j$, be the decomposition of $E^{(\lambda_j)}$ into irreducible components where each E_i is called an exceptional curve of the first kind.

Consequence(1) In order to study $V^{(t)}(f_j)$ under τ_t , we can find just one coordinate patch of the local coordinates for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$, where $1 \leq t \leq \lambda_j$ and $M^{(0)} = \mathbb{C}^2$.

Consequence(2) By Consequence(1), we can use the same τ_{λ_j} for the composition of the first finite number λ_j of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of both $V(f_j)$ and $V(f)$.

Consequence(3) In order to study each proper transform of both $V(f_j)$ and $V(f)$ under τ_t , without using a nonsingular change of coordinates, we can use the common one coordinate patch of the same local coordinates simultaneously, as it has been already used for each blow-up $\pi_t : M^{(t)} \rightarrow M^{(t-1)}$ in Consequence(1), where $1 \leq t \leq \lambda_j$.

After λ_j iterations of blow-ups, let $(v_{\lambda_j}, u_{\lambda_j})$ and $(v'_{\lambda_j}, u'_{\lambda_j})$ be the local coordinates for $M^{(\lambda_j)}$ where by Consequence(3) $\pi_{\lambda_j} : M^{(\lambda_j)} \rightarrow M^{(\lambda_j-1)}$ was defined to be the λ_j -th blow-up at some point of $M^{(\lambda_j-1)}$ with $u'_{\lambda_j} = 1/u_{\lambda_j}$ and $v'_{\lambda_j} = v_{\lambda_j} u_{\lambda_j}$. Note that $E_{\lambda_j} = \{v_{\lambda_j} = 0\} \cup \{v'_{\lambda_j} = 0\}$. For brevity of notation, write $(v, u) = (v_{\lambda_j}, u_{\lambda_j})$ and $(v', u') = (v'_{\lambda_j}, u'_{\lambda_j})$.

Note that $E_{\lambda_j} = \{(v, u) : v = 0\} \cup \{(v', u') : v' = 0\}$ is the j -th exceptional curve of the first kind. For notation, along E_{λ_j} , $(f_j \circ \tau_{\lambda_j})_{total} = 0$ and $(f_j \circ \tau_{\lambda_j})_{proper} = 0$ are called the local defining equations for the λ_j -th total transform of $f_j = 0$ and the λ_j -th proper transform of $f_j = 0$ under τ_{λ_j} , respectively.

At $(v, u + a) = (0, 0)$ along $v = 0$, $(f_j \circ \tau_{\lambda_j})_{total} = 0$ and $(f_j \circ \tau_{\lambda_j})_{proper} = 0$ can be written in the form, satisfying the following property:

$$(16.6.16) \quad \begin{aligned} (f_j \circ \tau_{\lambda_j})_{total} &= v^{n_j \theta_j(\alpha_{j,0,k})_{k=1}^j} (f_j \circ \tau_{\lambda_j})_{proper}, \\ (f_j \circ \tau_{\lambda_j})_{proper} &= (u + a + \varepsilon), \\ (\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \circ \tau_{\lambda_j})_{total} &= v^{\theta_{j+1}(\delta_k)_{k=1}^{j+1}} b(\delta_1, \dots, \delta_{j+1}), \end{aligned}$$

where a is a nonzero constant, ε is a nonunit along $v = 0$ and $b(\delta_1, \dots, \delta_{j+1})$ is a unit in $\mathbb{C}\{v, u + a\}$.

Moreover, as an application of the above consequences we have the following:

Consequence(4) By (16.6.12) or (16.6.14) and by the definition of a unique nonzero monomial $B_{j+1,i} \Pi_{k=1}^{j+1} f_{k-2}^{\beta_{j+1,i,k}}$ in $S_{j+1,i}$ with a constant $B_{j+1,i}$, it can be easily shown that $(S_{j+1,i} \circ \tau_{\lambda_j})_{total} = v^{\theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1}} b_{j+1,i}$ where $b_{j+1,i}$ is a unit in $\mathbb{C}\{v, u + a\}$.

Using $f = f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i$ in (16.6.14), then at $(v, u + a) = (0, 0)$ along $v = 0$, $(f \circ \tau_{\lambda_j})_{total} = 0$ and $(f \circ \tau_{\lambda_j})_{proper} = 0$ can be written as follows:

$$(16.6.17) \quad \begin{aligned} (f \circ \tau_{\lambda_j})_{total} &= v^{n_j d_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (f \circ \tau_{\lambda_j})_{proper}, \\ (f \circ \tau_{\lambda_j})_{proper} &= (u + a + \varepsilon)^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} W_{j+1,i} (u + a + \varepsilon)^i \quad \text{with} \\ W_{j+1,i} &= W_{j+1,i}(u, v) = b_{j+1,i} v^{M_{j+1,i}} \quad \text{and} \\ M_{j+1,i} &= \theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} - n_j(d_{j+1} - i) \theta_j(\alpha_{j,0,k})_{k=1}^j > 0, \end{aligned}$$

where each $b_{j+1,i}$ is a unit in $\mathbb{C}\{u+a, v\}$ for $0 \leq i \leq d_{j+1} - 2$, if exists, noting that $(S_{j+1,i} \circ \tau_{\lambda_j})_{total} = v^{n_j(d_{j+1}-i)} \theta_j(\alpha_{j,0,k})_{k=1}^j W_{j+1,i}$. \square

There is nothing to prove Sublemma 16.6.1 by Theorem 14.0 and Proposition 14.1.

In preparation for the induction proof of the theorem on the integer $(j+1)$, we must find the method for the construction of (f_{j+1}, f) of (16.6.15). For brevity of the proof, we may assume without loss of generality that $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) > 1$ and $j \leq \ell - 2$ from (16.6.14), which will be proved by three cases in the conclusion of the following sublemma.

Sublemma 16.6.2 Assumptions Suppose that the same assumption and notations of Sublemma 16.6.1 hold.

Conclusions For the proof of Theorem 16.6, it suffices to consider three cases, respectively. Moreover, for the induction proof of the theorem on the integer $j+1$, it remains to prove this theorem in Case(3) except for Case(1) and Case(2):

Case(1) If $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = 1$, we may assume that $f(y, z, f_1, \dots, f_j) = f_{j+1}(y, z, f_1, \dots, f_j)$ up to the notations, and so there is nothing to prove for the theorem because An algorithm for finding an irreducibility criterion for $f(y, z) \in \text{the type}[j+1]$ with $j+1 = \ell$ which satisfies two properties (1) and (2), is true, as we have seen in the conclusion of this theorem.

Case(2) If $j = \ell - 1$, we may assume that $f(y, z, f_1, \dots, f_j) = f_{j+1}(y, z, f_1, \dots, f_j)$ up to the notations, and so there is nothing to prove for the theorem because An algorithm for finding an irreducibility criterion for $f(y, z) \in \text{the type}[j+1]$ with $j+1 = \ell$ which satisfies two properties (1) and (2), is true, as in the conclusion of this theorem.

Case(3) After the proofs of Case(1) and Case(2) are done, in preparation for the induction proof of the theorem on the integer $j+1$, we may assume by Case(1) and Case(2) that

$$(16.6.18) \quad \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = \gcd(d_{j+1}, M_{j+1,0}) > 1 \quad \text{and} \quad j \leq \ell - 2,$$

where $M_{j+1,i} = \theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} - n_j(d_{j+1} - i)\theta_j(\alpha_{j,0,k})_{k=1}^j > 0$. \square

Proof of Sublemma 16.6.2. For the proof of the sublemma, it suffices to prove three cases, respectively.

The proof of Case(1). Let $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = 1$. Then, $\gcd(d_{j+1}, M_{j+1,0}) = 1$ where $M_{j+1,0} = \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1} - n_j d_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j > 0$ by (16.6.17).

So, $(f \circ \tau_{\lambda_j})_{proper}$ in (16.6.17) is irreducible in $\mathbb{C}\{u+a, v\}$ if and only if

$$(16.6.19) \quad \frac{M_{j+1,i}}{d_{j+1} - i} \geq \frac{M_{j+1,0}}{d_{j+1}} \quad \text{for } 0 \leq i \leq d_{j+1} - 2,$$

by Corollary 3.3. Also, an equality in (16.6.19) is equivalently rewritten as follows:

$$(16.6.20) \quad \frac{\theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1}}{d_{j+1} - i} \geq \frac{\theta_j(\beta_{j+1,0,k})_{k=1}^{j+1}}{d_{j+1}}.$$

First, note that $(f \circ \tau_{\lambda_j})_{proper} \stackrel{\text{multiseq}}{\sim} (u+a+\varepsilon)^{d_{j+1}} + v^{M_{j+1,0}}$ where $W_{j+1,0} = b_{j+1,0} v^{M_{j+1,0}}$ by (16.6.17), because $\gcd(d_{j+1}, M_{j+1,0}) = 1$ and $b_{j+1,0}$ is a unit in $\mathbb{C}\{u+a, v\}$.

Observe that if $M_{j+1,0} = 1$, that is, $(f \circ \tau_{\lambda_j})_{proper}$ has no singular point at $(u+a, v) = (0, 0)$, then in order to have a resolution for the curve $f = 0$ at $(0, 0)$ in the sense of Corollary 2.3, we need exactly one more exceptional curve which has three distinct intersection points with three components among additional exceptional curves including the exceptional curve $v = 0$ and the new proper transform, because the curve $v = 0$ and the last proper transform $(f \circ \tau_{\lambda_j})_{proper}$ in (16.6.17) meet tangentially at $(u+a, v) = (0, 0)$. Also, if $M_{j+1,0} > 1$, it is clear by Corollary 2.3 that $(f \circ \tau_{\lambda_j})_{proper}$ satisfies the same kind of properties as we have done in case that $M_{j+1,0} = 1$, as far as a resolution is concerned in the sense of Corollary 2.3. Now, replace d_{j+1} , $B_{j+1,i}$, $\beta_{j+1,i,k}$, $S_{j+1,i}$ by n_{j+1} , $A_{j+1,i}$, $\alpha_{j+1,i,k}$, $R_{j+1,i}$, respectively. Then, it is trivial that $f = f_{j+1}$ satisfies the desired properties up to the notations. Thus, the proof of Case(1) is done.

The proof of Case(2). If $j = \ell - 1$, then we will prove that $\gcd(d_{j+1}, M_{j+1,0}) = 1$. Assume the contrary, i.e., $\gcd(d_{j+1}, M_{j+1,0}) > 1$. Then, we would have three subcases (i), (ii) and (iii).

(i) $d_{j+1} < M_{j+1,0}$: Note that $1 < \gcd(d_{j+1}, M_{j+1,0}) < d_{j+1}$ and then $(f \circ \tau_{\lambda_j})_{proper} \in$ the type $[k]$ for some positive integer $k \geq 2$ by Hensel's lemma or Lemma 3.1 because $S_{j+1,i} = 0$ for $i = d_{j+1} - 1$, and so $f \in$ the type $[\ell']$ with $\ell' > \ell$. It would be impossible.

(ii) $d_{j+1} = M_{j+1,0}$: Then, $(f \circ \tau_{\lambda_j})_{proper}$ would not be irreducible in $\mathbb{C}\{u + a, v\}$ by Lemma 3.1, because $S_{j+1,i} = 0$ for $i = d_{j+1} - 1$.

(iii) $d_{j+1} > M_{j+1,0}$: It is enough to consider the following two cases.

(iiia) If $\gcd(d_{j+1}, M_{j+1,0}) < M_{j+1,0}$, then $(f \circ \tau_{\lambda_j})_{proper} \in$ the type $[k]$ for some positive integer $k \geq 2$ because $\gcd(d_{j+1}, M_{j+1,0}) > 1$, and so $f \in$ the type $[\ell']$ with $\ell' > \ell$. It would be impossible.

(iiib) If d_{j+1} is a positive multiple of $M_{j+1,0}$, then note that the last exceptional curve $v = 0$ and the proper transform $(f \circ \tau_{\lambda_j})_{proper} = 0$ meet tangentially and that $(f \circ \tau_{\lambda_j})_{proper} \in$ the type $[k]$ for $k \geq 1$, since $M_{j+1,0} > 1$. But, to resolve the total transform defined by $v(f \circ \tau_{\lambda_j})_{proper} = 0$ in the sense of Corollary 2.3, we need at least two more exceptional curves, each of which has three distinct intersection points with three components among additional exceptional curves including the exceptional curve $v = 0$ and the new proper transform because the curve $v = 0$ and the curve $(f \circ \tau_{\lambda_j})_{proper} = 0$ meet tangentially at $(u+a, v) = (0, 0)$. Therefore, $f \in$ the type $[\ell']$ with $\ell' > \ell$, which would imply a contradiction.

Thus, the proof of Case(2) is done, and so by the same kind of replacement as we have used in the proof of Case(1), f can be defined by f_{j+1} with the desired properties. Therefore, the proof of the sublemma is done.

The proof of Case(3). Using Case(1) and Case(2), there is nothing to prove. \square

Remark 16.6.2.1 for Sublemma 16.6.2. Hereafter, using Case(1), Case(2) and Case(3), in preparation for the induction proof of the theorem on the integer $j + 1$, we may assume without loss of generality that

$$(16.6.21.1) \quad \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = \gcd(d_{j+1}, M_{j+1,0}) > 1 \quad \text{and} \quad j \leq \ell - 2,$$

where $M_{j+1,i} = \theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} - n_j(d_{j+1} - i)\theta_j(\alpha_{j,0,k})_{k=1}^j > 0$.

(iii) Since f is irreducible in $\mathbb{C}\{y, z\}$, then by Hensel's lemma or Lemma 16.0, the equation $(f \circ \tau_{\lambda_j})_{proper}$ in (16.6.17) can be rewritten in the form

$$(16.6.21.2) \quad (f \circ \tau_{\lambda_j})_{proper} = [(u + a + \varepsilon)^{n_{j+1}} + \zeta v^{M'}]^{d_{j+2}} + \sum_{s,t \geq 0} a_{s,t} (u + a + \varepsilon)^s v^t,$$

where $d_{j+2} = \gcd(d_{j+1}, M_{j+1,0}) = \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) > 1$ with $d_{j+1} = n_{j+1}d_{j+2}$ and $M_{j+1,0} = M'd_{j+2}$ for some integers $n_{j+1} \geq 2$ and $M' \geq 1$, and ζ is a nonzero constant, and the $a_{s,t}$ are some nonzero constant such that $sM' + tn_{j+1} > n_{j+1}d_{j+2}M' = d_{j+1}M' = n_{j+1}M_{j+1,0}$ for $s \geq 0$ and $t \geq 0$.

Observe that $W_{j+1,d_j-n_{j+1}}$ is not zero, applying Hensel's lemma or Lemma 3.1 to the equation in (16.6.17) because $n_{j+1}(d_{j+2} - 1) = d_{j+1} - n_{j+1}$ in (16.6.21.2). \square

For the completeness of the proofs of this theorem, by the induction method on the integer j , $0 \leq j \leq \ell - 1$, the next aim is to construct three steps with proofs, denoted by Proposition 16.7(Step I), Proposition 16.8(Step II), and Proposition 16.9(Step III) called Theorem 16.6, by using the same kind of methods and notations as we have seen in Proposition 16.2(Step I), Proposition 16.3(Step II), and Theorem 16.4 called Proposition 16.4(Step III).

Proposition 16.7(Step I).

Assumptions By induction assumption on the integer j , suppose that (f_j, f) of (16.6.1) can be uniquely written in the form

$$(16.7.1) \quad \begin{cases} f_j &= f_{j-1}^{n_j} + \sum_{i=0}^{n_j-2} R_{j,i} f_{j-1}^i \in \mathbb{C}\{y, z, f_1, \dots, f_{j-2}\}[f_{j-1}] \\ f &= f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j], \end{cases}$$

satisfying the same properties and notations as in the conclusions of Theorem 16.6.

By induction assumption on the integer j , recall by (16.6.11-1) in the conclusion of this theorem that there exists a unique nonzero monomial

$$(16.7.2) \quad B_{j+1, d_{j+1}-n_{j+1}} \Pi_{k=1}^{j+1} f_{k-2}^{\beta_{j+1, d_{j+1}-n_{j+1}, k}}$$

in $S_{j+1, d_{j+1}-n_{j+1}}$ with a constant $B_{j+1, d_{j+1}-n_{j+1}}$ such that $\theta_{j+1}(\beta_{j+1, d_{j+1}-n_{j+1}, k})_{k=1}^{j+1} = \min\{\theta_{j+1}(\delta_k)_{k=1}^{j+1}\}$ for any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \in S_{j+1, d_{j+1}-n_{j+1}}$, if exists.

For the induction proof on the integer $j+1$, we may assume without proof that

$d_{j+2} = \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1, 0, k})_{k=1}^{j+1}) > 1$ with $d_{j+1} = n_{j+1} d_{j+2}$ and $j \leq \ell - 2$ as we have seen in Remark 16.6.2.1 for Sublemma 16.6.2.

Conclusions Then, (g_{j+1}, f) can be uniquely written in the following form:

$$(16.7.3) \quad \begin{cases} g_{j+1} &= f_j^{n_{j+1}} + \xi_{j+1} \Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k} \quad \text{with } f_{-1} = y \text{ and } f_0 = z, \\ f &= g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2, i} g_{j+1}^i, \end{cases}$$

where, considering y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at $0 \in \mathbb{C}^{j+2}$,

- (i) $n = d_{j+2} \Pi_{k=1}^{j+1} n_k$ with $d_{j+2} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j+1$, and $n = d_1$ if $j = 0$;
- (ii) $\sigma_k = \beta_{j+1, d_{j+1}-n_{j+1}, k}$ for $1 \leq k \leq j+1$ and $\xi_{j+1} = \frac{1}{d_{j+2}} B_{j+1, d_{j+1}-n_{j+1}}$ by (16.6.11-1);
- (iii) for each i with $0 \leq i \leq d_{j+2} - 1$, $T_{j+2, i} \in \mathbb{C}\{y, z, f_1, \dots, f_j\}$;
- (iv) $g_{j+1} = g_{j+1}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$;
- (v) $f = f(y, z, f_1, \dots, f_j, g_{j+1}) \in \mathbb{C}\{y, z, f_1, \dots, f_j\}[g_{j+1}]$,

satisfying two conditions, denoted by The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j+1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, each of which is represented respectively, as follows:

[1] The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j+1]$:

$g_{j+1} \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}[f_j]$ is an irreducible W-poly of degree n_{j+1} in f_j

with a coefficient of $f_j^{n_{j+1}-1}$ zero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$, and $g_{j+1}(y, z) \in \text{the type}[j+1]$ in the sense of Definition 2.5

Let j be arbitrary with $3 \leq j$, noting that $n = \Pi_{i=1}^r q_i$.

To find The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j+1]$, as an element in $\mathbb{C}\{y, z\}$ if necessary, it suffices to show that $g_{j+1} = f_j^{n_{j+1}} + \xi_{j+1} \Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k}$ of (16.7.3) satisfies two properties (1) and (2):

(1) Each $R_{j+1, 0} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$ of g_{j+1} in (16.7.3) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, n_{j+1} - 2$.

(1a) For a nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k}$ in g_{j+1} with $f_{-1} = y$ and $f_0 = z$,

$$(16.7.4) \quad \sigma_1 > 0 \quad \text{and} \quad \sigma_k < n_{k-1} \quad \text{for} \quad k = 2, 3, \dots, j+1.$$

(1b) Define a function $\theta_{j+1} : \mathbb{N}_0^{j+1} \rightarrow \mathbb{N}_0$ where \mathbb{N}_0^{j+1} is its $(j+1)$ -dimensional cartesian product by

$$(16.7.5) \quad \theta_{j+1}(t_k)_{k=1}^{j+1} = t_{j+1} \theta_j(\alpha_{j, 0, k})_{k=1}^j + n_j \theta(t_k)_{k=1}^j \quad \text{for each } (t_k)_{k=1}^{j+1} \in \mathbb{N}_0^{j+1}.$$

For any two nonzero monomials $\prod_{k=1}^{j+1} f_{k-2}^{\beta_k}$ and $\prod_{k=1}^{j+1} f_{k-2}^{\delta_k}$ where $\beta_1 > 0$, $\beta_k < n_{k-1}$ for $2 \leq k \leq j+1$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $2 \leq k \leq j+1$, we have

$$(16.7.6) \quad \theta_{j+1}(\beta_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1} \text{ if and only if } \beta_k = \delta_k \text{ for } k = 1, 2, \dots, j+1.$$

(1c) Then, g_{j+1} of (16.7.3) satisfies the following properties:

Let τ_{λ_j} be the composition of a finite number λ_j of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(f_j)$, as in Sublemma 16.6.1. After repeating the same number λ_j of blow-ups with the same local coordinates as we have used in the standard resolution of the singular point of $V(f_j)$, the local defining equation for the λ_j -th proper transform of the curve defined by $g_{j+1} = 0$ can be written in the form

$$(16.7.7) \quad (u + a + \varepsilon)^{n_{j+1}} + c_{j+1}v^{M'} \quad \text{with } M' = \theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j > 0$$

where $\sigma_k = \beta_{j+1, d_{j+1} - n_{j+1}, k}$ for $1 \leq k \leq j+1$.

from a uniquely defined nonzero monomial $B_{j+1, d_{j+1} - n_{j+1}} \prod_{k=1}^{j+1} f_{k-2}^{\beta_{j+1, d_{j+1} - n_{j+1}, k}}$ of $S_{j+1, d_{j+1} - n_{j+1}}$ as in $f(y, z, \dots, f_j) = f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1, i} f_j^i$ by induction assumption on the integer j and $c_{j+1} = c_{j+1}(u, v)$ is a unit in $\mathbb{C}\{u + a, v\}$ with $c_{j+1}(-a, 0) = \zeta$.

(1d) Also, an equation of (16.7.7) satisfies the following:

$$(16.7.8) \quad \gcd(n_{j+1}, \theta_{j+1}(\sigma_k)_{k=1}^{j+1}) = 1 \quad \text{with } \sigma_k = \beta_{j+1, d_{j+1} - n_{j+1}, k} \quad \text{for } 1 \leq k \leq j+1.$$

(2)(2a) $g_{j+1} = g_{j+1}(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z of multiplicity $\prod_{k=1}^{j+1} n_k$ at $0 \in \mathbb{C}^2$ with coefficients in $\mathbb{C}\{y\}$, and $g_{j+1} \in$ the type $[j+1]$.

(2b) $g_{j+1} = g_{j+1}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ of (16.7.3) is an irreducible W -poly in f_j with coefficients in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$ and with multiplicity n_{j+1} at $0 \in \mathbb{C}^{j+2}$.

[2] The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$:
 $f \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[g_{j+1}]$ is an irreducible W -poly of degree d_{j+2} in g_{j+1} with a coefficient of $g_{j+1}^{d_{j+2}-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$, and $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$

Note that $j+1 \leq \ell$ where j was already given by (16.6.1) and that $n = \prod_{i=1}^r q_i$.

To find the Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$, as an element in $\mathbb{C}\{y, z\}$ if necessary, it is enough to show that $f = g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2, i} g_{j+1}^i$ of (16.7.3) satisfies two properties (1) and (2): Note that either $\ell = j+2$ or $\ell > j+2$ and that $T_{j+2, d_{j+2}-1}$ may be nonzero.

(1) Each $T_{j+2, i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ of f in (16.7.8) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_{j+2} - 1$.

(1a) For any nonzero monomial $\prod_{k=1}^{j+2} f_{k-2}^{\delta_k}$ in $T_{j+2, i}$,

$$(16.7.9) \quad \delta_1 > 0 \quad \text{and} \quad \delta_k < n_{k-1} \quad \text{for } k = 2, 3, \dots, j+2.$$

In particular, if $i = d_{j+2} - 1$ for $T_{j+2, i}$, then $\delta_{j+2} \leq n_{j+1} - 2$.

(1b) Define a function $\bar{\theta}_{j+2} : \mathbb{N}_0^{j+2} \rightarrow \mathbb{N}_0$ by

$$(16.7.10) \quad \bar{\theta}_{j+2}(t_k)_{k=1}^{j+2} = t_{j+2} \theta_{j+1}(\sigma_k)_{k=1}^{j+1} + n_{j+1} \theta_{j+1}(t_k)_{k=1}^{j+1} \quad \text{for each } (t_k)_{k=1}^{j+2} \in \mathbb{N}_0^2.$$

For any two nonzero monomials $\prod_{k=1}^{j+2} f_{k-2}^{\gamma_k}$ and $\prod_{k=1}^{j+2} f_{k-2}^{\delta_k}$ in $T_{j+2, i}$,

$$(16.7.10^*) \quad \bar{\theta}_{j+2}(\gamma_k)_{k=1}^{j+2} = \bar{\theta}_{j+2}(\delta_k)_{k=1}^{j+2} \quad \text{if and only if } \gamma_k = \delta_k \text{ for } k = 1, 2, \dots, j+2.$$

So, there exists a unique nonzero monomial $C_{j+2, i} \prod_{k=1}^{j+2} f_{k-2}^{\beta_{j+2, i, k}}$ in $T_{j+2, i}$ with a nonzero constant $C_{j+2, i}$ such that $\bar{\theta}_{j+2}(\beta_{j+2, i, k})_{k=1}^{j+2} = \text{Min}\{\bar{\theta}_{j+2}(\delta_k)_{k=1}^{j+2}\}$ for any nonzero monomial $\prod_{k=1}^{j+2} f_{k-2}^{\delta_k}$ in $T_{j+2, i}$ with i fixed.

(1c) For all $i = 0, 1, \dots, d_{j+2} - 1$,

$$(16.7.11) \quad \bar{\theta}_{j+2}(\beta_{j+2,i,k})_{k=1}^{j+2} > (d_{j+2} - i)n_{j+1}\theta_{j+1}(\sigma_k)_{k=1}^{j+1}.$$

(1d) For all $i = 0, 1, \dots, d_{j+2} - 1$,

$$(16.7.12) \quad \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) \leq d_{j+2} \quad \text{and} \\ \frac{\bar{\theta}_{j+2}(\beta_{j+2,i,k})_{k=1}^{j+2}}{d_{j+2} - i} \geq \frac{\bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}}{d_{j+2}}.$$

Then, either $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = 1$ or $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2})$.

(1d-1) Let $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = 1$. Then f is irreducible in ${}_2\mathcal{O}_0$ with $f \in$ the type $[j+2]$ if and only if the inequality in (16.7.12) holds. and $f_1, f_2, \dots, f_j, g_{j+1}$ are irreducible in $\mathbb{C}\{y, z\}$ as above.

(1d-2) Let $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) \leq d_{j+2}$. If f is irreducible in ${}_2\mathcal{O}_0$ and $T_{j+2,d_{j+2}-1} = 0$, then $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) < d_{j+2}$, and so $f \in$ the type $[\ell]$ with $\ell \geq j+3$. But, if f is irreducible in ${}_2\mathcal{O}_0$ and $T_{j+2,d_{j+2}-1} \neq 0$, then $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2})$ may be equal to d_{j+2} .

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_{j+2}\Pi_{k=1}^{j+1}n_k$ at $0 \in \mathbb{C}^2$. Also, either $f \in$ the type $[j+2]$ or $f \in$ the type $[\ell]$ with $\ell \geq j+3$.

(2b) $f = f(y, z, f_1, \dots, f_j, g_{j+1}) \in \mathbb{C}\{y, z, f_1, \dots, f_j\}[g_{j+1}]$ of (16.7.8) is an irreducible W -poly of degree d_{j+2} in g_{j+1} with coefficients in $\mathbb{C}\{y, z, f_1, \dots, f_j\}$ and with multiplicity d_{j+2} at $0 \in \mathbb{C}^{j+3}$. \square

Remark 16.7.1. (1) Whenever g_{j+1} of (16.7.1) satisfies The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in$ the type $[j+1]$, then it is said that $g_{j+1}(y, z) \in$ the type $[j+1]$ in the sense of Definition 2.5.

(2) Assume that f of (16.7.1) satisfies The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$.

(i) If f satisfies $T_{j+2,d_{j+2}-1} = 0$ in addition, then it is said that f satisfies The Necessary Condition[A] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$.

(ii) If $T_{j+2,d_{j+2}-1} = 0$, and also $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = 1$, then it is said that $f(y, z) \in$ the type $[j+2]$ in the sense of Definition 2.5, noting that f of (16.7.1) satisfies The Necessary and Sufficient Condition[A] for $f(y, z) \in$ the type $[j+2]$ in the sense of Definition 2.5.

Proposition 16.8(Step II). Assumptions For the induction proof on the integer $j+1$, we may assume without loss of generality that $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) > 1$ and $j \leq \ell-2$ as we have seen in either Sublemma 16.6.2 or Sublemma 16.7.

Then, we may assume by Proposition 16.7 that (g_{j+1}, f) can be written as follows:

$$(16.8.1) \quad \begin{cases} g_{j+1} &= f_j^{n_{j+1}} + \xi_{j+1}\Pi_{k=1}^{j+1}f_{k-2}^{\sigma_k} \quad \text{with } f_{-1} = y \text{ and } f_0 = z, \\ f &= g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i}g_{j+1}^i, \end{cases}$$

$\sigma_k = \beta_{j+1,d_{j+1}-n_{j+1},k}$ for $1 \leq k \leq j+1$, satisfying two conditions as we have seen in Proposition 16.7, denoted by The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in$ the type $[j+1]$ and The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j+2$.

Conclusions The main aim is to construct a unique pair (f_{j+1}, f) such that (f_{j+1}, f) can be written in the form

$$(16.8.2) \quad \begin{cases} f_{j+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}f_j^i \quad \text{with } f_{-1} = y \text{ and } f_0 = z, \\ f &= f_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} S_{j+2,i}f_{j+1}^i, \end{cases}$$

where $y, z, f_1, \dots, f_{j+1}$ are considered as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+3} if necessary, satisfying the following properties:

(i) The first problem is how to construct $f_{j+1} = f_{j+1}(y, z)$ satisfying the condition in $\widehat{[1]}$ such that $f_{j+1}(y, z) \stackrel{\text{multiseq}}{\sim} g_{j+1}(y, z)$ under the standard resolutions.

[1] The Necessary and Sufficient Condition[A] for $f_{j+1}(y, z) \in \text{the type}[j+1]$:
 $f_{j+1} \in \mathbb{C}\{f_{-1}, \dots, f_{j-1}\}[f_j]$ is an irreducible W-poly of degree n_{j+1} in f_j with a coefficient of $f_j^{n_{j+1}-1}$ zero in $\mathbb{C}\{f_{-1}, \dots, f_{j-1}\}$, and $f_{j+1}(y, z) \in \text{the type}[j+1]$ in the sense of Definition 2.5

(ii) The second problem is to prove that $f = f(y, z, f_1, \dots, f_j, f_{j+1})$ satisfies the condition in $\widehat{[2]}$ which is defined by the same kind of property as $f(y, z, f_1, \dots, f_j, g_{j+1})$ have done in The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$.

[2] The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$:
 $f \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[f_{j+1}]$ is an irreducible W-poly of degree d_{j+2} in f_{j+1} with a coefficient of $f_{j+1}^{d_{j+2}-1}$ zero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$

For the construction of a pair (f_{j+1}, f) in (16.8.2), it suffices to consider the following two cases, depending on the fact that $T_{j+2, d_{j+2}-1}$ in a pair (g_{j+1}, f) of (16.8.1) is zero or not. For brevity of notations, let $h_1 = g_{j+1}$ and $T_{j+2, i}^{(1)} = T_{j+2, i}$.

Case(1): Let $T_{j+2, d_{j+2}-1}^{(1)} = 0$. By (16.8.1), put $f_{j+1} = g_{j+1}$, $R_{j+1, 0} = \xi_{j+1} \prod_{k=1}^{j+1} f_{k-2}^{\sigma_k}$, and $S_{j+2, i} = T_{j+2, i}^{(1)}$ for $0 \leq i \leq d_{j+2} - 2$. Then, it is clear by Proposition 16.7 that (f_{j+1}, f) of the main aim and (g_{j+1}, f) of (16.8.1) are the same pairs in the sense of Definition 16.2.2.

Case(2): Let $T_{j+2, d_{j+2}-1}^{(1)} \neq 0$. Then, there is a sequence of pairs of W-polys in z , $\{(h_p, f) : p = 1, 2, \dots\}$, such that $h_1 = g_{j+1}$ and $(f_{j+1}, f) = (h_{\nu+1}, f) = (h_{\nu+2}, f) = \dots$ for some integer $\nu \leq \frac{n_{j+1}+1}{2}$, each pair of which can be written in the form

$$(16.8.3) \quad \begin{cases} h_1 &= f_j^{n_{j+1}} + \xi_{j+1} \prod_{k=1}^{j+1} f_{k-2}^{\sigma_k} = f_j^{n_{j+1}} + R_{j+1, 0}^{(1)}, \\ f &= h_1^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2, i}^{(1)} h_1^i, \end{cases}$$

and for $p = 2, 3, \dots$

$$(16.8.4) \quad \begin{cases} h_p &= h_{p-1} + \frac{1}{d_{j+2}} T_{j+2, d_{j+2}-1}^{(p-1)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1, i}^{(p)} f_j^i, \\ f &= h_p^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2, i}^{(p)} h_p^i, \end{cases}$$

with $T_{j+2, d_{j+2}-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{j+2, d_{j+2}-1}^{(\nu+1)} = T_{j+2, d_{j+2}-1}^{(\nu+2)} = \dots = 0$,

where, considering $f_{-1}, f_0, \dots, f_j, h_p$ as independent complex $(j+3)$ -variables at $0 \in \mathbb{C}^{j+3}$,

- (i) $n = d_{j+2} \prod_{k=1}^{j+1} n_k$ with $d_{j+2} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j+1$, and $n = d_1$ if $j = 0$;
- (ii) $R_{j+1, i}^{(p)} = R_{j+1, i}^{(p)}(f_{-1}, f_0, \dots, f_{j-1}) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$ for $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$;
- (iii) $T_{j+2, i}^{(p)} = T_{j+2, i}^{(p)}(f_{-1}, f_0, \dots, f_j) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$ for $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$;
- (iv) $h_p = h_p(f_{-1}, f_0, \dots, f_j) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}[f_j]$ for $p \geq 1$;
- (v) $f = f(f_{-1}, f_0, \dots, f_j, h_p) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[h_p]$,

satisfying two conditions, denoted by The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[j+1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, each of which is represented respectively, as follows:

In more detail, for any fixed (h_p, f) of (16.8.4), h_p of (h_p, f) satisfies The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[j+1]$ and f of (h_p, f) satisfies The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$. In particular, if p_1 is a given integer with $p_1 \geq \frac{n_{j+1}+1}{2}$, define $(f_{j+1}, f) = (h_{p_1}, f)$ by Definition 16.2.2. Then, f_{j+1} of (f_{j+1}, f) of (16.8.2) satisfies The Necessary and Sufficient Condition[A] for $f_{j+1}(y, z) \in \text{the type}[j+1]$

in $\widehat{[1]}$, and f of (f_{j+1}, f) of (16.8.2) satisfies The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j + 2$ in $\widehat{[2]}$ by the same way as (h_p, f) of (16.8.4) does up to the change of notations: Recall that $f_{-1} = y$ and $f_0 = z$.

[1] The Necessary and Sufficient Condition[A] for $h_p(y, z) \in$ the type $[j + 1]$:
 $h_p \in \mathbb{C}\{f_{-1}, \dots, f_{j-1}\}[f_j]$ is an irreducible W-poly of degree n_{j+1} in f_j with a coefficient of $f_j^{n_{j+1}-1}$ zero in $\mathbb{C}\{f_{-1}, \dots, f_{j-1}\}$, and $h_p \in$ the type $[j+1]$ in the sense of Definition 2.5

Let j be arbitrary with $j \geq 0$, noting that $n = d_{j+2} \prod_{k=1}^{j+1} n_k$ with $d_{j+2} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j + 1$, and $n = d_1$ if $j = 0$.

To find The Necessary and Sufficient Condition[A] for $h_p(y, z) \in$ the type $[j + 1]$, it suffices to show that $h_p = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(p)} f_j^i$ of (16.8.4) satisfies two properties (1) and (2):

(1) Let p be fixed with $p \geq 1$. Each $R_{j+1,i}^{(p)} \neq 0$ satisfies the properties (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, n_{j+1} - 2$. Also, for each $p \geq 1$, $h_p \stackrel{\text{multiseq}}{\sim} h_1$ and $h_p \in \mathbb{C}\{y\}[z]$ is an irreducible W-poly in z of multiplicity $\prod_{k=1}^{j+1} n_k$.

For notation, it may be said that for each $p \geq 1$, h_p satisfies The Necessary and Sufficient Condition[A] for $h_p(y, z) \in$ the type $[j + 1]$.

(1a) For any nonzero monomial $\prod_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $R_{j+1,i}^{(p)}$,

$$(16.8.5) \quad \delta_1 > 0 \text{ and } \delta_k < n_{k-1} \text{ for } k = 2, 3, \dots, j + 1.$$

(1b) For any two nonzero monomials $\prod_{k=1}^{j+1} f_{k-2}^{\alpha_k}$ and $\prod_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $R_{j+1,i}^{(p)}$,

$$(16.8.6) \quad \theta_{j+1}(\alpha_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1} \text{ if and only if } \alpha_k = \delta_k \text{ for } k = 1, 2, \dots, j + 1.$$

So, there exists a unique nonzero monomial $A_{j+1,i}^{(p)} \prod_{k=1}^{j+1} f_{k-2}^{\alpha_{j+1,i,k}^{(p)}}$ in $R_{j+1,i}^{(p)}$ with a constant $A_{j+1,i}^{(p)}$ such that $\theta_{j+1}(\alpha_{j+1,i,k}^{(p)})_{k=1}^{j+1} = \min\{\theta_{j+1}(\delta_k)_{k=1}^{j+1}\}$ for any nonzero monomial $\prod_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $R_{j+1,i}^{(p)}$.

(1c) For all $i = 0, 1, \dots, n_{j+1} - 2$,

$$(16.8.7) \quad \theta_{j+1}(\alpha_{j+1,i,k}^{(p)})_{k=1}^{j+1} > (n_{j+1} - i) n_j \theta_j(\alpha_{j,0,k})_{k=1}^j.$$

(1d) For all $i = 0, 1, \dots, n_{j+1} - 2$ and for $k = 1, 2, \dots, j + 1$,

$$(16.8.8) \quad \gcd(n_{j+1}, \theta_{j+1}(\alpha_{j+1,0,k}^{(p)})_{k=1}^{j+1}) = 1 \text{ with } \sigma_k = \alpha_{j+1,0,k}^{(p)},$$

$$\frac{\theta_{j+1}(\alpha_{j+1,i,k}^{(p)})_{k=1}^{j+1}}{n_{j+1} - i} \geq \frac{\theta_{j+1}(\alpha_{j+1,0,k}^{(p)})_{k=1}^{j+1}}{n_{j+1}}.$$

(2)(2a) For each $p \geq 1$, $h_p = h_p(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W-poly in z with coefficients in $\mathbb{C}\{y\}$ and with $h_p \stackrel{\text{multiseq}}{\sim} h_1 = g_{j+1}$, and $h_p \in$ the type $[j + 1]$.

(2b) $h_p = h_p(f_{-1}, f_0, \dots, f_j) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}[f_j]$ of (16.8.4) is an irreducible W-poly in f_j with coefficients in $\mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$ and with multiplicity n_{j+1} at $0 \in \mathbb{C}^{j+2}$.

[2] The Necessary Condition[B] for $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j + 2$:
 $f \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[h_p]$ is an irreducible W-poly of degree d_{j+2} in h_p with a coefficient of $h_p^{d_{j+2}-1}$ either zero or nonzero in $\mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$, and $f(y, z) \in$ the type $[\ell]$ with $\ell \geq j + 2$

Note that $j + 1 \leq \ell$ where j was already given by (16.6.1) and that $n = \prod_{i=1}^r q_i$.

To find The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, it suffices to show that $f = h_p^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i}^{(p)} h_p^i$, of (16.8.4) satisfies two properties (1) and (2): Note that either $\ell = j+2$ or $\ell > j+2$ and that $T_{j+2,d_{j+2}-1}^{(p+1)}$ may be nonzero.

(1) Each $T_{j+2,i}^{(p)} \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$ of f in (16.8.4) satisfies (1a), (1b), (1c) and (1d) for $i = 0, 1, \dots, d_{j+2} - 1$.

(1a) For any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\gamma_k}$ in $T_{j+2,i}^{(p)}$,

$$(16.8.9) \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_k < n_{k-1} \quad \text{for } k = 2, 3, \dots, j+2.$$

In particular, if $i = d_{j+2} - 1$ for $T_{j+2,i}^{(p)}$ then $\gamma_{j+2} \leq n_{j+1} - 2$.

(1b) Define $\bar{\theta}_{j+2}(t_k)_{k=1}^{j+2} = t_{j+2}\theta_{j+1}(\sigma_k)_{k=1}^{j+1} + n_{j+1}\theta_{j+1}(t_k)_{k=1}^{j+1}$ for any $(t_k)_{k=1}^{j+2} \in N_0^{j+2}$ by the same way as we have seen in Proposition 16.7, (1b) of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$.

For any two nonzero monomials $\Pi_{k=1}^{j+2} f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^{j+2} f_{k-2}^{\gamma_k}$ in $T_{j+2,i}^{(p)}$,

$$(16.8.10) \quad \bar{\theta}_{j+2}(\beta_k)_{k=1}^{j+2} = \bar{\theta}_{j+2}(\gamma_k)_{k=1}^{j+2} \text{ if and only if } \beta_k = \gamma_k \text{ for } k = 1, 2, \dots, j+2.$$

So, there is a unique nonzero monomial $C_{j+2,i}^{(p)} \Pi_{k=1}^{j+2} f_{k-2}^{\beta_{j+2,i,k}^{(p)}}$ in $T_{j+2,i}^{(p)}$ with a constant $C_{j+2,i}^{(p)}$ such that $\bar{\theta}_{j+2}(\beta_{j+2,i,k}^{(p)})_{k=1}^{j+2} = \min\{\bar{\theta}_{j+2}(\gamma_k)_{k=1}^{j+2}\}$ for any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\gamma_k}$ in $T_{j+2,i}^{(p)}$.

(1c) For all $i = 0, 1, \dots, d_{j+2} - 1$,

$$(16.8.11) \quad \bar{\theta}_{j+2}(\beta_{j+2,i,k}^{(p)})_{k=1}^{j+2} > (d_{j+2} - i)n_{j+1}\theta_{j+1}(\sigma_k)_{k=1}^{j+1}.$$

(1d) For all $i = 0, 1, \dots, d_{j+2} - 1$,

$$(16.8.12) \quad \begin{aligned} \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2}) &\leq d_{j+2}, \\ \frac{\bar{\theta}_{j+2}(\beta_{j+2,i,k}^{(p)})_{k=1}^{j+2}}{d_{j+2} - i} &\geq \frac{\bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2}}{d_{j+2}}. \end{aligned}$$

Then, either $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2}) = 1$ or $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2})$.

(1d-1) Let $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2}) = 1$. Then, f is irreducible in ${}_2\mathcal{O}_0$ if and only if the inequality in (16.8.12) holds. In this case, $f \in \text{the type } [j+2]$, but note that $T_{j+2,d_{j+2}-1}^{(p)}$ may not be zero where

$$(16.8.13) \quad h_p = h_{p-1} + \frac{1}{d_{j+2}} T_{j+2,d_{j+2}-1}^{(p-1)} \quad \text{and} \quad f = h_p^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i}^{(p)} h_p^i.$$

(1d-2) Let $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(p)})_{k=1}^{j+2}) \leq d_{j+2}$. There is a positive integer ν with $\nu \leq \frac{n_{j+1}+1}{2}$ such that $T_{j+2,d_{j+2}-1}^{(\nu+1)} = 0$ and $T_{j+2,d_{j+2}-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu$. In this case, $f \in \text{the type } [\ell]$ with $\ell \geq j+3$ and note that

$$(16.8.14) \quad 1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(\nu+1)})_{k=1}^{j+2}) < d_{j+2}.$$

(2)(2a) $f = f(y, z) \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z with coefficients in $\mathbb{C}\{y\}$ and with multiplicity $n = d_{j+2}\Pi_{k=1}^{j+1} n_k$ at $0 \in \mathbb{C}^2$. Also, either $f \in \text{the type } [j+2]$ or $f \in \text{the type } [\ell]$ with $\ell \geq j+3$.

(2b) $f = f(f_{-1}, f_0, \dots, f_j, h_p) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[h_p]$ of (16.8.4) is an irreducible W -poly in h_p with coefficients in $\mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$ and with multiplicity d_{j+2} at $0 \in \mathbb{C}^{j+3}$.

[3] The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$:
 $f \in \mathbb{C}\{f_{-1}, \dots, f_j\}[f_{j+1}]$ is an irreducible W -poly of degree d_{j+2} in f_{j+1} with a coefficient of $f_{j+1}^{d_{j+2}-1}$ zero in $\mathbb{C}\{f_{-1}, \dots, f_j\}$, and $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$

Note that $j + 1 \leq \ell$ where j was already given by (16.6.1) and that $n = \Pi_{i=1}^r q_i$. \square

Remark 16.8.0. (i) For notation, if $p = \nu + 1$, it is said that $f = f(f_{-1}, f_0, \dots, f_j, h_p) \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[h_p]$ satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ because h_p of (h_p, f) in (16.8.4) satisfies The Necessary Condition and sufficient condition for $h_p \in \text{the type}[j + 1]$. Assuming that $p = \nu + 1$, if $\gcd(d_{j+2}, \theta_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = 1$, then The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ may be replaced by The Necessary and sufficient Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell = j + 2$.

(ii) Assuming that $\gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) = 1$, $\gcd(d_{j+2}, \theta_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = 1$ with $p = \nu + 1$, and $\gcd(n_j, \theta_j(\alpha_{j,0,k})_{k=1}^j) = 1$ for each $j = 1, 2, \dots, m$, then note that an equality in (16.6.9), an equality in (16.8.8) and an equality in (16.8.12) are the necessary and sufficient condition for f to be irreducible in $\mathbb{C}\{y, z\}$.

(iii) As soon as the construction in Case(1) and Case(2) is done, then f of (f_{j+1}, f) in (16.8.2) satisfies the same kind of condition for The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$, as f of (f_j, f) in (16.7.1) does for The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 1$, as we have seen in the proof of the theorem.

Corollary 16.8.1. For a given integer $j + 1$ in the theorem, we can prove by Theorem 15.4, Proposition 16.7 and Proposition 16.8 that there exists a unique sequence of irreducible W -polys in z , $\{f_0 = z, f_1, \dots, f_{j+1}\}$ with $f_k \in \mathbb{C}\{y\}[z]$ and $f_k \in \text{the type}[k]$ for $1 \leq k \leq j + 1$ and $f_{j+1} \neq f$, satisfying the desired properties and notations as in the conclusion of the theorem. \square

Proposition 16.9(Step III). Assumptions For the induction proof on the integer $(j + 1)$ of this proposition, we may assume by the induction on the integer $(j + 1)$ that the assumptions and the conclusions of Proposition 16.7(Step(I)) and Proposition 16.8(Step(II)) have been already shown.

Conclusions We are going to prove the remaining part of Theorem 16.6, equivalently, the following:

Step III-1 We prove by Step I and Step II that for a given integer $j + 1 \leq \ell - 1$ in this theorem we can construct a sequence of irreducible W -polys in z , $\{f_0 = z, f_1, \dots, f_{j+1}\}$ with $f_k \in \mathbb{C}\{y\}[z]$ and $f_k \in \text{the type}[k]$ for $1 \leq k \leq j + 1$ and $f_{j+1} \neq f$, such that (i) each f_k satisfies An algorithm for finding an irreducibility criterion for $f_k(y, z) \in \text{the type}[k]$ and (ii) f satisfies An algorithm for finding an irreducibility criterion for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ as we have seen in the conclusion of the theorem, respectively.

Step III-2 A sequence of irreducible W -polys satisfying the desired properties and notations as in Step III-1 must be unique. \square

The Proof of Proposition 16.7. We prove that g_{j+1} of (g_{j+1}, f) in (16.7.3) satisfies The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j + 1]$ and that f of (g_{j+1}, f) in (16.7.3) satisfies The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$, respectively.

The proof of The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j + 1]$:

(1)(1a) The proof is clear by (16.6.10).

(1b) Let $\Pi_{k=1}^{j+1} f_{k-2}^{\beta_k}$ and $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ be arbitrary nonzero monomials in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$ where $\beta_1 > 0$, $\beta_k < n_{k-1}$ for $2 \leq k \leq j + 1$, $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $2 \leq k \leq j + 1$. Then, note by the definition of θ_{j+1} that $\theta_{j+1}(\beta_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1}$ if and only if

$$(16.7.13) \quad \beta_{j+1}\theta_j(\alpha_{j,0,k})_{k=1}^j + n_j\theta_j(\beta_k)_{k=1}^j = \delta_{j+1}\theta_j(\alpha_{j,0,k})_{k=1}^j + n_j\theta_j(\delta_k)_{k=1}^j.$$

Since $\gcd(n_j, \theta_j(\alpha_{j,0,k})_{k=1}^j) = 1$, $0 \leq \beta_{j+1} < n_j$ and $0 \leq \delta_{j+1} < n_j$, then $(\beta_{j+1} - \delta_{j+1})\theta_j(\alpha_{j,0,k})_{k=1}^j = n_j(\theta_j(\delta_k)_{k=1}^j - \theta_j(\beta_k)_{k=1}^j)$ if and only if $\beta_{j+1} = \delta_{j+1}$ and $\theta_j(\beta_k)_{k=1}^j = \theta_j(\delta_k)_{k=1}^j$. Next, $\theta_j(\beta_k)_{k=1}^j = \theta_j(\delta_k)_{k=1}^j$ if and only if $\beta_j\theta_{j-1}(\alpha_{j-1,0,k})_{k=1}^{j-1} + n_{j-1}\theta_{j-1}(\beta_k)_{k=1}^{j-1} = \delta_j\theta_{j-1}(\alpha_{j-1,0,k})_{k=1}^{j-1} + n_{j-1}\theta_{j-1}(\delta_k)_{k=1}^{j-1}$ by the definition of θ_{j-1} .

Since $\gcd(n_{j-1}, \theta_{j-1}(\alpha_{j-1,0,k})_{k=1}^{j-1}) = 1$, $0 \leq \beta_j < n_{j-1}$ and $0 \leq \delta_j < n_{j-1}$, then $(\beta_j - \delta_j)\theta_{j-1}(\alpha_{j-1,0,k})_{k=1}^{j-1} = n_{j-1}(\theta_{j-1}(\delta_k)_{k=1}^{j-1} - \theta_{j-1}(\beta_k)_{k=1}^{j-1})$ if and only if $\beta_j = \delta_j$ and $\theta_{j-1}(\beta_k)_{k=1}^{j-1} = \theta_{j-1}(\delta_k)_{k=1}^{j-1}$.

Continuing the above process inductively, it can be easily shown that $\theta_{j+1}(\beta_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1}$ if and only if for each $k = 1, 2, \dots, j+1$, $\beta_k = \delta_k$, by using the fact that $\gcd(n_p, \theta_p(\alpha_{j,0,k})_{k=1}^p) = 1$ for $p = 1, 2, \dots, j$. Thus the proof of (1b) is done.

(1c) For brevity of notation let $\{g_k : k = 1, 2, \dots, j+1\}$ be in $\mathbb{C}\{y, z\}$ such that $g_k = f_k$ for $1 \leq k \leq j$ where $f_k = f(y, z, f_1, \dots, f_{k-1}) \in \mathbb{C}\{y, z, f_1, \dots, f_{k-2}\}[f_{k-1}]$ for $k = 1, 2, \dots, j$ was already defined in this theorem and $g_{j+1} = g_{j+1}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ was already constructed by an equation in (16.7.1). Then, it is clear without any need of proof that g_1, g_2, \dots, g_j are irreducible elements in $\mathbb{C}\{y, z\}$, which satisfy the same kind of properties and notations as in the assumption of Theorem 14.0 and Proposition 14.1. In order to apply Sublemma 16.6.1 or Theorem 14.0 to a sequence $\{g_k : k = 1, 2, \dots, j+1\}$, it suffices to show that g_{j+1} satisfies the following inequality:

$$(16.7.14) \quad \frac{\theta_{j+1}(\sigma_k)_{k=1}^{j+1}}{n_{j+1}} > n_j \theta_j(\alpha_{j,0,k})_{k=1}^j.$$

which is equivalently rewritten as follows:

$$(16.7.15) \quad \frac{\theta_{j+1}(\beta_{j+1,d_{j+1}-n_{j+1},k})_{k=1}^{j+1}}{d_{j+1} - (d_{j+1} - n_{j+1})} > n_j \theta_j(\alpha_{j,0,k})_{k=1}^j.$$

Since the inequality in (16.7.15) was already proved by (16.6.12) in the induction assumption of this theorem, there is nothing to prove for the inequality in (16.7.14). Following the same method and notations as we have used in the conclusion of Sublemma 16.6.1, let τ_{λ_j} be the composition of a finite number λ_j of successive blow-ups which is needed only to get the standard resolution of the singular point of $V(f_j)$.

After the same number λ_j of blow-ups with the same local coordinates as we have done in (16.6.17) and (16.7.7), the local defining equation $(g_{j+1} \circ \tau_{\lambda_j})_{proper}$ for the λ_j -th proper transform of the curve defined by $g_{j+1} = 0$ is given analytically by

$$(16.7.16) \quad \begin{aligned} (g_{j+1} \circ \tau_{\lambda_j})_{total} &= v^{n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (g_{j+1} \circ \tau_{\lambda_j})_{proper} \\ (g_{j+1} \circ \tau_{\lambda_j})_{proper} &= (u + a + \varepsilon)^{n_{j+1}} + c_{j+1}(u, v) v^{M'}, \end{aligned}$$

where a is a nonzero constant, ε is a nonunit along $v = 0$ and $c_{j+1}(u, v)$ is a unit in $\mathbb{C}\{u+a, v\}$ with $c_{j+1}(-a, 0) = \zeta$ and $M' = \theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_{j+1} n_j \theta_j(\alpha_{j,0,k})_{k=1}^j$.

(1d) It is clear by (16.6.17) and (16.6.21.2) that $\gcd(n_{j+1}, \theta_{j+1}(\sigma_k)_{k=1}^{j+1}) = \gcd(n_{j+1}, M') = 1$ and

$$(16.7.17) \quad \begin{aligned} M' &= \theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j \\ &= \theta_{j+1}(\beta_{j+1,d_{j+1}-n_{j+1},k})_{k=1}^{j+1} - n_j (d_{j+1} - (d_{j+1} - n_{j+1})) \theta_j(\alpha_{j,0,k})_{k=1}^j \\ &= M_{j+1,d_{j+1}-n_{j+1}}. \end{aligned}$$

(2)(2a) Since $\gcd(n_{j+1}, M') = 1$ with $M' = M_{j+1,d_{j+1}-n_{j+1}}$, then it is trivial by (16.7.16) that $(g_{j+1} \circ \tau_{\lambda_j})_{proper}$ is irreducible in $\mathbb{C}\{u+a, v\}$, and so $g_{j+1} \in \mathbb{C}\{y\}[z]$ is an irreducible W -poly in z by construction as we have done in (16.7.1) and Theorem 15.4. Now, for any given nonzero monomial $\prod_{k=1}^{j+1} f_{k-2}^{\beta_{j+1,i,k}}$ in $S_{j+1,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, apply (ii) of Theorem 15.2 to $f = f_j^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} S_{j+1,i} f_j^i$. Then, $S_{j+1,i} \in \mathbb{C}\{y\}[z]$ has a multiplicity

$m_i \geq (d_{j+1} - i) \prod_{k=1}^j n_k$ at $0 \in \mathbb{C}^2$ for each i . If $i = d_{j+1} - n_{j+1}$, then $m_i \geq \prod_{k=1}^{j+1} n_k$, and so the proof is done because $f_j \in \mathbb{C}\{y, z\}$ has a multiplicity $\prod_{k=1}^j n_k$ at $0 \in \mathbb{C}^2$.

(2b) Observe that $0 < n_1 < \alpha_{1,0,1}$ and $2 \leq n_k$ for $1 \leq k \leq j+2$, and $\sigma_1 > 0$ and $0 \leq \sigma_k < n_{k-1}$ for $2 \leq k \leq j+1$. Since g_{j+1} is irreducible in $\mathbb{C}\{y, z\}$, then by Lemma 3.1 $g_{j+1}(y, z)$ of (16.7.1) can be represented in the form

$$(16.7.18) \quad g_{j+1} = (z^{n_1} + \xi y^{\alpha_{1,0,1}})^d + \sum_{p,q \geq 0} c_{p,q} y^p z^q \quad \text{with } n_1 p + \alpha_{1,0,1} q > n_1 \alpha_{1,0,1} d,$$

where the $c_{p,q}$ are some nonzero complex numbers, and $p \geq 0$ and $q \geq 0$ are integers such that $n_1 p + \alpha_{1,0,1} q > n_1 \alpha_{1,0,1} d$ with $d = n_2 n_3 \cdots n_{j+1}$, and ξ is a nonzero number.

Whenever y, z, f_1, \dots, f_j is viewed as independent complex $(j+2)$ -variables, in order to prove for $g_{j+1} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ to have multiplicity n_{j+1} at the origin in \mathbb{C}^{j+2} , it suffices to observe the following computation:

$$(16.7.19) \quad \begin{aligned} & \alpha_{1,0,1} n_1 n_2 \cdots n_j (\sigma_1 + \sigma_2 + \cdots + \sigma_{j+1}) \\ & > n_1 \sigma_1 + \alpha_{1,0,1} (\sigma_2 + n_1 \sigma_3 + n_1 n_2 \sigma_4 + \cdots + n_1 n_2 \cdots n_{j-1} \sigma_{j+1}) \\ & = n_1 \sigma_1 + \alpha_{1,0,1} \tau \geq \alpha_{1,0,1} n_1 n_2 \cdots n_{j+1}, \end{aligned}$$

by (16.7.18) because $y^{\sigma_1} z^{\tau}$ belongs to $\prod_{k=1}^{j+1} f_{k-2}^{\sigma_k} \in \mathbb{C}\{y\}[z]$ where $\tau = \sigma_2 + n_1 \sigma_3 + n_1 n_2 \sigma_4 + \cdots + n_1 n_2 \cdots n_{j-1} \sigma_{j+1}$.

Thus, we proved by (16.7.19) that $\sigma_1 + \sigma_2 + \cdots + \sigma_{j+1} > n_{j+1}$, and so $g_{j+1} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ has a multiplicity n_{j+1} at the origin in \mathbb{C}^{j+2} . Also, it is trivial by (2a) that $g_{j+1} \in \mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}[f_j]$ is an irreducible W -poly in f_j with coefficients in $\mathbb{C}\{y, z, f_1, \dots, f_{j-1}\}$ because ${}_n\mathcal{O}_0$ is a unique factorization domain.

Therefore, we finished the proof of The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j+1]$.

The proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$

(1) (1a) To prove it, apply the WDT with a divisor g_{j+1} to f . By Theorem 15.2, f may be written uniquely in the form

$$(16.7.20) \quad f = g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} g_{j+1}^i \quad \text{with } T_{j+2,i} \in \mathbb{C}\{y, z\},$$

where $g_{j+1} \in \mathbb{C}\{y, z\}$ and $T_{j+2,i} \in \mathbb{C}\{y, z\}$.

Whenever y, z, f_1, \dots, f_j are viewed as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} , in order to prove that $T_{j+2,i} \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$ and $T_{j+2,i}$ satisfies the property (1a), first of all note that g_{j+1} satisfies the facts corresponding to the facts Fact(A) and Fact(B) in the conclusions of Sublemma 15.4.α of Theorem 15.4 which have been already satisfied by $h_{j+1,1}$ of (15.4.3) in the conclusions of Sublemma 15.4.α.

Also, as we have seen in the proof of Sublemma 15.4.α, whenever a pair $(h_{j+1,1}, f)$ in (15.4.3) satisfies the facts Fact(A) and Fact(B), recall that the pair $(h_{j+1,1}, f)$ satisfies the facts Fact(C), Fact(D) and Fact(E) as we have seen in the proof of Sublemma 15.4.α. So, replace $(h_{j+1,1}, f)$ by (g_{j+1}, f) in the sense of Definition 16.2.2, and then there is nothing to prove for (1a).

(1b) The proof of the uniqueness of a function $\bar{\theta}_{j+2} : \mathbb{N}_0^{j+2} \rightarrow \mathbb{N}_0$ of (16.7.10) can be easily done, by using the same method as we have used in the proof of the uniqueness of a function $\theta_{j+1} : \mathbb{N}_0^{j+1} \rightarrow \mathbb{N}_0$ of (16.7.5) for (1b) of The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j+1]$.

(1c) In preparation for the proof of an inequality in (16.7.11), it is needed to show by (1a) and (1b) that for each fixed i , $T_{j+2,i}$ of (16.7.3) can be rewritten in the form

$$(16.7.21) \quad T_{j+2,i} = \sum_{q=0}^{n_{j+1}-1} T_{j+2,i,q}^{(1)} f_j^q,$$

with two properties (i) and (ii):

(i) Let i be fixed with $0 \leq i \leq d_{j+2} - 1$. For each $q = 0, 1, \dots, n_{j+1} - 1$ and for any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $T_{j+2,i,q}^{(1)} \subseteq \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\delta_1 > 0$ and $0 \leq \delta_k < n_{k-1}$ for $k = 2, 3, \dots, j+1$.

In particular, if $i = d_{j+2} - 1$, then $q \leq n_{j+1} - 2$.

(ii) Let i and q be fixed. For any two nonzero monomials $\Pi_{k=1}^{j+1} f_{k-2}^{\gamma_k}$ and $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $T_{j+2,i,q}^{(1)} \subseteq \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, then

$$(16.7.22) \quad \theta_{j+1}(\gamma_k)_{k=1}^{j+1} = \theta_{j+1}(\delta_k)_{k=1}^{j+1} \text{ if and only if } \gamma_k = \delta_k \text{ for } k = 1, 2, \dots, j+1.$$

So, there exists a unique nonzero monomial $C_{j+2,i,q}^{(1)} \Pi_{k=1}^{j+1} f_{k-2}^{\beta_{i,q,k}^{(1)}}$ in $T_{j+2,i,q}^{(1)}$ with a constant $C_{j+2,i,q}^{(1)}$ such that $\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} = \min\{\theta_{j+1}(\delta_k)_{k=1}^{j+1}\}$ for any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k}$ in $T_{j+2,i,q}^{(1)}$ with q fixed.

It is trivial that the above summation for $T_{j+2,i,q}^{(1)}$ in (16.7.21) and (16.7.22) can be clearly constructed by the similar method as we have seen in the proof of (1a) and (1b). Thus, f can be rewritten in the form

$$(16.7.23) \quad f = g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} g_{j+1}^i \\ = (f_j^{n_{j+1}} + \xi_{j+1} \Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k})^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} \sum_{q=0}^{n_{j+1}-1} T_{j+2,i,q}^{(1)} f_j^q (f_j^{n_{j+1}} + \xi_{j+1} \Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k})^i$$

for $0 \leq i \leq d_{j+2} - 1$ and $0 \leq q \leq n_{j+1} - 1$, as an element of $\mathbb{C}\{y\}[z, f_1, \dots, f_j]$, noting that if $i = d_{j+2} - 1$ then $q = n_{j+1} - 2$.

By (16.6.17) and (16.7.16), recall that the equation $(f \circ \tau_{\lambda_j})_{total}$ for $f(y, z)$ of (16.7.23) can be written in the form

$$(16.7.24) \quad (f \circ \tau_{\lambda_j})_{total} = v^{n_j d_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (f \circ \tau_{\lambda_j})_{proper}, \\ (f \circ \tau_{\lambda_j})_{proper} = (u + a + \varepsilon)^{d_{j+1}} + \sum_{i=0}^{d_{j+1}-2} W_{j+1,i} (u + a + \varepsilon)^i \quad \text{with} \\ W_{j+1,i} = W_{j+1,i}(u, v) = b_{j+1,i} v^{M_{j+1,i}} \quad \text{and} \\ M_{j+1,i} = \theta_{j+1}(\beta_{j+1,i,k})_{k=1}^{j+1} - n_j(d_{j+1} - i) \theta_j(\alpha_{j,0,k})_{k=1}^j > 0, \\ (g_{j+1} \circ \tau_{\lambda_j})_{total} = v^{n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (g_{j+1} \circ \tau_{\lambda_j})_{proper}, \\ (g_{j+1} \circ \tau_{\lambda_j})_{proper} = (u + a + \varepsilon)^{n_{j+1}} + c_{j+1}(u, v) v^{M'},$$

where $d_{j+2} = \gcd(d_{j+1}, M_{j+1,0}) = \gcd(d_{j+1}, \theta_{j+1}(\beta_{j+1,0,k})_{k=1}^{j+1}) > 1$ with $d_{j+1} = n_{j+1} d_{j+2}$ and $M_{j+1,0} = M' d_{j+2}$ for some integers $n_{j+1} \geq 2$ and $M' \geq 1$, and a is a nonzero constant, ε is a unit along $v = 0$ and $c_{j+1}(u, v)$ is a unit in $\mathbb{C}\{u + a, v\}$ with $c_{j+1}(-a, 0) = \zeta$.

By (16.6.16), at $(v, u + a) = (0, 0)$ along $v = 0$, $(f_j \circ \tau_{\lambda_j})_{total} = 0$ with $(f_j \circ \tau_{\lambda_j})_{proper} = 0$, $(\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \circ \tau_{\lambda_j})_{total} = 0$ for any nonzero monomial $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \in T_{j+2,i,q}^{(1)}$, and so $(T_{j+2,i,q}^{(1)} \circ \tau_{\lambda_j})_{total} = 0$ can be written as follows:

$$(16.7.25) \quad (f_j \circ \tau_{\lambda_j})_{total} = v^{n_j \theta_j(\alpha_{j,0,k})_{k=1}^j} (f_j \circ \tau_{\lambda_j})_{proper}, \\ (f_j \circ \tau_{\lambda_j})_{proper} = (u + a + \varepsilon), \\ (\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \circ \tau_{\lambda_j})_{total} = v^{\theta_{j+1}(\delta_k)_{k=1}^{j+1}} b(\delta_1, \dots, \delta_{j+1}), \\ (T_{j+2,i,q}^{(1)} \circ \tau_{\lambda_j})_{total} = v^{\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1}} b_{j+2,i,q},$$

where a is a nonzero constant, ε is a nonunit along $v = 0$ and $b(\delta_1, \dots, \delta_{j+1})$ is a unit in $\mathbb{C}\{v, u + a\}$ and $b_{j+2,i,q}$ is a unit in $\mathbb{C}\{v, u + a\}$.

In preparation for the proof of (16.7.11), whenever any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \in T_{j+2,i}$ is chosen arbitrary, for brevity of notation put $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} f_j^q \in T_{j+2,i,q}^{(1)} f_j^q$ with $q = \delta_{j+2}$. For any $\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} f_j^q \in T_{j+2,i,q}^{(1)} f_j^q$, note by (16.6.16) and (16.6.17) that for a unit $b = b(\delta_1, \dots, \delta_{j+1})$ in $\mathbb{C}\{v, u + a\}$,

$$(16.7.26) \quad ((\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} f_j^q) \circ \tau_{\lambda_j})_{total} = b v^{\theta_{j+1}(\delta_k)_{k=1}^{j+1}} v^{q n_j \theta_j(\alpha_{j,0,k})_{k=1}^j} (u + a + \varepsilon)^q.$$

Then, it can be easily shown by (16.7.22), (16.7.23), \dots , (16.7.26) that for each $i = 0, 1, \dots, d_{j+2} - 1$ and for any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \in T_{j+2,i}$ with $q = \delta_{j+2}$,

$$(16.7.27) \quad \theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} + n_j(q + i n_{j+1} - d_{j+1}) \theta_j(\alpha_{j,0,k})_{k=1}^j > 0.$$

So, by the induction assumption on the integer j and by Sublemma 16.6.1, after the same number λ_j of blow-ups with the same coordinates as we have used in (16.6.16) and (16.6.17), then the local defining equation $(f \circ \tau_{\lambda_j})_{total} = 0$ for the λ_j -th total transform of $f(y, z)$ in (16.7.23) can be rewritten in the following form:

$$(16.7.28) \quad \begin{aligned} (f \circ \tau_{\lambda_j})_{total} &= v^{n_j d_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (f \circ \tau_{\lambda_j})_{proper} \\ (f \circ \tau_{\lambda_j})_{proper} &= [(u + a + \varepsilon)^{n_{j+1}} + b_{j+1} v^{M'}]^{d_{j+2}}, \\ &\quad + \sum_i \sum_q b_{i,q} v^{M'_{i,q}} (u + a + \varepsilon)^q [(u + a + \varepsilon)^{n_{j+1}} + b_{j+1} v^{M'}]^i, \\ M' &= \theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j \quad \text{with } \gcd(n_{j+1}, M') = 1, \\ M'_{i,q} &= \theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} + n_j(q + i n_{j+1} - d_{j+1}) \theta_j(\alpha_{j,0,k})_{k=1}^j > 0, \end{aligned}$$

noting by (16.7.27) that $M'_{i,q} > 0$ where $0 \leq i \leq d_{j+2} - 1$ and $0 \leq q = \beta_{i,q,j+2}^{(1)} \leq n_{j+1} - 1$, and a is a nonzero constant, and ε is a nonunit along $v = 0$, and b_{j+1} and $b_{i,q}$ are units in $\mathbb{C}\{u + a, v\}$ if exist. Note that c_{j+1} is a nonunit in $\mathbb{C}\{u + a, v\}$ with $b_{j+1} = c_{j+1}$.

So, when $(f \circ \tau_{\lambda_j})_{proper}$ is viewed as an element in $\mathbb{C}\{u + a, v\}$, then $n_{j+1} M'_{i,q} + q M' + i n_{j+1} M' > n_{j+1} M' d_{j+2} = M' d_{j+1}$ because $(f \circ \tau_{\lambda_j})_{proper}$ is irreducible in $\mathbb{C}\{u + a, v\}$, which is equivalently rewritten as

$$(16.7.29) \quad \frac{M'_{i,q}}{d_{j+1} - (i n_{j+1} + q)} > \frac{M' d_{j+2}}{n_{j+1} d_{j+2}} = \frac{M'}{n_{j+1}},$$

by Lemma 16.0 because $0 \leq q < n_{j+1}$ and $\gcd(n_{j+1}, M') = 1$.

By $M'_{i,q}$ and M' of (16.7.28), an inequality in (16.7.29) can be easily simplified as

$$(16.7.30) \quad \frac{\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1}}{d_{j+1} - (i n_{j+1} + q)} > \frac{\theta_{j+1}(\sigma_k)_{k=1}^{j+1}}{n_{j+1}}.$$

Thus, by the definition of $\bar{\theta}_{j+2}$ in (16.7.10) with $q = \beta_{i,q,j+2}^{(1)}$ and by (16.7.30), we have

$$(16.7.31) \quad \begin{aligned} \bar{\theta}_{j+2}(\beta_{i,q,k}^{(1)})_{k=1}^{j+2} &= q \theta_{j+1}(\sigma_k)_{k=1}^{j+1} + n_{j+1} \theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} \\ &> q \theta_{j+1}(\sigma_k)_{k=1}^{j+1} + (d_{j+1} - i n_{j+1} - q) \theta_{j+1}(\sigma_k)_{k=1}^{j+1} \\ &= n_{j+1} (d_{j+2} - i) \theta_{j+1}(\sigma_k)_{k=1}^{j+1}, \end{aligned}$$

because $d_{j+1} = n_{j+1} d_{j+2}$. Since q was chosen arbitrary for any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \in T_{j+2,i,q}^{(1)} f_j^{\delta_j}$ with $q = \delta_{j+2} < n_{j+1}$, we proved an inequality in (16.7.11).

(1d) For the proof, consider the curve defined by $v(f \circ \tau_{\lambda_j})_{proper} = 0$ in $\mathbb{C}\{u+a, v\}$ as we have seen in (16.7.28). As far as the standard resolution of the singular point $(u+a, v) = (0, 0)$ of $v(f \circ \tau_{\lambda_j})_{proper} = 0$ is concerned, for brevity of notation, we can easily construct a local biholomorphic mapping ϕ from $(u, v) = (-a, 0)$ to $(\bar{u}, \bar{v}) = (0, 0)$ as follows:

$$(16.7.32) \quad \phi(u, v) = (\bar{u}, \bar{v})$$

such that $\bar{u} = u + a + \varepsilon$ and $\bar{v} = (b_{j+1})^{\frac{1}{M'}} v$.

Let $V(h)$ and $V(H)$ be defined by $h(\bar{u}, \bar{v}) = (f \circ \tau_{\lambda_j})_{proper} \circ \phi^{-1}$ and $H(\bar{u}, \bar{v}) = (f \circ \tau_{\lambda_j})_{total} \circ \phi^{-1}$, respectively in terms of the new coordinates.

Then, $(f \circ \tau_{\lambda_j})_{total}$ of (16.7.28) can be rewritten in terms of the new coordinates, as follows:

$$(16.7.33) \quad \begin{aligned} H &= b_0 \bar{v}^{n_j d_{j+1} \theta_j (\alpha_{j,0,k})_{k=1}^j} h, \\ h &= [\bar{u}^{n_{j+1}} + \bar{v}^{M'}]^{d_{j+2}}, \\ &\quad + \sum_i \sum_q b'_{i,q} \bar{v}^{M'_{i,q}} \bar{u}^q [\bar{u}^{n_{j+1}} + \bar{v}^{M'}]^i, \\ M' &= \theta_{j+1} (\sigma_k)_{k=1}^{j+1} - n_j n_{j+1} \theta_j (\alpha_{j,0,k})_{k=1}^j \quad \text{with } \gcd(n_{j+1}, M') = 1, \\ M'_{i,q} &= \theta_{j+1} (\beta_{i,q,k}^{(1)})_{k=1}^{j+1} - n_j (d_{j+1} - i n_{j+1} - q) \theta_j (\alpha_{j,0,k})_{k=1}^j > 0, \end{aligned}$$

where $0 \leq i \leq d_{j+2} - 1$ and $0 \leq q = \beta_{i,q,j+2}^{(1)} \leq n_{j+1} - 1$, and b_0 and $b'_{i,q}$ are units in $\mathbb{C}\{\bar{u}, \bar{v}\}$ if exist.

As an application of Theorem 16.1(Theorem 3.7), we can easily get the following consequences:

Let $\mu_\omega = \pi_{\lambda_{j+1}} \circ \pi_{\lambda_{j+2}} \circ \cdots \circ \pi_{\lambda_{j+\omega}} : M^{(\lambda_{j+\omega})} \rightarrow M^{(\lambda_j)}$ be the composition of a finite number ω of successive blow-ups which is needed only to get the standard resolution of the singular point $(\bar{u}, \bar{v}) = (0, 0)$ of $V(G) = \{(\bar{u}, \bar{v}) \in M^{(\lambda_j)} : G = 0\}$ where $G = \bar{v}(\bar{u}^{n_{j+1}} + \bar{v}^{M'})$, because M' may be equal to one.

(i)(ia) We can use just one coordinate patch of the local coordinates for each blow-up π_i of μ_ω with $1 \leq i \leq \omega$ in the sense of Lemma 2.12.

(ib) Just as above, we can use the same μ_ω for the composition of the first finite number ω of successive blow-ups in preparation for the standard resolution of the singular point $(0, 0)$ of $V(h)$ where $h(\bar{u}, \bar{v}) = (f \circ \tau_{\lambda_j})_{proper} \circ \phi^{-1}$.

(ic) Also, we can use just the common one coordinate patch of the given local coordinates for each blow-up π_i of the above μ_ω in (ia), in order to study any of $V^{(i)}(h)$ for all $i = 1, 2, \dots, \omega$ in the sense of Lemma 2.14.

(ii) For simplicity of notations, let (r, s) be the common one of the local coordinates for the $\omega - th$ blow-up $\pi_m : M^{(\omega)} \rightarrow M^{(\omega-1)}$ at $(0, 0)$ which is the quasisingular point of $V^{(\omega-1)}(G)$. Being viewed as an analytic mapping, $\mu_\omega : M^{(\omega)} \rightarrow M^0$ can be written in the form

$$(16.7.34) \quad \mu_\omega(s, r) = (\bar{u}, \bar{v}) = (s^{M'} r^\zeta, s^{n_{j+1}} r^\eta),$$

where

(iia) ζ and $\eta > 0$ are nonnegative integers such that $\eta M' - \zeta n_{j+1} = 1$,

(iib) $E_\omega = \{s = 0\}$ is defined by the $\omega - th$ exceptional curve of the first kind.

(iii) By (ii), along $s = 0$, $(h \circ \mu_\omega)_{total}$ for the ω -th total transform of the curve defined by $h = 0$ is given analytically as follows:

$$(16.7.35) \quad \begin{aligned} (H \circ \mu_\omega)_{total} &= s^{d_{j+1} \theta_{j+1} (\sigma_k)_{k=1}^{j+1}} (h \circ \mu_\omega)_{proper} \\ (h \circ \mu_\omega)_{total} &= s^{n_{j+1} d_{j+2} M'} (h \circ \mu_\omega)_{proper} \\ (h \circ \mu_\omega)_{proper} &= (1 + r)^{d_{j+2}} + \sum_i \sum_q b''_{i,q} s^{M''_{i,q}} (1 + r)^i, \\ M''_{i,q} &= n_{j+1} M'_{i,q} - M' (d_{j+1} - i n_{j+1} - q) > 0, \end{aligned}$$

noting that $M''_{i,q} > 0$, where for $0 \leq i \leq d_{j+2} - 1$ and $0 \leq q \leq n_{j+1} - 1$, the $b''_{i,q}$ are units in $\mathbb{C}\{r+1, s\}$, if exist.

It is clear by (16.7.28), (16.7.33) and the definition of $\bar{\theta}_{j+2}$ with $q = \beta_{i,q,j+2}$ that

$$\begin{aligned}
 (16.7.36) \quad M''_{i,q} &= n_{j+1}M'_{i,q} - M'(d_{j+1} - in_{j+1} - q) \\
 &= n_{j+1}\{\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} - n_j(d_{j+1} - in_{j+1} - q)\theta_j(\alpha_{j,0,k})_{k=1}^j\} \\
 &\quad - \{\theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_j n_{j+1}\theta_j(\alpha_{j,0,k})_{k=1}^j\}(d_{j+1} - in_{j+1} - q) \\
 &= n_{j+1}\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} - (d_{j+1} - in_{j+1} - q)\theta_{j+1}(\sigma_k)_{k=1}^{j+1} \\
 &= \bar{\theta}_{j+2}(\beta_{i,q,k}^{(1)})_{k=1}^{j+2} - n_{j+1}(d_{j+2} - i)\theta_{j+1}(\sigma_k)_{k=1}^{j+1},
 \end{aligned}$$

because $\bar{\theta}_{j+2}(\beta_{i,q,k}^{(1)})_{k=1}^{j+2} = n_{j+1}\theta_{j+1}(\beta_{i,q,k}^{(1)})_{k=1}^{j+1} + q\theta_{j+1}(\sigma_k)_{k=1}^{j+1}$ with $q = \beta_{i,q,j+2}$ and $d_{j+1} = n_{j+1}d_{j+2}$.

So, it is clear by (16.7.22) that $M''_{i,q} = M''_{i,q'}$ if and only if $\beta_{i,q,k}^{(1)} = \beta_{i,q',k}^{(1)}$ for $1 \leq k \leq j+2$ with $q = \beta_{i,q,j+2}$. For each fixed i with $0 \leq i \leq d_{j+2} - 1$, let M''_i be defined by $\min\{M''_{i,q} : 0 \leq q \leq n_{j+1} - 1\}$. Then, by the definition of M''_i , for each $i = 0, 1, \dots, d_{j+2} - 1$,

$$(16.7.37) \quad M''_i = \bar{\theta}_{j+2}(\beta_{j+2,i,k}^{(1)})_{k=1}^{j+2} - n_{j+1}(d_{j+2} - i)\theta_{j+1}(\sigma_k)_{k=1}^{j+1},$$

because $\bar{\theta}_{j+2}(\beta_{i,k}^{(1)})_{k=1}^{j+2} = \min\{\bar{\theta}_{j+2}(\beta_{i,q,k}^{(1)})_{k=1}^{j+2} : 0 \leq \beta_{i,q,j+2} = q \leq n_{j+1} - 1\}$.

Now, since $(h \circ \mu_\omega)_{proper}$ is irreducible in $\mathbb{C}\{r+1, s\}$, then by Theorem 3.2

$$(16.7.38) \quad \frac{M''_i}{d_{j+2} - i} \geq \frac{M''_0}{d_{j+2}} \quad \text{for } 1 \leq i \leq d_{j+2} - 1.$$

Therefore, by (16.7.37) and (16.7.38), we get

$$(16.7.39) \quad \frac{\bar{\theta}_{j+2}(\beta_{j+2,i,k}^{(1)})_{k=1}^{j+2}}{d_{j+2} - i} \geq \frac{\bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(1)})_{k=1}^{j+2}}{d_{j+2}} \quad \text{for } 1 \leq i \leq d_{j+2} - 1.$$

Thus, the proof of (16.7.12) is done.

(1d-1) If $\gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(1)})_{k=1}^{j+2}) = 1$, then it is trivial by Corollary 3.3 that f is irreducible in $\mathbb{C}\{y, z\}$ with $f \in$ the type $[j+2]$ if and only if (16.7.39) holds and $f_1, f_2, \dots, f_j, g_{j+1}$ are irreducible in $\mathbb{C}\{y, z\}$ just as above, because $g_{j+1} \in$ the type $[j+1]$.

(1d-2) If $1 < \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{j+2,0,k}^{(1)})_{k=1}^{j+2}) \leq d_{j+2}$, the proof is trivial by Theorem 3.2, Theorem 3.5 and Theorem 3.6.

Thus we proved (1a), (1b), (1c) and (1d).

(2)(2a) The proof is trivial by (1d) of (1).

(2b) To show that $f = g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} g_{j+1}^i \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}[g_{j+1}]$ has a multiplicity d_{j+2} at the origin in \mathbb{C}^{j+3} , consider any nonzero monomial $\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k}$ in $T_{j+2,i}$ with $\delta_1 > 0$ and $\delta_k < n_{k-1}$ for $k = 2, 3, \dots, j+2$. Since $\alpha_{1,0,1} > n_1 > 0$ and $\delta_1 > 0$, then

$$\begin{aligned}
 (16.7.40) \quad &\alpha_{1,0,1}n_1n_2 \cdots n_{j+1}(\delta_1 + \delta_2 + \cdots + \delta_{j+2} + i) \\
 &> n_1\delta_1 + \alpha_{1,0,1}(\delta_2 + n_1\delta_3 + \cdots + n_1n_2 \cdots n_j\delta_{j+2} + n_1n_2 \cdots n_{j+1}i) \\
 &\geq \alpha_{1,0,1}n_1n_2 \cdots n_{j+1}d_{j+2} = n_1\alpha_{1,0,1}d_2,
 \end{aligned}$$

by Lemma 3.1 or (16.7.18) because $y^{\delta_1}z^\tau$ belongs to either $\sum_{i=0}^{d_{j+2}-1} T_{j+2,i}g_{j+1}^i$ or $g_{j+1}^{d_{j+2}}$ as a convergent power series expansion in $\mathbb{C}\{y, z\}$ where $\tau = \delta_2 + n_1\delta_3 + \cdots + n_1n_2 \cdots n_j\delta_{j+2} + n_1n_2 \cdots n_{j+1}i$, and so $n_1\delta_1 + \alpha_{1,0,1}\tau\alpha_{1,0,1} \geq n_1n_2 \cdots n_{j+1}d_{j+2}$. Thus, we proved that $\delta_1 + \delta_2 + \cdots + \delta_{j+1} + i > d_{j+2}$, what we wanted. So, $f \in \mathbb{C}\{y\}[z, f_1, \dots, f_j, g_{j+1}]$ has a multiplicity d_{j+2}

at the origin in \mathbb{C}^{j+3} . Also, it is trivial that $f \in \mathbb{C}\{y\}[z, f_1, \dots, f_j, g_{j+1}]$ is an irreducible W -poly of degree d_{j+2} in g_{j+1} because ${}_n\mathcal{O}_0$ is a unique factorization domain and f is irreducible in $\mathbb{C}\{y, z\}$.

Therefore, we finished the proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, and so the proof of this proposition is completed. \square

Remark 16.7.2.

(1) Note by (16.7.16) and (16.7.26) that the local defining equation $(g_{j+1} \circ \tau_{\lambda_j})_{proper}$ for the λ_j -th proper transform of the curve defined by $g_{j+1} = 0$ is analytically written in the form

$$(16.7.43) \quad \begin{aligned} (g_{j+1} \circ \tau_{\lambda_j})_{total} &= v^{n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} (g_{j+1} \circ \tau_{\lambda_j})_{proper} \\ (g_{j+1} \circ \tau_{\lambda_j})_{proper} &= (u + a + \varepsilon)^{n_{j+1}} + c_{j+1}(u, v) v^{M'}. \end{aligned}$$

By (16.7.34) and (16.7.36), it is easy to compute that

$$(16.7.44) \quad \begin{aligned} (g_{j+1} \circ \tau_{\lambda_j} \circ \mu_\omega)_{total} &= s^{n_{j+1} \theta_{j+1}(\sigma_k)_{k=1}^{j+1}} (g_{j+1} \circ \tau_{\lambda_j} \circ \mu_\omega) \\ (g_{j+1} \circ \tau_{\lambda_j} \circ \mu_\omega) &= (r+1). \end{aligned}$$

Let $\tau_{\lambda_{j+1}} = \tau_{\lambda_j} \circ \mu_\omega$, and then $\tau_{\lambda_{j+1}}$ is the standard resolution of the singular point $(0, 0)$ of $V(g_{j+1}) = \{(y, z) \in \mathbb{C}^2 : g_{j+1}(y, z) = 0\}$.

(2) By (1), at $(s, r) = (0, 0)$ along $s = 0$, $(\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \circ \tau_{\lambda_{j+1}})_{total} = 0$ and $(f_j \circ \tau_{\lambda_j})_{proper} = 0$ can be written in the form, satisfying the following property:

$$(16.7.45) \quad \begin{aligned} (\Pi_{k=1}^{j+2} f_{k-2}^{\delta_k} \circ \tau_{\lambda_{j+1}})_{total} &= [(\Pi_{k=1}^{j+1} f_{k-2}^{\delta_k} \circ \tau_{\lambda_{j+1}})] [(f_j)^{\delta_{j+2}} \circ \tau_{\lambda_{j+1}}] \\ &= [(\bar{v}^{\theta_{j+1}(\delta_k)_{k=1}^{j+1}} b) \circ \mu_\omega] [(\bar{v}^{\delta_{j+2} n_j \theta_j(\alpha_{j,0,k})_{k=1}^j} \bar{u}^{\delta_{j+2}}) \circ \mu_\omega], \\ &= s^{n_{j+1} \theta_{j+1}(\delta_k)_{k=1}^{j+1}} s^{\delta_{j+2} n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j} s^{\delta_{j+2} M'} b' \end{aligned}$$

where b' is a unit in $\{\bar{u}, \bar{v}\}$.

Since $M' = \theta_{j+1}(\sigma_k)_{k=1}^{j+1} - n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j$, then it is easy to prove that

$$(16.7.46) \quad \begin{aligned} n_{j+1} \theta_{j+1}(\delta_k)_{k=1}^{j+1} + \delta_{j+2} n_j n_{j+1} \theta_j(\alpha_{j,0,k})_{k=1}^j &+ \delta_{j+2} M' \\ &= n_{j+1} \theta_{j+1}(\delta_k)_{k=1}^{j+1} + \delta_{j+2} \theta_{j+1}(\sigma_k)_{k=1}^{j+1} \\ &= \theta_{j+2}(\delta_k)_{k=1}^{j+2}. \end{aligned}$$

(3) From the proof of the next proposition, denoted by Proposition 16.8, it will be shown that $\tau_{\lambda_{j+1}}$, which is the standard resolution of the singular point $(0, 0)$ of $V(g_{j+1})$, is also the standard resolution of the singular point $(0, 0)$ of $V(f_{j+1})$ with $f_{j+1}(y, z) \stackrel{\text{multiseq}}{\sim} g_{j+1}(y, z)$. If proved, then by using (2), it can be said that $\tau_{\lambda_{j+1}}$ is well-defined, satisfying an additional property (d) for Fact[4] on the integer $j+1$.

(4) By (16.7.37),

$$(16.7.47) \quad \begin{aligned} (f \circ \tau_{\lambda_{j+1}})_{total} &= s^{d_{j+1} \theta_{j+1}(\sigma_k)_{k=1}^{j+1}} (f \circ \tau_{\lambda_{j+1}})_{proper}, \\ (f \circ \tau_{\lambda_{j+1}})_{proper} &= (h \circ \mu_\omega)_{proper}. \end{aligned}$$

Proof of Proposition 16.8. For the construction of a pair (f_{j+1}, f) in (16.8.2), it suffices to consider the following two cases, depending on the fact that $T_{j+2, d_{j+2}-1}$ of (16.8.1) is either zero or not. For brevity of notations, let $h_1 = g_{j+1}$ and $T_{j+2, i}^{(1)} = T_{j+2, i}$ for $0 \leq i \leq d_{j+2} - 1$.

Case(1) Let $T_{j+2, d_{j+2}-1}^{(1)}$ be zero. It is clear.

Case(2) Let $T_{j+2, d_{j+2}-1}^{(1)}$ be nonzero. It has been already shown by Sublemma 15.5 and Sublemma 15.6 in the proof of Theorem 15.4 that the following assertion is true:

There is a sequence of W-polys in z of pairs, $\{(h_p, f) : p = 1, 2, \dots\}$ such that

$$(16.8.15) \quad (h_{\nu+1}, f) = (h_{\nu+2}, f) = \dots \quad \text{for some integer } \nu \leq \frac{n_{j+1}+1}{2},$$

each pair of which can be written in the form

$$(16.8.16) \quad \begin{cases} h_1 &= f_j^{n_{j+1}} + \xi_{j+1} \Pi_{k=1}^{j+1} f_{k-2}^{\sigma_k} = f_j^{n_{j+1}} + R_{j+1,0}^{(1)}, \\ f &= h_1^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i}^{(1)} h_1^i, \end{cases}$$

and for $p = 2, 3, \dots$

$$(16.8.17) \quad \begin{cases} h_p &= h_{p-1} + \frac{1}{d_{j+2}} T_{j+2,d_{j+2}-1}^{(p-1)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1,i}^{(p)} f_j^i, \\ f &= h_p^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i}^{(p)} h_p^i, \end{cases}$$

with $T_{j+2,d_{j+2}-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{j+2,d_{j+2}-1}^{(\nu+1)} = T_{j+2,d_{j+2}-1}^{(\nu+2)} = \dots = 0$ where $T_{j+2,i}^{(p)} = T_{j+2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ for $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$, and $R_{j+1,i}^{(p)} = R_{j+1,i}^{(p)}(y) \in \mathbb{C}\{y\}$ for $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$, if exist, satisfying the same kind of the properties and notations as we have seen in Sublemma 15.5 of Theorem 15.4, as follows:

(16.8.18)(16.8.18-1) Property(1) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$. Then $R_{j+1,i}^{(p)} = R_{j+1,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree $< \Pi_{t=1}^j n_t$ and has a multiplicity $\geq (n_{j+1} - i) \Pi_{t=1}^j n_t$ at $0 \in \mathbb{C}^2$.

(16.8.18-2) Property(2) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$. Then $T_{j+2,i}^{(p)} = T_{j+2,i}^{(p)}(y, z) \in \mathbb{C}\{y\}[z]$ is a polynomial in z of degree $< \Pi_{t=1}^{j+1} n_t$ and has a multiplicity $\geq (d_{j+2} - i) \Pi_{t=1}^{j+1} n_t$ at $0 \in \mathbb{C}^2$.

Consider y, z, f_1, \dots, f_j as independent complex $(j+2)$ -variables at the origin in \mathbb{C}^{j+2} .

(16.8.18-3) Property(3) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq n_{j+1} - 2$. Then for any nonzero monomial $\Pi_{t=1}^{j+1} f_{t-2}^{\delta_t}$ in $R_{j+1,i}^{(p+1)} = R_{j+1,i}^{(p+1)}(y, z, f_1, \dots, f_{j-1}) \in \mathbb{C}\{y\}[z, f_1, \dots, f_{j-1}]$, $\delta_1 > 0$ and $\delta_t < n_{t-1}$ for $t = 2, 3, \dots, j+1$.

(16.8.18-4) Property(4) Let p and i be fixed with $p \geq 1$ and $0 \leq i \leq d_{j+2} - 1$. Then for any nonzero monomial $\Pi_{t=1}^{j+2} f_{t-2}^{\gamma_t}$ in $T_{j+2,i}^{(p)} = T_{j+2,i}^{(p)}(y, z, f_1, \dots, f_j) \in \mathbb{C}\{y\}[z, f_1, \dots, f_j]$, $\gamma_1 > 0$ and $\gamma_t < n_{t-1}$ for $t = 2, 3, \dots, j+2$.

(16.8.18-5) Property(5) In particular, if $i = d_{j+2} - 1$ for $T_{j+2,i}^{(p)}$ of Property(4), then $\gamma_{j+2} \leq n_{j+1} - 2$.

(16.8.18-6) Property(6) There is a pair $(h_{\nu+1}, f) \in H$ which satisfies the following:

There is an integer $\nu \leq \frac{n_{j+1}+1}{2}$ such that $T_{j+2,d_{j+2}-1}^{(p)} \neq 0$ for $p = 1, 2, \dots, \nu$ and $T_{j+2,d_{j+2}-1}^{(\nu+1)} = T_{j+2,d_{j+2}-1}^{(\nu+2)} = \dots = 0$. That is, $(h_\nu, f) \neq (f_{j+1}, f)$ and $(h_{\nu+1}, f) = (f_{j+1}, f)$ for an integer $\nu \leq \frac{n_{j+1}+1}{2}$.

Remark. Without any need of proof, Property(1), Property(2), \dots , Property(6), which are mentioned just above, follow clearly from Sublemma 15.5 of Theorem 15.4, which belongs to Case (II) in the conclusion of Sublemma 15.5 of Theorem 15.4. In Sublemma 15.5, note that $f_{-1} = y$ and $f_0 = z$.

For the proof of this proposition in Case(2), it suffices to show that two properties, denoted by, The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[j+1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, can be satisfied respectively. Then, the proof will be by induction on the integer $p \geq 1$.

Now, it is enough to consider the following two subcases for Case(2), respectively:

Subcase(A) $p = 1$, and Subcase(B) $p \geq 1$.

Subcase(A) of Case(2) Let $p = 1$. Then it suffices to show that (h_1, f) , given by (16.8.3), satisfies The Necessary and Sufficient Condition[A] for $h_1(y, z) \in \text{the type}[j+1]$, and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, which was already proved by Proposition 16.7.

Subcase(B) of Case(2) Let $p \geq 1$. For the proof of this subcase, it suffices to show by Subcase(A) for Case(2) that the following sublemma is true:

Sublemma 16.8.1 for Subcase(B) of Case(2).

Assumptions For the induction proof, suppose we have shown on the integer $p \geq 1$ that The Necessary and Sufficient Condition[A] for $h_p(y, z) \in \text{the type}[j+1]$ and

The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$ are true for (h_p, f) , following the same notations and properties as we have seen in (16.8.4), (16.8.5), \dots , (16.8.14) of the statement of this proposition.

Conclusions Then, (h_{p+1}, f) can be written by

$$(16.8.19) \quad \begin{cases} h_{p+1} &= h_p + \frac{1}{d_{j+2}} T_{j+2, d_{j+2}-1}^{(p)} = f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1, i}^{(p+1)} f_j^i, \\ f &= h_{p+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2, i}^{(p+1)} h_{p+1}^i, \end{cases}$$

with $T_{j+2, d_{j+2}-1}^{(p)} \neq 0$ for $1 \leq p \leq \nu$ and $T_{j+2, d_{j+2}-1}^{(\nu+1)} = T_{j+2, d_{j+2}-1}^{(\nu+2)} = \dots = 0$ where $T_{j+2, i}^{(p)} \in \mathbb{C}\{f_{-1}, f_0, \dots, f_j\}$ for $p \geq 1$ and $0 \leq i \leq d_{j+2}-1$, and $R_{j+1, i}^{(p+1)} \in \mathbb{C}\{f_{-1}, f_0, \dots, f_{j-1}\}$ for $p \geq 1$ and $0 \leq i \leq n_{j+1}-2$, if exist, satisfying the same kind of properties and notations, denoted by The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[j+1]$ and The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$, inductively as we have seen in the conclusion of Proposition 16.8. \square

Proof of Sublemma 16.8.1 for Subcase(B) of Case(2) Let $p \geq 1$ with $T_{j+2, d_{j+2}-1}^{(p)} \neq 0$.

[1] The proof of The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[j+1]$

(1) The proof can be easily proved by Sublemma 15.5 of Theorem 15.4 and the induction assumption on the integer p .

(2) The proofs of (2a) and (2b) are clear by (1), noting that ${}_n\mathcal{O}$ is a unique factorization domain. Thus, the proof of The Necessary and Sufficient Condition[A] for $h_{p+1}(y, z) \in \text{the type}[j+1]$ is done.

[2] The proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$

Using the same method as we have used in the proof of Proposition 16.7 and following the same kind of properties and notations as we have seen in (16.7.11), (16.7.12), (16.7.13), \dots , (16.7.37), (16.7.38), (16.7.39), the proof of The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$ can be easily done. Thus, the proof of Sublemma 16.8.1 can be done.

[3] The proof of The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j+2$

It is trivial to prove. Therefore, the proof of Proposition 16.8 is completely finished. \square

Proof of Proposition 16.9(Step III).

The Proof of Step III-1 For a given integer $j+1 \leq \ell-1$ and (g_{j+1}, f) in Proposition 16.7, the main aim is to construct a unique pair (f_{j+1}, f) which can be uniquely written in the form

$$(16.9.1) \quad \begin{cases} f_{j+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_{j+1, i} f_j^i \\ f &= f_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} S_{j+2, i} f_{j+1}^i, \end{cases}$$

where $y, z, f_1, \dots, f_{j+1}$ are considered as independent complex $(j+3)$ -variables at the origin in \mathbb{C}^{j+3} if necessary, satisfying the following properties:

(16.9.1-a) The first problem is to prove that we can construct $f_{j+1} = f_{j+1}(y, z, f_1, \dots, f_j)$ satisfying The Necessary and Sufficient Condition[A] for $f_{j+1}(y, z) \in \text{the type}[j+1]$ such that $f_{j+1}(y, z) \stackrel{\text{multiseq}}{\sim} g_{j+1}(y, z)$ in the sense of Proposition 16.8.

(16.9.1-b) The second problem is to prove that $f = f(y, z, f_1, \dots, f_{j+1})$ satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$, using the same way as $f = f(y, z, f_1, \dots, f_j)$ does The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 1$ in the sense of Proposition 16.8.

In preparation for finding the proofs in (16.9.1-a) and (16.9.1-b), it was already shown by (16.7.3) of Proposition 16.7 that (g_{j+1}, f) can be uniquely written in the following form:

$$(16.9.2) \quad \begin{cases} g_{j+1} &= f_j^{n_{j+1}} + \xi_{j+1} \prod_{k=1}^{j+1} f_{k-2}^{\sigma_k} \quad \text{with } f_{-1} = y \text{ and } f_0 = z, \\ f &= g_{j+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-1} T_{j+2,i} g_{j+1}^i, \end{cases}$$

where $n = d_{j+2} \prod_{k=1}^{j+1} n_k$ with $d_{j+2} \geq 2$ and $n_k \geq 2$ for $1 \leq k \leq j + 1$, and $\sigma_k = \beta_{j+1, d_{j+1}-n_{j+1}, k}$ for $1 \leq k \leq j + 1$ and $\xi_{j+1} = \frac{1}{d_{j+2}} B_{j+1, d_{j+1}-n_{j+1}}$, such that g_{j+1} of (g_{j+1}, f) satisfies The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j + 1]$ and f of (g_{j+1}, f) does The Necessary Condition[B] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ in the sense of Proposition 16.7 or Proposition 16.8.

To solve the problems in (16.9.1-a) and (16.9.1-b), it suffices to consider two cases:
Case(1) Let $T_{j+2, d_{j+2}-1} = 0$. Then there is nothing to solve, because g_{j+1} of (g_{j+1}, f) satisfies The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j + 1]$ and f of (g_{j+1}, f) does The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ in the sense of Proposition 16.8.

Case(2) Let $T_{j+2, d_{j+2}-1} \neq 0$. note that g_{j+1} of (g_{j+1}, f) satisfies The Necessary and Sufficient Condition[A] for $g_{j+1}(y, z) \in \text{the type}[j + 1]$, but note that f of (g_{j+1}, f) does not satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ in the sense of Proposition 16.8.

So, for finding the proofs in (16.9.1-a) and (16.9.1-b), it remains to consider Case(2). Since $T_{d_{j+2}-1} \neq 0$ from (16.9.2), then f of (g_{j+1}, f) of (16.9.2) does not satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[j + 2]$ in the sense of Proposition 16.7.

In preparation for the construction of the pair (f_{j+1}, f) in (16.9.1), which satisfies the corresponding properties as we have seen in Theorem 16.6, it was already known by Proposition 16.8 and Theorem 15.4 that the pair $(h_{\nu+1}, f)$ can be rewritten in the form

$$(16.9.3) \quad \begin{cases} h_{\nu+1} &= f_j^{n_{j+1}} + \sum_{i=0}^{n_{j+1}-2} R_i^{(\nu+1)} f_j^i, \\ f &= h_{\nu+1}^{d_{j+2}} + \sum_{i=0}^{d_{j+2}-2} T_i^{(\nu+1)} h_{\nu+1}^i, \end{cases}$$

such that $h_{\nu+1}$ of $(h_{\nu+1}, f)$ in (16.9.3) satisfies The Necessary and Sufficient Condition[A] for $h_{\nu+1}(y, z) \in \text{the type}[j + 1]$ and f of $(h_{\nu+1}, f)$ in (16.9.3) satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$. Note that $T_{d_{j+2}-1}^{(\nu+1)}$ was shown to be zero.

Now, to construct such a pair (f_{j+1}, f) , define f_{j+1} by $h_{\nu+1}$ of (16.9.3). Then, to find a representation of (f_{j+1}, f) in terms of Theorem 16.6, let $R_{j+1,i}$ be $R_i^{(\nu+1)}$ for $0 \leq i \leq n_{j+1}-2$, and $S_{j+2,i}$ be $T_i^{(\nu+1)}$ for $0 \leq i \leq d_{j+1}-2$, following the notations in Proposition 16.8.

Then, by all the definitions of $A_{j+1,i} \prod_{k=1}^{j+1} f_{k-2}^{\alpha_{j+1,i,k}}$ and $A_i^{(\nu+1)} \prod_{k=1}^{j+1} f_{k-2}^{\alpha_{i,k}^{(\nu+1)}}$ for $0 \leq i \leq n_{j+1}-2$, we get that $\alpha_{j+1,i,k} = \alpha_{i,k}^{(\nu+1)}$ with $\alpha_{0,k}^{(\nu+1)} = \sigma_k$ for $1 \leq k \leq j + 1$ and $A_{j+1,i} = A_i^{(\nu+1)}$ for $0 \leq i \leq n_{j+1}-2$. So, $\bar{\theta}_{j+2}$ is well defined, that is, $\bar{\theta}_{j+2} = \theta_{j+2}$. Similarly, by all the definitions of $B_{j+2,i} \prod_{k=1}^{j+2} f_{k-2}^{\beta_{j+2,i,k}}$ and $C_i^{(\nu+1)} \prod_{k=1}^{j+2} f_{k-2}^{\beta_{i,k}^{(\nu+1)}}$ for $0 \leq i \leq n_{j+1}-2$, it is clear that $\beta_{j+2,i,k} = \beta_{i,k}^{(\nu+1)}$ and $B_{j+2,i} = C_i^{(\nu+1)}$ for $0 \leq i \leq d_{j+2}-2$ and $1 \leq k \leq j + 2$. Also, we already proved that $\gcd(d_{j+2}, \theta_{j+2}(\beta_{j+2,0,k})_{k=1}^{j+2}) = \gcd(d_{j+2}, \bar{\theta}_{j+2}(\beta_{0,k}^{(\nu+1)})_{k=1}^{j+2}) < d_{j+2}$.

Thus, the main problem in this theorem, equivalently, to compute such a pair (f_{j+1}, f) is completely solved, whether $T_{j+2, d_{j+2}-1}$ of (16.9.2) is zero or not.

Step III-2 We are going to prove that for a given integer $j + 1 \leq \ell - 1$ in this theorem we can construct a unique sequence of irreducible W -polys in z , $\{f_0 = z, f_1, \dots, f_{j+1}\}$ with $f_k \in \mathbb{C}\{y\}[z]$ for $1 \leq k \leq j + 1$ and $f_{j+1} \neq f$, such that each f_k satisfies The Necessary and Sufficient Condition[A] for $f_k(y, z) \in \text{the type}[k]$ and $f = f(y, z, f_1, \dots, f_{j+1})$ satisfies The Necessary Condition[A] for $f(y, z) \in \text{the type}[\ell]$ with $\ell \geq j + 2$ as we have seen in the conclusion of the theorem. The proof just follows from Theorem 15.4, Step I, Step II and Step III-1. Thus, the proof of Step III(Proposition 16.9) is completely finished.

Therefore, we can complete the proof of Theorem 16.6. \square

Corollary 16.10. Assumptions Under the same assumptions and results in Theorem 16.6, for coincidence of notations f may be rewritten as follows: Note that $f_0 = z$.

$$(16.10.1) \quad f = z^n + \sum_{i=0}^{n-2} a_i y^{\alpha_i} z^i = f_0^n + \sum_{i=0}^{n-2} S_{1i} z^i$$

where $S_{1,i} = a_i y^{\alpha_i}$ with a_i units in $\mathbb{C}\{y\}$, $B_{1i} = a_i(0)$ and $\beta_{1,i,1} = \alpha_i$.

Conclusions Then, we find the following computational algorithm with $d_1 = n$:

(1) Let $d_2 = \gcd(d_1, \beta_{1,0,1})$ with $d_1 = n$ and $\beta_{1,0,1} = \alpha_0$.

Then we can find $n_1, \alpha_{1,0,1}$ and $A_{1,0}$ such that $n = d_1 = n_1 d_2$, $\beta_{1,0,1} = \alpha_{1,0,1} d_2$, $\alpha_{1,0,1} = \beta_{1,d_1-n_1,1}$ and $d_2 A_{1,0} = B_{1,d_1-n_1}$. Note that $d_2 \theta_1(\alpha_{1,0,1}) = d_2 \alpha_{1,0,1} = \beta_{1,0,1} = \theta_1(\beta_{1,0,1}) = \alpha_0$.

(2) Recall that $B_{2,i} y^{\beta_{2,i,1}} z^{\beta_{2,i,2}}$ is a unique nonzero monomial in $S_{2,i}$ with a nonzero constant $B_{2,i}$ such that $\theta_2(\beta_{2,i,k})_{k=1}^2 = \min\{\theta_2(\delta_k)_{k=1}^2\}$ for any nonzero monomial $y^{\delta_1} z^{\delta_2}$ in $S_{2,i}$, and then let $d_3 = \gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2)$.

Then we can find $n_2, (\alpha_{2,0,k})_{k=1}^2$ and $A_{2,0}$ such that $d_2 = n_2 d_3$, $\alpha_{2,0,k} = \beta_{2,d_2-n_2,k}$ for $k = 1, 2$ and $d_3 A_{2,0} = B_{2,d_2-n_2}$. Note that $d_3 \theta_2(\alpha_{2,0,k})_{k=1}^2 = \theta_2(\beta_{2,0,k})_{k=1}^2$.

(3)(i) Let m be such that $3 \leq m \leq j$. Recall that $B_{m,i} \Pi_{k=1}^m f_{k-2}^{\beta_{m,i,k}}$ is a unique nonzero monomial in $S_{m,i}$ with a nonzero constant $B_{m,i}$ such that $\theta_m(\beta_{m,i,k})_{k=1}^m = \min\{\theta_m(\delta_k)_{k=1}^m\}$ for any nonzero monomial $\Pi_{k=1}^m f_{k-2}^{\delta_k}$ in $S_{m,i}$, and let $d_{m+1} = \gcd(d_m, \theta_m(\beta_{m,0,k})_{k=1}^m)$.

Then we can find $n_m, (\alpha_{m,0,k})_{k=1}^m$ and $A_{m,0}$ such that $d_m = n_m d_{m+1}$, $\alpha_{m,0,k} = \beta_{m,d_m-n_m,k}$ for $1 \leq k \leq m$, and $d_{m+1} A_{m,0} = B_{m,d_m-n_m}$.

Moreover, $\{\alpha_{m,0,k} : k = 1, 2, \dots\}$ is a unique sequence such that $d_m \theta_m(\alpha_{m,0,k})_{k=1}^m = \theta_m(\beta_{m,0,k})_{k=1}^m$, and $\alpha_{m,0,1} > 0$, $\alpha_{m,0,k} < n_{k-1}$ for $2 \leq k \leq m$, and $\beta_{m,0,1} > 0$, $\beta_{m,0,k} < n_{k-1}$ for $2 \leq k \leq m$.

(ii) Note that $n = d_1 > d_2 = \gcd(d_1, \beta_{1,0,1}) > d_3 = \gcd(d_2, \theta_2(\beta_{2,0,k})_{k=1}^2) > d_4 = \gcd(d_3, \theta_3(\beta_{3,0,k})_{k=1}^3) > \dots > d_{j+1} = \gcd(d_j, \theta_j(\beta_{j,0,k})_{k=1}^j)$. Thus, $\{d_1, d_2, \dots, d_{j+1}\}$ is a strictly decreasing positive integer sequence.

(4) Let $H_1 = z^{n_1} + y^{\alpha_{1,0,1}}$, $H_2 = H_1^{n_2} + y^{\alpha_{2,0,1}} z^{\alpha_{2,0,2}}$, $H_3 = H_2^{n_3} + \Pi_{k=1}^3 H_{k-2}^{\alpha_{3,0,k}}$, \dots , and $H_m = H_{m-1}^{n_m} + \Pi_{k=1}^m H_{k-2}^{\alpha_{m,0,k}}$ for $1 \leq m \leq j$ where $H_{-1} = y$ and $H_0 = z$.

Then, $f_p \stackrel{\text{multiseq}}{\sim} H_p$ for $1 \leq p \leq j$. \square

Remark 16.11. Without assuming that f is irreducible in ${}_2\mathcal{O}_0$, suppose that f satisfies the same assumption as in Theorem 16.6. For a fixed j with $0 \leq j \leq \ell - 1$ suppose also that there exists such a unique sequence of irreducible W -polys in z , $\{f_0, f_1, \dots, f_j\}$, satisfying the same conclusion as in Theorem 16.6. Again, except that F_{λ_j} is irreducible in $\mathbb{C}\{u+a, v\}$ as in additional condition (d), assume that F_{λ_j} with the property (16.6) satisfies the construction of $g_{j+1}(y, z, f_1, \dots, f_j)$ and $f(y, z, f_1, \dots, f_j, g_{j+1})$ as in Step I and II. Then we can prove by Step III that each h_p is irreducible in ${}_2\mathcal{O}_0$ with $h_1 = g_{j+1}$ and $p \geq 1$. Note that f may not be irreducible in ${}_2\mathcal{O}_0$.

Chapter XI: The 2nd Algorithm for computing irreducible W-polys from all the W-polys of two complex variables and The 3rd algorithm for computing the corresponding standard Puiseux expansion from any irreducible W-poly of two complex variables

§19. The 2nd Algorithm for computing completely irreducible W-polys from all the W-polys of two complex variables with proofs

Theorem 19.1(The 2nd Algorithm).

Assumptions *Suppose that Theorem 19.1 and Theorem 1.15 have the same assumption.*

Conclusions *Then, they have the same conclusion. \square*

Proof. There is nothing to prove for Theorem 19.1 because it was already proved by Theorem 16.5 and by Theorem 16.6 with Proposition 16.7 and Proposition 16.8. \square

Example 19.1.1 for The 2nd algorithm in Theorem 19.1: Example 19.1.1 and Example 1.10.1 of §1.10 are the same. \square

§20. The 3rd Algorithm for computing the corresponding standard Puiseux expansion from any irreducible W-poly of two complex variables with respect to the multiplicity sequences with proofs

Theorem 20.1(The 3rd Algorithm).

Assumptions *Suppose that Theorem 20.1 and Theorem 1.16 have the same assumption.*

Conclusions *Then, they have the same conclusion. \square*

Proof. There is nothing to prove for Theorem 20.1 because it was already proved by Theorem 16.5 and by Theorem 16.6 with Proposition 16.7 and Proposition 16.8. \square

Example 20.1.1 for The 3rd algorithm in Theorem 20.1: Example 20.1.1 and Example 1.10.2 of §1.10 are the same. \square

Appendix

It is very interesting to study what problems can be computed in irreducible plane curve singularities in algebraic geometry. Appendix consists of Appendix A, Appendix B and Appendix C. As an application of The 1st Algorithm, The 2nd Algorithm and The 3rd Algorithm of Part[A], the aim is to find three algorithms, called The α -algorithm, The β -algorithm, The γ -algorithm in Appendix A, Appendix B and Appendix C, respectively, completely and rigorously in an elementary way, as follows:

[1] In Appendix A, the aim is to find an explicit algorithm for finding a one-to-one function from Family(2)(the family of the standard Puiseux expansion) onto Family(3)(the family of all the multiplicity sequences of irreducible plane curves with isolated singularity under the standard resolution), called The α -algorithm. By Theorem A.4 and Theorem A.5, we can find The α -algorithm completely and rigorously in an elementary way.

[2] In Appendix B, the aim is to find an explicit algorithm for finding a one-to-one function from Family(2) onto Family(4)(the family of the divisors defined by the total transforms of irreducible plane curves with isolated singularity under the standard resolution), called The β -algorithm. By Theorem B.2 and Theorem B.3, we can find The β -algorithm completely and rigorously in an elementary way.

[3] In Appendix C, the aim is to find an explicit algorithm for finding a one-to-one function from Family(2) onto Family(5)(the family of the singular parts of the divisors defined by the total transforms of irreducible plane curves with isolated singularity under the standard resolution), called The γ -algorithm. By Theorem C.2 and Theorem C.3, we can find The γ -algorithm completely and rigorously in an elementary way.

Remark. Whenever any element of Family(4) is viewed as a finite sequence of positive integers which is strictly increasing by either Definition 1.2 of Part[A] or Definition A.2, it will be proved by Remark A.2.2 that each element of Family(5) can be viewed a proper subsequence of some element of Family(4), satisfying an additional property.

Appendix A

The α -algorithm for finding a one-to-one function from Family(2) onto Family(3)

§A.0. Introduction

It was already proved by The 1st Algorithm of PART[A] that we can find a one-to-one function ϕ from **Family(1)** onto **Family(2)**. In Appendix A, the aim is to find an algorithm for computing a one-to-one function from **Family(2)** onto **Family(3)** in the beginning of Appendix, called the α -algorithm, for notation.

§A.1. The terminology and notations in preparation for finding the α -algorithm

In preparation for the explicit and rigorous representation of the α -algorithm, the β -algorithm and the γ -algorithm as we have seen in the beginning of Appendix, recall the definitions of four Families, that is, Family(k) ($1 \leq k \leq 4$) with equivalence relations, as we have seen in Definition 1.2 and Definition 2.4 of Part[A]. Moreover, Family(4) with some additional properties will be said to be a new family in Definition A.2, called Family(5). It will be proved by Appendix C later that Family(5) is well-defined and that we can find the γ -algorithm for finding a one-to-one function from Family(2) onto Family(5).

Definition A.1. For brevity, let **Family(0)** be the 0-th family, consisting of all the convergent power series $f \in \mathbb{C}\{y, z\}$ such that f is irreducible in $\mathbb{C}\{y, z\}$ with isolated singularity at $0 \in \mathbb{C}^2$, as in Definition 1.2 and Definition 2.4 of Part[A].

(1) **Family(1)** is the 1stfamily, consisting of all the standard Puiseux W-polys $f \in \mathbb{C}\{y, z\}$ of the recursive type with isolated singularity at $0 \in \mathbb{C}^2$, denoted by

$$(A.1.1) \quad \text{Family(1)} = \{f \text{ is arbitrary standard Puiseux W-poly of the recursive r-type: } f \in \text{Family(0) and } r \text{ are arbitrary positive integers}\},$$

with an equivalence relation for any two standard Puiseux W-polys of the recursive type in Family(1), as in Definition 1.2 of Part[A].

(2) **Family(2)** is the 2nd family, consisting of all the irreducible plane curve singularities with the standard Puiseux expansions, denoted by

$$(A.1.2) \quad \text{Family(2)} = \{C_r(t) : C_r(t) \text{ is the standard Puiseux expansion of the } r\text{-type for any } r \in \mathbb{N}\},$$

with an equivalence relation for any two standard Puiseux expansions in Family(2), as in Definition 1.2 of Part[A].

(3) **Family(3)** is the 3rd family, consisting of all the multiplicity sequences of irreducible plane curves with isolated singularity under the standard resolution, denoted by

$$(A.1.3) \quad \text{Family(3)} = \{\text{Multiseq}(V(f)) : f \in \text{Family(0) and } f \text{ is irreducible in } {}_2\mathcal{O}\}$$

with an equivalence relation for any two multiplicity sequences in Family(3), as in Definition 1.2 of Part[A].

(4) **Family(4)** is the 4-th family, consisting of all the divisors of $(f \circ \tau)$ defined by the total transform of $V(f)$, denoted by $(f \circ \tau)_{\text{divisor}}$ where $\tau : M \rightarrow \mathbb{C}^2$ is the standard resolution of the singularity of $V(f)$ with $f \in \text{Family(0)}$. Then, Family(4) is denoted by

$$(A.1.4) \quad \text{Family(4)} = \{(f \circ \tau)_{\text{divisor}} \text{ under } \tau : f \in \text{Family(0) and } \tau : M \rightarrow \mathbb{C}^2 \text{ is the standard resolution of the singularity of } V(f)\}.$$

As in Definition 1.2 of Part[A], in more detail, we define $(f \circ \tau)_{\text{divisor}}$, and also an equivalence relation for any two divisors $(f \circ \tau)_{\text{divisor}}$ and $(g \circ \mu)_{\text{divisor}}$ in Family(4), by the following:

(4-a) For any $f \in \text{Family(0)}$, let $\tau_\xi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ with $\tau = \tau_\xi$ be the composition of a finite number ξ of successive blow-ups π_i at $0 \in \mathbb{C}^2$, which is needed only to get the standard resolution of the singularity of $V(f)$, the zero set of f . Let $V(\tilde{f})$ be

the zero set of \tilde{f} where $\tilde{f} = f \circ \tau_\xi$. As a subset of $M^{(\xi)}$, $V(\tilde{f})$ consists of the exceptional curve $E = \tau_\xi^{-1}(0,0)$ and the proper transform $V^{(\xi)}(f)$ of $V(f)$. Let $\tau_\xi^{-1}(0,0) = \cup_{i=1}^\xi E_i$ be the decomposition into irreducible components, where each E_i is called an exceptional curve of the first kind. Then, $V(\tilde{f})$ vanishes along each irreducible component E_i of E with a certain multiplicity e_i . Define the total transform of $V(f)$ by the divisor of $f \circ \tau_\xi$, denoted by $(f \circ \tau_\xi)_{divisor}$, which is written in the form

$$(A.1.5) \quad (f \circ \tau_\xi)_{divisor} = V^{(\xi)}(f) + \sum_{i=1}^\xi e_i E_i,$$

where each e_i is the multiplicity of $f \circ \tau_\xi$ along E_i for $1 \leq i \leq \xi$ and $e_{i+1} > e_i$.

(4-b) For any f of (A.1.5) of (4-a), $\{(f \circ \tau_\xi)_{divisor}\}_{seq.} = \{e_i : i = 1, 2, \dots, \xi\}$ is called a sequence whose element consists of a coefficient of each irreducible component of the exceptional curve $E = \tau_\xi^{-1}(0,0)$. Then it is clear by (A.1.5) that $(f \circ \tau_\xi)_{divisor}$ in Family(4) can be viewed as a finite sequence $\{(f \circ \tau_\xi)_{divisor}\}_{seq.}$ of positive integers strictly increasing.

So, Family(4) of (A.1.4) can be equivalently rewritten by

$$(A.1.6) \quad \text{Family(4)}_{seq.} = \{ \{ (f \circ \tau)_{divisor} \}_{seq.} : f \in \text{Family(0)} \text{ is arbitrary where } \tau : M \rightarrow \mathbb{C}^2 \text{ is the standard resolution of the singularity of } V(f) \}, \text{ viewed as a family of sequences.}$$

(4-c) For any $g \in \text{Family(0)}$, using the same method as in both (4-a) and (4-b), let $\mu_\eta = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \dots \circ \bar{\pi}_\eta : \bar{M}^{(\eta)} \rightarrow \mathbb{C}^2$ be the composition of a finite number η of successive blow-ups at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singularity of $V(g)$. Define the total transform of $V(g)$ by the divisor of $g \circ \mu_\eta$, denoted by $(g \circ \mu_\eta)_{divisor}$, which is written in the form $(g \circ \mu_\eta)_{divisor} = V^{(\eta)}(g) + \sum_{i=1}^\eta \bar{e}_i \bar{E}_i$, where each \bar{e}_i is the multiplicity of $g \circ \mu_\eta$ along \bar{E}_i for $1 \leq i \leq \eta$ and $\bar{e}_{i+1} > \bar{e}_i$. Then, we can define $\{(g \circ \mu_\eta)_{divisor}\}_{seq.}$ by a sequence of $(g \circ \mu_\eta)_{divisor}$ such that $\{(g \circ \mu_\eta)_{divisor}\}_{seq.} = \{\bar{e}_i \in N : i = 1, 2, \dots, \eta\}$.

(4-d) For any two f of (4-a) and g of (4-c), it is said that $(f \circ \tau_\xi)_{divisor}$ and $(g \circ \mu_\eta)_{divisor}$ are equivalent (denoted by either $f \stackrel{divisor}{\sim} g$ under the standard resolutions or $(f \circ \tau_\xi)_{divisor} = (g \circ \mu_\eta)_{divisor}$ under the standard resolutions) if either of the following condition is satisfied:

$$(A.1.7) \quad \begin{aligned} &\text{either } \{(f \circ \tau_\xi)_{divisor}\}_{seq.} \equiv \{(g \circ \mu_\eta)_{divisor}\}_{seq.} \quad \text{as sequence,} \\ &\text{or } \{e_i : i = 1, 2, \dots, \xi\} \equiv \{\bar{e}_i \in N : i = 1, 2, \dots, \eta\} \quad \text{as sequence,} \\ &\text{or } \{e_i = \bar{e}_i : i = 1, 2, \dots, \xi = \eta\}. \quad \square \end{aligned}$$

Definition A.2. In preparation for construction of the new family, called Family(5), we can use the same properties and notations as in the definition of Family(4). Firstly, for convenience of notation, we define a subset of Family(4), which may be rewritten by

$$(A.2.0) \quad \text{Subfamily(4)} = \{(g_r \circ \tau)_{divisor} : \text{either } g_r \in \text{Family(1)} \text{ or } g_r \text{ is a Puiseux convergent power series of the recursive r-type in } \mathbb{C}\{y, z\} \subseteq \text{Family(4)}\},$$

where $\tau = \tau_\xi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ is the composition of a finite number ξ of successive blow-ups π_i at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singularity of $V(g_r)$, the zero set of g_r , noting by Theorem 1.4 and Theorem 1.6 of Part[A] and by Theorem 7.3 and Theorem 10.2 of Part[B] that $\text{Family(4)}_{seq.}$ and $\text{Subfamily(4)}_{seq.}$ are the same in the sense of (A.1.7) of Definition A.1.

Secondly, we will construct the following three definitions, denoted by Subdefinition A.2.2, Subdefinition A.2.4 and Subdefinition A.2.5, which will be proved to be well-defined, later. After then, it suffices to apply the above three definitions to the construction of Family(5) with an equivalence relation, as follows:

(a) Sublemma A.2.1 (Assumptions) By the definition of Subfamily(4), let $g_r \in \text{Family(1)}$ be such that $(g_r \circ \tau_\xi)_{divisor} = V^{(\xi)}(g_r) + \sum_{i=1}^\xi e_i E_i$ where $\tau_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ is the composition of a finite number ξ of successive blow-ups at the origin in \mathbb{C}^2 , which is the standard resolution of the singularity of $V(g_r)$ as we have seen in (A.1.5) and (A.2.0) and $\tau_\xi^{-1}(0,0) = E = \cup_{i=1}^\xi E_i$ is the decomposition into irreducible components.

(Conclusions) It was already proved by Theorem 14.0 of §14 of Part[C] that the standard representation of the divisor of $g_r \circ \tau_\xi$, denoted by $(g_r \circ \tau_\xi)_{\text{divisor}}$, is defined by the following: For any $A \subset M^{(\xi)}$, \bar{A} is called the closure of A in $M^{(\xi)}$.

$$(A.2.1) \quad \begin{aligned} (g_r \circ \tau_\xi)_{\text{divisor}} &= V^{(\xi)}(g_r) + \sum_{i=1}^{\lambda_r} e_i E_i \quad \text{with} \quad \lambda_r = \xi \\ &= V^{(\xi)}(g_r) + \sum_{i=1}^{\lambda_1} e_i E_i + \sum_{i=\lambda_1+1}^{\lambda_2} e_i E_i + \cdots + \sum_{i=\lambda_{r-1}+1}^{\lambda_r} e_i E_i, \end{aligned}$$

where each e_i is the multiplicity of $g_r \circ \tau_\xi$ along E_i for $1 \leq i \leq \xi = \lambda_r$ and $V^{(\xi)}(g_r)$ is the proper transform of $V(g_r)$ under τ_ξ and write $\lambda_0 = 0$ for notation, satisfying two properties:

Write $L = V^{(\xi)}(g_r) \cup (\bigcup_{i=1}^{\xi} E_i)$, and for any $A \subset M^{(\xi)}$ \bar{A} is the closure of A in $M^{(\xi)}$.

- (i) For each $i = 1, \dots, \xi$, $E_i \cap \overline{(L - E_i)}$ has at most three distinct points under τ_ξ in L .
- (ii)(ia) There is a strictly increasing finite sequence $\{\lambda_i : 1 \leq i \leq r\}$ such that $E_{\lambda_i} \cap \overline{(L - E_{\lambda_i})}$ has exactly three distinct points under τ_{λ_r} in L for each $i = 1, 2, \dots, r$.
- (iib) For any $j \notin \{\lambda_i : 1 \leq i \leq r\}$ where $1 \leq j \leq \xi$, $E_j \cap \overline{(L - E_j)}$ has at most two distinct points under τ_{λ_r} in L .

(a-1) Subdefinition A.2.2 Suppose that the same properties and notations as in the assumptions and conclusions of Sublemma A.2.1 hold. It is said that E_{λ_j} is called the j -th Puiseux exceptional curve of the first kind. Note that $1 < \lambda_1 < \lambda_2 < \cdots < \lambda_r = \xi$. It can be proved by Theorem 7.3 and Theorem 10.2 of Part[B] that a finite number r of Puiseux exceptional curves of the first kind for any $g_r \in \text{Family}(1)$ is invariant under an equivalence relation of Family(4).

(b) Sublemma A.2.3 (Assumptions) By Subdefinition A.2.1 with Subfamily(4), study all the exceptional curves of the first kind of $E = \bigcup_{i=1}^{\xi} E_i$ with $\xi = \lambda_r$ in more detail, and let E_{λ_j} be the j -th Puiseux exceptional curve of the first kind. For convenience of notation, let $\Omega^{(1)} = \bigcup_{i=1}^{\lambda_1} E_i$, $\Omega^{(2)} = \bigcup_{i=\lambda_1+1}^{\lambda_2} E_i$, \dots , and $\Omega^{(r)} = \bigcup_{i=\lambda_{r-1}+1}^{\lambda_r} E_i$. Applying the conclusion of Theorem 3.7 of Part[B] to the equations in both (14.2.5) and (14.3.1) of §14 of Part[C], it is trivial to prove that the following in the conclusions are true: For any $A \subset M^{(\xi)}$, \bar{A} is the closure of A in $M^{(\xi)}$.

(Conclusions) Let w be with $0 \leq w \leq r-1$. Then $\Omega^{(w+1)} = \bigcup_{i=\lambda_w+1}^{\lambda_{w+1}} E_i$ satisfies two properties : If necessary, we write $\lambda_0 = 0$.

Let E_t be an arbitrary exceptional curve of the first kind for $1 \leq t \leq \xi = \lambda_r$.

Property(1) For any $E_t \subset \Omega^{(w+1)}$ with $w+1 \leq r$, $E_t \cap \overline{\Omega^{(w+1)} - E_t}$ have at most two distinct points in $\Omega^{(w+1)}$.

Property(2) Let w be with $w+1 \leq r$, and $\Omega^{(w+1)} = \bigcup_{i=\lambda_w+1}^{\lambda_{w+1}} E_i$.

(i) There are two distinct exceptional curves of the first kind in $\Omega^{(w+1)}$, denoted by $E_{\lambda_{w+1}}$ and $E_{\lambda_w+s_w}$ with $1 < s_w \leq \lambda_{w+1} - \lambda_w$, each of which satisfies the following property: Note that $\Omega^{(w+1)} = \bigcup_{i=\lambda_w+1}^{\lambda_{w+1}} E_i$.

- (ia) $E_{\lambda_{w+1}} \cap \overline{\Omega^{(w+1)} - E_{\lambda_{w+1}}}$ has one and only one intersection point in $\Omega^{(w+1)}$.
- (ib) $E_{\lambda_w+s_w} \cap \overline{\Omega^{(w+1)} - E_{\lambda_w+s_w}}$ has one and only one intersection point in $\Omega^{(w+1)}$.
- (ii) $E_{\lambda_w+j} \cap \overline{\Omega^{(w+1)} - E_{\lambda_w+j}}$ has two distinct intersection points in $\Omega^{(w+1)}$ for any j where $1 < j \leq \lambda_{w+1} - \lambda_w$ and $j \neq s_w$.

(b-1) Subdefinition A.2.4. Suppose that the same properties and notations as in the assumptions and conclusions of Sublemma A.2.3 hold.

(a) If E_j satisfies (i) of Property(2) in Sublemma A.2.3, then E_j is called a singular exceptional curve of the first kind for $E = \bigcup_{i=1}^{\xi} E_i$.

(b) If E_j satisfies (ii) of Property(2), then E_j is called a regular exceptional curve of the first kind for $E = \bigcup_{i=1}^{\xi} E_i$.

Note that if $r \geq 2$, then $1 < s_w \leq \lambda_{w+1} - \lambda_w$ for each $w = 0, 1, \dots, r-1$, and $1 < s_0 < \lambda_1$. Also, if $r = 1$ then $1 < s_0 < \lambda_1$.

Remark Note by Subdefinition A.2.3 that the Puiseux exceptional curve of the first kind in Subdefinition A.2.2 may be a singular exceptional curve of the first kind in Subdefinition A.2.4, by computing the exceptional curves for the standard resolution of the singularity of $\{g(y, z) = (z^2 + y^3)^2 + y^5 z = 0\}$ at $0 \in \mathbb{C}^2$.

(c) **Subdefinition A.2.5 (Definition of singular part of the divisor).** Suppose that the same properties and notations as in the assumptions and conclusions of Sublemma A.2.3 hold. Using the same properties and notations as in Sublemma A.2.3 and Subdefinition A.2.4, $(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}$ is defined by the following:

$$(A.2.2) \quad (g_r \circ \tau_\xi)_{\text{singular part of the divisor}} = V^{(\xi)}(g_r) + \sum_{1,1} \\ \text{with} \quad \sum_{1,1} = \sum_{i=0}^{r-1} \{e_{\lambda_i+1} E_{\lambda_i+1} + e_{\lambda_i+s_i} E_{\lambda_i+s_i}\},$$

where $\lambda_0 = 0$ and each e_j is the multiplicity of $g_r \circ \tau_\xi$ along E_j for $1 \leq j \leq \xi$ and $V^{(\xi)}(g_r)$ is the proper transform of $V(g_r)$ under the standard resolution τ_ξ of the singularity of $V(g_r)$.

Note that E_{λ_i+1} and $E_{\lambda_i+s_i}$ are singular exceptional curves of the first kind for each $i = 1, 2, \dots, r$.

(d) **Family(5)** is the 5-th family, consisting of the singular part of the divisor defined by the total transform of $V(g_r)$ as we have seen in (A.2.2), denoted by $(g_r \circ \tau)_{\text{singular part of the divisor}}$,

$$(A.2.3) \quad \text{Family(5)} = \{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}} : g_r \in \text{Family(1)} \text{ where } \tau_\xi : M \rightarrow \mathbb{C}^2 \text{ is the standard resolution of the singularity of } V(g_r), \text{ the zero set of } g_r\}.$$

As we have seen in $\{\text{Family(4)}\}_{\text{seq.}}$ of (A.1.6), $(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}$ can be equivalently defined by a sequence of 2r elements, denoted by $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$,

$$(A.2.4) \quad \{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}} \\ = \{e_{\lambda_0+1}, e_{\lambda_0+s_0}; e_{\lambda_1+1}, e_{\lambda_1+s_1}; e_{\lambda_2+1}, e_{\lambda_2+s_2}; \dots; e_{\lambda_{r-1}+1}, e_{\lambda_{r-1}+s_{r-1}}\},$$

which is strictly increasing where $\lambda_0 = 0$ and $1 < s_{j-1} \leq \lambda_j - \lambda_{j-1}$ for $1 \leq j \leq r$.

Also, $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$ is called a sequence of $(g_r \circ \tau)_{\text{singular part of the divisor}}$. So, viewed as a family of sequences, **Family(5)** of (A.2.3) can be identified with **Family(5)**_{seq.},

$$(A.2.5) \quad \underline{\text{Family(5)}_{\text{seq.}} = \{ \{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}} : g_r \in \text{Family(1)} \\ \text{where } \tau_\xi : M \rightarrow \mathbb{C}^2 \text{ is the standard resolution of the singularity of } V(g_r) \}.$$

Observe that $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$ is a proper subsequence of $\{(g_r \circ \tau)_{\text{divisor}}\}_{\text{seq.}}$ where $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$ consists of $2r$ elements. As far as the computation for algorithms is concerned, it may be proved that the computation method of $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$ is much simpler than that of $\{(g_r \circ \tau)_{\text{divisor}}\}_{\text{seq.}}$.

(e) **In order to define an equivalence relation on Family(5)**, let $\phi_\rho \in \mathbb{C}\{y, z\}$ be the standard Puiseux polynomial of the recursive ρ -type as we have used in the Family(1) of Definition 1.2 of Part[A]. By the same method as we have used in the construction of $\{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}}$, as we have seen in (A.2.2), let $\mu_\eta = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \dots \circ \bar{\pi}_\eta : \bar{M}^{(\eta)} \rightarrow \mathbb{C}^2$ be the composition of a finite number η of successive blow-ups at $0 \in \mathbb{C}^2$, which is needed only to get the standard resolution of the singular point of $V(\phi_\rho)$.

First, by the same method as we have used in (A.2.1), first we can define $\{(\phi_\rho \circ \mu_\eta)_{\text{divisor}}\}_{\text{seq.}}$ by a sequence of $(\phi_\rho \circ \mu_\eta)_{\text{divisor}}$ where

$$(\phi_\rho \circ \mu_\eta)_{\text{divisor}} = V^{(\eta)}(\phi_\rho) + \sum_{i=1}^{\eta} \bar{e}_i \bar{E}_i \text{ and} \\ \{(\phi_\rho \circ \mu_\eta)_{\text{divisor}}\}_{\text{seq.}} = \{\bar{e}_i \in N : i = 1, 2, \dots, \eta = \psi_\rho\}, \text{ satisfying two properties:}$$

(i) Each \bar{E}_i of ψ_ρ exceptional curves of the first kind has at most three distinct intersection points with other exceptional curves of the first kind and the proper transform under μ_η .

(ii) There exists a sequence $\{\psi_i : 1 \leq i \leq \rho\}$ under μ_η such that $\bar{E}_{\psi_1}, \bar{E}_{\psi_2}, \dots, \bar{E}_{\psi_\rho}$ are the only ρ exceptional curves of the first kind, each of which has three distinct intersection points with other exceptional curves of the first kind and the proper transform.

Next, by the same method as we have used in (A.2.2) and by (A.2.4), define the following subsequence:

$$(A.2.6) \quad (\phi_\rho \circ \mu_\eta)_{\text{singular part of the divisor}} = V^{(\xi)}(\phi_\rho) + \sum_{i=1}^{\rho} \{e_{\psi_i+1} \bar{E}_{\psi_i+1} + e_{\psi_i+t_i} \bar{E}_{\psi_i+t_i}\},$$

$$\{(\phi_\rho \circ \mu_\eta)_{\text{singular part of the divisor}}\}_{\text{seq.}}$$

$$= \{\bar{e}_{\psi_0+1}, \bar{e}_{\psi_0+t_0}; \bar{e}_{\psi_1+1}, \bar{e}_{\psi_1+t_1}; \bar{e}_{\psi_2+1}, \bar{e}_{\psi_2+t_2}; \dots; \bar{e}_{\psi_{\rho-1}+1}, \bar{e}_{\psi_{\rho-1}+t_{\rho-1}}\},$$

where $\psi_0 = 0$ and $1 < t_{i-1} \leq \psi_i - \psi_{i-1}$ for $1 \leq i \leq \rho$.

As in (A.2.2) of Subdefinition A.2.5, note that \bar{E}_{ψ_i+1} and $\bar{E}_{\psi_i+t_i}$ are singular exceptional curves of the first kind for each $i = 1, 2, \dots, \rho$.

Then it is said that $(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}$ and $(\phi_\rho \circ \mu_\eta)_{\text{singular part of the divisor}}$ are equivalent if either (a) or (b) is satisfied:

$$(A.2.7) \quad (a) \quad \{(g_r \circ \tau_\xi)_{\text{singular part of the divisor}}\}_{\text{seq.}} \equiv \{(\phi_\rho \circ \mu_\eta)_{\text{singular part of the divisor}}\}_{\text{seq.}} \text{ as sequence.}$$

$$(b) \quad (b1) \quad \lambda_j = \psi_j \text{ and } s_j = t_j \text{ for } j = 0, \dots, r-1 = \rho-1 \text{ with } \lambda_r = \psi_\rho, \text{ and}$$

$$(b2) \quad e_{\lambda_j+1} = \bar{e}_{\lambda_j+1} \text{ and } e_{\lambda_j+s_j} = \bar{e}_{\lambda_j+s_j} \text{ for } j = 0, \dots, r-1. \quad \square$$

Remark A.2.6. We may assume that Family(4)_{seq.} and Subfamily(4)_{seq.} are the same in the sense of (A.1.7) of Definition A.1. Since any element of Family(4) is viewed as a finite sequence of positive integers which is strictly increasing by Definition A.1, note that each element of Family(5) can be viewed a proper subsequence of some element of Family(4), satisfying an additional property. \square

The solution of The α -algorithm can be given by Theorem A.4 and Theorem A.5. In preparation for the representation of The α -algorithm and other algorithms in the beginning of this appendix, we need the following definition with the new notation.

Definition A.3. The representation of an equivalence of any two elements of both Family(1) and Family(2), and also the representation of an equivalence of any two elements of both Family(1) and Family(4) in terms of multiplicity sequences of irreducible plane curve singularities can be defined as follows.

Definition A.3.0(The Euclidean multiplicity sequence for two positive integers).

Definition A.3.0 has the same statement as Definition 9.2 of Part[B] does in §9 of Part[B].

Definition A.3.1 [I] Let $S = \{t_i \in \mathbb{C} : i = 1, 2, \dots, \lambda\}$ be a finite sequence of complex numbers with $\lambda > 0$. It is said that S is the join of a finite number r of subsequences B_i of S in order, denoted by $S = \text{Join}(B_1, B_2, \dots, B_r)$, if the following are satisfied:

$$(A.3.1) \quad B_1 = \{t_i : i = 1, 2, \dots, q_1\}, \quad B_2 = \{t_{q_1+i} : i = 1, 2, \dots, q_2\}, \dots,$$

$$B_r = \{t_{q_{r-1}+i} : i = 1, 2, \dots, q_r\}$$

where each B_i is a subsequence of S such that $q_i > 0$ and $q_1 + q_2 + \dots + q_r = \lambda$.

Namely, whenever $t_\alpha \in B_i$ and $t_\beta \in B_j$ for any two B_i and B_j with $1 \leq i < j \leq r$, then $1 \leq \alpha < \beta \leq \lambda$.

[II] Let $g_r \in \text{Family}(1)$ be given arbitrary. By the same properties and notations as in (A.2.1) of Sublemma A.2.1 of Definition A.2 and by (A.2.0) of Definition A.2, $\{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}} \in \text{Family}(2)$ is well-defined by

$$(A.3.2) \quad \{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}} \equiv \{e_i : i = 1, 2, \dots, \lambda_r\} \equiv \text{Join}(B_1, B_2, \dots, B_r), \text{ as sequence}$$

for $j = 1, 2, \dots, r$, the j -th subsequence B_j of which is written respectively as follows:

$$(A.3.3) \quad \begin{aligned} B_1 &= \{e_i : i = 1, 2, \dots, \lambda_1\} \quad \text{with } 1 < \lambda_1 < \lambda_2 < \dots < \lambda_r, \\ B_j &= \{e_{\lambda_{j-1}+i} : i = 1, 2, \dots, (\lambda_j - \lambda_{j-1})\} \quad \text{for } j = 2, 3, \dots, r. \end{aligned}$$

Definition A.3.2 [I] Without any need of computation, we may assume by Theorem 1.4 and Theorem 1.6 of Part[A] that we can find the standard Puiseux polynomial $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type such that $V(g_r) \equiv C_r(t)$ (Multiseq) for given the standard Puiseux expansion $C_r(t)$ of the r -type of the curve, and conversely.

[II] For notation, we may assume without loss of generality that $g_r \in \text{Family}(1)$ satisfies the same notations and properties as in Definition 1.1 of Part[A], and also $C_r(t) \in \text{Family}(2)$ satisfies the same notations and properties as in Definition 1.2 of Part[A]. Then, it was already proved by Theorem 1.4 and Theorem 1.6 of Part[A] that $V(g_r) \equiv C_r(t)$ (Multiseq) if and only if (i) and (ii) of (A.3.4) are true, and in addition (iii) is just the computational formula for finding e_{λ_j} in this case. Thus, the following may be viewed as a definition for $V(g_r) \equiv C_r(t)$ (Multiseq):

$$(A.3.4) \quad \begin{aligned} &V(g_r) \equiv C_r(t) \text{ (Multiseq)} \\ \iff & \quad \begin{aligned} & \text{(i)} \quad n = n_1 d_1 \text{ and } \alpha_1 = \beta_{1,1} d_1 \text{ with } d_1 = \gcd(n, \alpha_1) \text{ and,} \\ & \text{(ii)} \quad d_{j-1} = n_j d_j \text{ and } \alpha_j - \alpha_{j-1} = \hat{\Delta}_j d_j \text{ with } d_j = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1}), \\ & \quad \text{where } \hat{\Delta}_j = \Delta_j(\beta_{j,k})_{k=1}^j - n_j n_{j-1} \Delta_{j-1}(\beta_{j-1,k})_{k=1}^{j-1} \text{ for } 2 \leq j \leq r. \end{aligned} \end{aligned}$$

Remark A.3.2.1. (iii) $e_{\lambda_1} = d_1 n_1 \beta_{1,1}$ and $e_{\lambda_j} = e_{\lambda_{j-1}} n_j + d_j n_j \hat{\Delta}_j$ for $2 \leq j \leq r$.

Definition A.3.3 For convenience of notation, we may assume that $g_r \in \text{Family}(1)$ satisfies the same properties and notations as in Definition A.3.1. Note that $\{\lambda_j : 1 < \lambda_1 < \lambda_2 < \dots < \lambda_r, 1 \leq j \leq r\}$ is a finite integer sequence in Subdefinition A.2.2. By the same way as we have used in (A.3.2) of Definition A.3.1, Multiseq($V(g_r)$) in Family(3) is well-defined by

$$(A.3.5) \quad \text{Multiseq}(V(g_r)) \equiv \{c_i : i = 1, 2, \dots, \lambda_r\} \equiv \text{Join}\{P_1, P_2, \dots, P_r\}, \quad \text{as sequence}$$

for $j = 1, 2, \dots, r$, each subsequence P_j of which can be uniquely written as follows:

$$(A.3.6) \quad \begin{aligned} P_1 &= \{c_i : i = 1, 2, \dots, \lambda_1\} \quad \text{and} \\ P_j &= \{c_{\lambda_{j-1}+i} : i = 1, 2, \dots, (\lambda_j - \lambda_{j-1})\} \quad \text{for } j = 2, 3, \dots, r. \quad \square \end{aligned}$$

§ A.2 and § A.3. The α -algorithm(The algorithm for finding a one-to-one function between Family(2) and Family(3)) and its examples

§ A.2. The 1st half of the α -algorithm(Theorem A.4)

Theorem A.4(The 1st half of the α -algorithm: an algorithm for finding a one-to-one function from Family(2) into Family(3)).

Assumptions Let the standard Puiseux expansion $C_r(t)$ of the r -type for the curve C be given as follows:

$$(A.4.1) \quad \begin{aligned} C_r(t) &:= \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}, \end{cases} \\ &\text{where } 2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_r \quad \text{and} \\ &\quad n > d_1 > d_2 > \dots > d_r = 1 \quad \text{with } d_i = \gcd(n, \alpha_1, \dots, \alpha_i), 1 \leq i \leq r. \end{aligned}$$

Conclusions

(1)(1a) For a good representation, use Definition A.3.3 with (A.3.5) and Definition A.3.0. Then, Multiseq($C_r(t)$), called the multiplicity sequence of $C_r(t)$, can be clearly represented by the following algorithm:

(The 1st half of the α -algorithm(Theorem A.4))

$$(A.4.2)^* \quad \text{Multiseq}(C_r(t)) = \text{Join}(P_1, P_2, \dots, P_r),$$

where $P_1 = \{[n : \alpha_1]\}, P_2 = \{[d_1 : \alpha_2 - \alpha_1]\}, \dots, P_r = \{[d_{r-1} : \alpha_r - \alpha_{r-1}]\}.$

Note. As we have seen in either (9.2.3) or (9.4.3) of Part[B], we use the same kind of notations as in (A.4.2)*:

Assuming that $\gcd(n_1, k_1) = 1$ and $\{[n_1, k_1]\} = \{c_1, c_2, \dots, c_t\}$ by Definition A.3.0, for brevity of notation we may define $\{[dn_1, dk_1]\} = \{d[n_1, k_1]\} = \{dc_1, dc_2, \dots, dc_t\}$ for any positive integer d .

(1b) The representation of $\text{Multiseq}(C_r(t))$ in (A.4.2) is unique. That is, if $\text{Multiseq}(C_r(t))$ has another representation with $\text{Multiseq}(C_r(t)) = \text{Join}(Q_1, Q_2, \dots, Q_s)$ for an integer $s > 0$ in the sense of Definition A.3.3, then $P_i \equiv Q_i$ as sequence for $1 \leq i \leq r = s$ by Definition A.2.

(2) Assuming for notation that any standard Puiseux expansion has the same multiplicity sequence as we have seen in (A.4.2), it is trivial to compute the corresponding standard Puiseux expansion with the same multiplicity sequence. \square

Remark A.4.1. (i) For any $C_r(t) \in \text{Family}(2)$, we may assume that $V(g_r) \equiv C_r(t)$ (multi. seq.) for some $g_r \in \text{Family}(1)$. Using the properties and notations of Definition A.3 for $(g_r \circ \tau_\xi)_{\text{divisor}} \in \text{Family}(4)$, then the number of elements in $\{[d_{j-1} : \alpha_j - \alpha_{j-1}]\}$, as a sequence, is equal to $\lambda_j - \lambda_{j-1}$ for $2 \leq j \leq r$, which must be uniquely determined.

(ii) The proof of Theorem A.4 is clear.

Example A.4.2 for Theorem A.4: As in Example 1.6.3 for Theorem 1.6 of Part[A], let the standard Puiseux expansion $C_4(t)$ of the 4-th type for the curve be given by

$$(A.4.3) \quad C_4(t) := \begin{cases} y = t^{100} \\ z = t^{250} + t^{375} + t^{410} + t^{417}. \end{cases}$$

By (The 1st half of the α -algorithm(Theorem A.4)), $\text{Multiseq}(C_4(t))$ can be written as follows:

$$(A.4.4) \quad \text{Multiseq}(C_4(t)) = \text{Join}(P_1, P_2, P_3, P_4),$$

where $P_1 = \{[100 : 250]\}, P_2 = \{[50 : 125]\}, P_3 = \{[25 : 35]\}$ and $P_4 = \{[5 : 7]\}.$

because of the following elementary computations:

- (i) $n = 100$ and $\alpha_1 = 250 \implies P_1 = \{[n : \alpha_1]\} = \{[100 : 250]\}.$
- (ii) $P_2 = \{[d_1 : \alpha_2 - \alpha_1]\}$ with $d_1 = \gcd(n, \alpha_1) = \gcd(100, 250) = 50$ and $\alpha_2 - \alpha_1 = 125.$
- (iii) $P_3 = \{[d_2 : \alpha_3 - \alpha_2]\}$ with $d_2 = \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(50, 125) = 25$ and $\alpha_3 - \alpha_2 = 35.$
- (iv) $P_4 = \{[d_3 : \alpha_4 - \alpha_3]\}$ with $d_3 = \gcd(d_2, \alpha_3 - \alpha_2) = \gcd(25, 35) = 5$ and $\alpha_4 - \alpha_3 = 7.$

§A.3 The 2nd half of The α -algorithm(Theorem A.5)

The 2nd half of the α -algorithm is an algorithm for finding a function from Family(2) onto Family(3). We will compute The 2nd half of the α -algorithm in more detail.

Theorem A.5(The 2nd half of The α -algorithm: an algorithm for finding the standard Puiseux expansion from either any multiplicity sequence of irreducible plane curve singularities or any finite sequence of positive integers).

Assumptions Assume that E is a finite sequence of positive integers as follows:

$$(A.5.1) \quad E = \{a_1, a_2, \dots, a_t\},$$

where $a_1 \geq a_2 \geq \dots \geq a_{t-1} \geq a_t = 1$ are all positive integers.

(i) The 1st problem is to determine whether or not E is equal to the multiplicity sequence of an irreducible plane curve with isolated singularity.

(ii) The 2nd problem is to find the standard Puiseux expansion $C_r(t)$ of the r -type for the irreducible plane curve such that $\text{Multiseq}(C_r(t)) \equiv E$ as sequence if the solution for The 1st problem is positive.

To solve The 1st problem and The 2nd problem at the same time, we may assume by the same kind of notations as in (A.4.2)* that $\text{Multiseq}(C(t))$ can be represented as follows:

$$(A.5.2) \quad \text{Multiseq}(C(t)) = \text{Join}(P_1, P_2, \dots, P_r),$$

with $P_1 = \{[n : \alpha_1]\}$ and $P_i = \{[d_{i-1} : \alpha_i - \alpha_{i-1}]\}$ for $2 \leq i \leq r$,

where the standard Puiseux expansion of $C(t)$ is given by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_r}$ for some integer $r \geq 1$, with the following properties:

- (a) $2 \leq n < \alpha_1 < \alpha_2 < \dots < \alpha_r$.
- (b) $n > d_1 > \dots > d_r = 1$, and $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$ for $1 \leq i \leq r$.

Conclusions For each $k = 1, 2, \dots, r$, the aim is to find an elementary computational algorithm for the construction of the k -th Euclidean multiplicity subsequence $P_k = \{[d_{k-1} : \alpha_k - \alpha_{k-1}]\}$ of E for $1 \leq k \leq r$ (**The 2nd half of The α -algorithm**). For notation write $P_1 = \{[n : \alpha_1]\}$ with $d_0 = n$ and $\alpha_0 = 0$, if necessary.

The first step for The 2nd half of The α -algorithm An elementary computational algorithm formula for $P_1 = \{[n : \alpha_1]\}$ with d_1 can be represented as follows:

$$(A.5.3)$$

$$\begin{aligned} n &= a_1 = \text{the largest element in } E \quad \text{and} \\ \tau_0 &= \text{the finite number of the element } n \text{ in } E, \\ \alpha_1 &= \tau_0 n + \max \{a_i \in E : a_i < n\} \quad \text{with} \\ d_1 &= \gcd(n, \alpha_1) \in E, \end{aligned}$$

where note by definition that $\max \{a_i \in E : a_i < n\}$ is equal to the second largest positive integer in E . Then, observe that n and α_1 were already computed by (A.5.3).

Now, it remains only to determine whether or not the problem is completely solved.

(i) If $d_1 = \gcd(n, \alpha_1) = 1$, then the standard Puiseux expansion of $C(t)$ is completely solved by $y = t^n$ and $z = t^{\alpha_1}$.

(ii) If $d_1 = \gcd(n, \alpha_1) > 1$, then take the next step.

Assuming that $d_1 > 1$, for $k = 2, 3, \dots, r$, an elementary computational algorithm formula with $(\alpha_k - \alpha_{k-1})$ and d_k can be represented as follows:

The k -th step for The 2nd half of The α -algorithm Let $d_{k-1} = \gcd(n, \alpha_1, \dots, \alpha_{k-1}) > 1$ with $k \geq 2$, and let $\alpha_{k-1} - \alpha_{k-2}$ be given with $\alpha_0 = 0$. Then, an elementary computational algorithm formula for $P_k = \{[d_{k-1} : \alpha_k - \alpha_{k-1}]\}$ with d_k can be represented as follows:

$$(A.5.4)$$

$$\begin{aligned} \sigma_{k-1} &= \frac{\min \{a_i \in E : a_i > d_{k-1}\}}{d_{k-1}} \quad \text{and,} \\ \tau_{k-1} &= \text{the finite number of the element } d_{k-1} \text{ in } E, \\ \alpha_k - \alpha_{k-1} &= (\tau_{k-1} - \sigma_{k-1})d_{k-1} + \max \{a_i \in E : a_i < d_{k-1}\} \quad \text{with} \\ d_k &= \gcd(d_{k-1}, \alpha_k - \alpha_{k-1}) \in E. \end{aligned}$$

Then, we can compute the following: $\alpha_k = \alpha_{k-1} + (\alpha_k - \alpha_{k-1})$.

Now, it remains only to determine whether or not the problem is completely solved.

(i) If $d_k = \gcd(d_{k-1}, \alpha_k - \alpha_{k-1}) = 1$, then the standard Puiseux expansion of $C(t)$ is completely solved by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_k}$.

(ii) If $d_k = \gcd(d_{k-1}, \alpha_k - \alpha_{k-1}) > 1$, then take the next step. \square

Remark A.5.1. Note that $\sigma_{k-1} = \frac{\min \{a_i \in E : a_i > d_{k-1}\}}{d_{k-1}} = \frac{\text{the 2nd smallest element in } P_{k-1}}{d_{k-1}}$ is the finite number of the element d_{k-1} in P_{k-1} because $d_{k-1} = \min P_{k-1} = \min \{a \in P_{k-1} : a \text{ is arbitrary}\}$ for each fixed $k \geq 2$, and so $\tau_{k-1} - \sigma_{k-1} \geq 0$.

Example A.6 and Example A.7 for The 2nd half of The α - algorithm(Theorem A.5): Here are two finite sequences G_i of positive integers for $i = 1, 2$, as it has been seen in (A.5.1). By The 2nd half of The α - algorithm, we show that the following are true:

Example A.6: We prove that (i) G_1 is a multiplicity sequence, which is equal to that of some irreducible plane curve singularity defined by the standard Puiseux expansion $C_r(t)$ of the r -type and (ii) compute $C_r(t)$.

Example A.7: We can prove that G_2 cannot be a multiplicity sequence defining any irreducible plane curve singularity.

Example A.6: If the finite sequence G_1 is given below, find the standard Puiseux expansion for an irreducible curve $C(t)$ which has the same multiplicity sequence as in G_1 , if exists:

$$(A.6.1) \quad \begin{aligned} G_1 &= \{a_i : i = 1, 2, \dots, 31\} \\ &= \{12600, 12600; 7200; 5400; 1800, 1800, 1800, 1800; \\ &\quad 500, 500, 500; 300; 200; 100, 100, 100, 100; \\ &\quad 55; 45; 10, 10, 10, 10; 5, 5, 5, 5; 3; 2; 1, 1\}. \end{aligned}$$

To get the solution, suppose that G_1 is equal to the multiplicity sequence which is defined by the standard Puiseux expansion for the curve $C(t)$, and then take the following steps.

The first step Find the construction for $P_1 = \{[n : \alpha_1]\}$, the Euclidean multiplicity sequence for two given positive integers n and α_1 . Note that $\min\{P_1\} = d_1 = \gcd(n, \alpha_1)$.

(I) By (A.5.3), an easy computation says that

$$\begin{aligned} n &= 12600 = \text{the largest element in } G_1 \quad \text{and} \\ \tau_0 &= \text{the finite number of the element } n \text{ in } G_1 = 2, \\ \alpha_1 &= \tau_0 n + \max \{a_i \in G_1 : a_i < n\} = 2 \cdot 12600 + 7200 = 32400 \quad \text{with} \\ d_1 &= \gcd(n, \alpha_1) = \gcd(12600, 32400) = \gcd(12600, 7200) = 1800 \in G_1. \end{aligned}$$

By Definition A.3.0, an easy computation says that $d_1 > 1$ and

$$(A.6.2) \quad \begin{aligned} P_1 &= \{[n : \alpha_1]\} = \{[12600 : 32400]\} = \{12600, 12600; 7200; 5400; 1800, 1800, 1800\}, \\ G_1 &= \text{Join}(P_1, Q) \text{ for some subsequence } Q \text{ of } G_1. \end{aligned}$$

So, we can take the next step.

The second step Find the construction for $P_2 = \{[d_1 : \alpha_2 - \alpha_1]\}$, as the second Euclidean multiplicity subsequence of G_1 . Note that $\min P_2 = d_2 = \gcd(d_1, \alpha_2 - \alpha_1)$.

(I) By (A.5.4), an easy computation says that

$$\begin{aligned} \sigma_1 &= \frac{\min \{a_i \in G_1 : a_i > d_1\}}{d_1} = \frac{5400}{1800} = 3 \quad \text{and} \\ \tau_1 &= \text{the counting number of the element } d_1 \text{ in } G_1 = 4, \\ \alpha_2 - \alpha_1 &= (\tau_1 - \sigma_1)d_1 + \max \{a_i \in G_1 : a_i < d_1\} = (4 - 3)1800 + 500 = 2300 \quad \text{with} \\ d_2 &= \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(1800, 2300) = \gcd(1800, 500) = 100 \in G_1. \end{aligned}$$

Thus, $\alpha_2 = \alpha_1 + (\alpha_2 - \alpha_1) = 32400 + 2300 = 34700$ with $d_2 = 100 > 1$.

(II) By (I), note that $\alpha_2 - \alpha_1 = 2300$ and $d_1 = 1800$ with $d_2 = 100 > 1$.

For the necessity of the next step with $d_2 > 1$, by Definition A.3.0, it is clear that

$$(A.6.3) \quad \begin{aligned} P_2 &= \{[d_1 : \alpha_2 - \alpha_1]\} = \{[1800 : 2300]\} \\ &= \{1800; 500, 500, 500; 300; 200; 100, 100\}. \end{aligned}$$

Then, $G_1 = \text{Join}(P_1, P_2, Q)$ for a subsequence Q of G_1 , and so we can take the next step.

The third step Find the construction for $P_3 = \{[d_2 : \alpha_3 - \alpha_2]\}$, as the third Euclidean multiplicity subsequence of G_1 . Note that $\min P_3 = d_3 = \gcd(d_2, \alpha_3 - \alpha_2)$.

(I) By the same method as in (A.5.4), we can compute the following:

$$\alpha_3 = \alpha_2 + (\alpha_3 - \alpha_2) = 34700 + 255 = 34955 \text{ with } d_3 = 5 > 1.$$

(II) By (I), note that $\alpha_3 - \alpha_2 = 255$ and $d_2 = 100$ with $d_3 = 5 > 1$.

For the necessity of the next step with $d_3 > 1$, by Definition A.3.0, it is clear that

$$(A.6.4) \quad P_3 = \{[d_2 : \alpha_3 - \alpha_2]\} = \{[100 : 255]\} = \{100, 100; 55; 45; 10, 10, 10, 10; 5, 5\}.$$

Then, $G_1 = \text{Join}(P_1, P_2, P_3, Q)$ for some subsequence Q of G_1 , and so we can take the next step.

The fourth step Find the construction for $P_4 = \{[d_3 : \alpha_4 - \alpha_3]\}$, as the fourth Euclidean multiplicity subsequence of G_1 . Note that $\min P_4 = d_4 = \gcd(d_3, \alpha_4 - \alpha_3)$.

(I) By the same method as in (A.5.4), we can compute the following:

$$\alpha_4 = \alpha_3 + (\alpha_4 - \alpha_3) = 34955 + 13 = 34968 \text{ with } d_4 = 1.$$

(II) By (I), note that $\alpha_4 - \alpha_3 = 13$ and $d_3 = 5$ with $d_4 = 1$.

Since $d_4 = 1$, then by Definition A.3.0, an easy computation says that

$$(A.6.5) \quad \begin{aligned} P_4 &= \{[d_3 : \alpha_4 - \alpha_3]\} = \{[5 : 13]\} = \{5, 5; 3; 2, 2; 1, 1\} \text{ with } P_4 = Q, \text{ and so} \\ G_1 &= \text{Join}(P_1, P_2, P_3, P_4) \text{ by Theorem A.5.} \end{aligned}$$

Summarizing four steps, we proved that G_1 is a multiplicity sequence being equivalent to some irreducible plane curve $C(t)$ with $G_1 = \text{Join}(P_1, P_2, P_3, P_4)$ where $n = 12600$, $\alpha_1 = 32400$, $\alpha_2 = 34700$, $\alpha_3 = 34955$ and $\alpha_4 = 34968$.

So, the standard Puiseux expansion $C_4(t)$ of the 4-type for the curve $C(t)$ is given as follows:

$$(A.6.6) \quad C_4(t) := \begin{cases} y = t^{12600} \\ z = t^{32400} + t^{34700} + t^{34955} + t^{34968}. \end{cases}$$

Example A.7: If the finite sequence G_2 is given below, find the standard Puiseux expansion for an irreducible plane curve $C(t)$ whose multiplicity sequence is equal to G_2 , if exists:

$$(A.7.1) \quad \begin{aligned} G_2 &= \{b_i : i = 1, 2, \dots, 30\} \\ &= \{12600, 12600; 7200; 5400; 1800, 1800, 1800, 1800; \\ &\quad 500, 500; 300; 200; 100, 100, 100, 100; \\ &\quad 55; 45; 10, 10, 10, 10; 5, 5, 5, 5; 3; 2; 1, 1\}. \end{aligned}$$

In order to find whether G_2 is equivalent to a multiplicity sequence defining an irreducible plane curve singularity or not, apply the same method and notations, as we have used in Example A.6, to Example A.7. Then, we can compute that G_2 cannot be a multiplicity sequence for any irreducible plane curve singularity.

§A.4. The proof for Theorem A.5(The 2nd half of the α -algorithm)

Proof of Theorem A.5. It is clear that Theorem A.4 is true. For the proof, we use the same properties and notations as in Theorem A.4 of §A.2 and Theorem A.5 of §A.3. Moreover, we may start to assume by Definition A.3.0, and Theorem 11.2 and Corollary

11.3 of Part[B] that E in the assumption of Theorem A.5 satisfies two properties (i) and (ii):

(i) As a multiplicity sequence, E can be written as follows:

$$(A.5.5) \quad E = \{\mu_{1,1}, \mu_{1,2}, \dots, \mu_{1,q_1}; \mu_{2,1}, \mu_{2,2}, \dots, \mu_{2,q_2}; \dots; \mu_{w-1,1}, \mu_{w-1,2}, \dots, \mu_{w-1,q_{w-1}}; \mu_{w,1}, \mu_{w,2}, \dots, \mu_{w,q_w}\} \\ = \{Q_1; Q_2; \dots; Q_{w-1}; Q_w\} \quad \text{with} \quad Q_i = P_i \quad \text{for } 1 \leq i \leq w = r+1,$$

where each subsequence $\{Q_i\} = \{\mu_{i,1}, \mu_{i,2}, \dots, \mu_{i,q_i}\}$ is called the i -th Euclidean divisor subsequence of E , consisting of q_i elements for $i = 1, 2, \dots, w$, and $q_1 + q_2 + \dots + q_w = t$.

(ii) We have the following:

$$(A.5.6) \quad \begin{aligned} \mu_{1,j} &= a_j & \text{for } 1 \leq j \leq q_1. \\ \mu_{2,j} &= a_{q_1+j} & \text{for } 1 \leq j \leq q_2. \\ &\dots\dots\dots \\ \mu_{w,j} &= a_{q_1+q_2+\dots+q_{w-1}+j} & \text{for } 1 \leq j \leq q_w. \end{aligned}$$

Now, by induction on the positive integer, we are going to find an elementary computational algorithm for the solution of the main problem. First of all, note that $n = a_1 = \mu_{1,1}$.

[I] The 1st step: Let $n = a_1$. In this step, the aim is to find an elementary computational algorithm for α_1 , in construction of the Euclidean divisor sequence $\{P_1\} = \{[n : \alpha_1]\}$.

Then, it is enough to consider the following in order:

- (a) Let τ_0 be the counting number of n in E . First, find τ_0 .
- (b) Next, compute α_1 by (a) and by Definition A.3.0.
- (c) If $\gcd(n, \alpha_1) = 1$, then the algorithm is completely finished.
If $\gcd(n, \alpha_1) > 1$, then take the 2nd step.

After we solve (a) and (b), then we have the following algorithm for the 1st step.

The algorithm for the 1st step: An elementary computational algorithm formula for $P_1 = [n : \alpha_1]$ with α_1 can be represented as follows:

$$(A.5.7) \quad \boxed{\begin{aligned} n &= a_1 = \text{the largest element in } E \quad \text{and} \\ \tau_0 &= \text{the counting number of the element } n \text{ in } E, \\ \alpha_1 &= \tau_0 n + \max \{a_i \in E : a_i < n\} \quad \text{with} \\ d_1 &= \gcd(n, \alpha_1) \in E, \end{aligned}}$$

where $\max \{a_i \in E : a_i < n\}$ is equal to the second largest positive integer in E .

Thus, observe that n and α_1 were already computed by (A.5.7).

- (c) Moreover, to finish this step, it is enough to consider the following: Note that $\gcd(n, \nu_2) = \gcd(n, \alpha_1) = d_1$ where $\alpha_1 = \tau_0 n + \nu_2$.
- (c1) If $\gcd(n, \alpha_1) = 1$, then the problem is completely solved.
- (c2) If $\gcd(n, \alpha_1) > 1$, take the next step.

[II] The 2nd step: Let $d_1 = \gcd(n, \alpha_1) > 1$. Note by the first step that $a_{q_1} = \mu_{1,q_1} = \gcd(n, \alpha_1) = \min\{P_1\}$ from (A.5.5) and (A.5.6).

In this step, the aim is to find an elementary computational algorithm for $\alpha_2 - \alpha_1$, in construction of the Euclidean divisor sequence $\{P_2\} = \{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}$.

Then, it is enough to consider the following in order:

- (a) Let σ_1 be the counting number of $\min\{P_1\} = \gcd(n, \alpha_1)$ in $\{[n, \alpha_1]\} = \{P_1\}$, and τ_1 be the counting number of the element $\gcd(n, \alpha_1)$ in E . First, find σ_1 and τ_1 , respectively. It is clear that $\tau_1 \geq \sigma_1$.
- (b) Next, compute $\alpha_2 - \alpha_1$, using the result of (a) and Definition A.3.0.
- (c) If $\gcd(n, \alpha_1, \alpha_2) = 1$, then the algorithm is completely finished.
If $\gcd(n, \alpha_1, \alpha_2) > 1$, then take the third step.

Now, let us solve (a), (b) and (c) in order.

(a) We are going to compute σ_1 and τ_1 , respectively.

(a1) Then, $\gcd(n, \alpha_1) = \mu_{1,q_1}$ in $\{[n, \alpha_1]\} = \{P_1\}$ for coincidence of notation in (A.5.5). Then, by Definition A.3, we can compute σ_1 as follows:

Let μ_{1,s_1} be the second smallest element in $\{P_1\}$. First, it is trivial to compute σ_1 such that $\mu_{1,s_1} = \sigma_1 \mu_{1,q_1}$ for $1 \leq s_1 < q_1$.

(a2) It is clear that τ_1 is directly countable in E .

(b) In order to compute $\alpha_2 - \alpha_1$, there are two cases: Note that $\gcd(n, \alpha_1) = a_{q_1}$.

(b1) $\tau_1 = \sigma_1$ and (b2) $\tau_1 > \sigma_1$.

(b1) Let $\tau_1 = \sigma_1$. Then, observe by Definition A.3 that $\alpha_2 - \alpha_1 < \gcd(n, \alpha_1)$.

We compute $\alpha_2 - \alpha_1$ as follows:

$$(*) \quad \alpha_2 - \alpha_1 = \text{the largest among the elements in } E \\ \text{any of which is less than } \gcd(n, \alpha_1) = d_1 = \mu_{2,1} = a_{q_1+1}$$

Therefore, $a_{q_1+1} \in E$ can be easily found from (A.5.5) and (A.5.6).

(b2) Let $\tau_1 > \sigma_1$. Observe by Definition A.3 that $\alpha_2 - \alpha_1 > \gcd(n, \alpha_1)$. So, by Definition A.3,

$$\alpha_2 - \alpha_1 = \gamma_1 \gcd(n, \alpha_1) + \text{the second largest element in } P_2,$$

where $\max\{P_2\} = \gcd(n, \alpha_1)$ and γ_1 is the number of the element $\gcd(n, \alpha_1)$ in $\{P_2\}$.

Observe that $\gamma_1 = \tau_1 - \sigma_1$, and also that the second largest among the elements in P_2 is the same as the largest among the elements in E , any of which is less than $\gcd(n, \alpha_1)$.

Find the largest among the elements in E , each of which is less than $\gcd(n, \alpha_1) = \min\{[n, \alpha_1]\}$.

Therefore, $\alpha_2 - \alpha_1$ can be computed as follows:

$$(**) \quad \alpha_2 - \alpha_1 = (\tau_1 - \sigma_1) \gcd(n, \alpha_1) + \text{the largest among the elements in } E \\ \text{any of which is less than } \gcd(n, \alpha_1).$$

Note that $\gcd(n, \alpha_1) =$ the largest among the elements in $\{[\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]\}$ with $\gcd(n, \alpha_1) < \alpha_2 - \alpha_1$.

The algorithm for the second step: For either of two cases (b1) and (b2), an elementary computational algorithm formula for $P_2 = [\gcd(n, \alpha_1) : \alpha_2 - \alpha_1]$ with $\alpha_2 - \alpha_1$ can be represented as follows, at the same time:

(A.5.8)

$\sigma_1 = \frac{\min\{a_i \in E : a_i > d_1\}}{d_1} \quad \text{and}$ $\tau_1 = \text{the counting number of the element } d_1 \text{ in } E,$ $\alpha_2 - \alpha_1 = (\tau_1 - \sigma_1)d_1 + \max\{a_i \in E : a_i < d_1\} \quad \text{with}$ $d_2 = \gcd(d_1, \alpha_2 - \alpha_1) \in E.$

Note that $\sigma_1 = \frac{\min\{a_i \in E : a_i > d_1\}}{d_1} = \frac{\text{the 2nd smallest element in } \{P_1\}}{\min\{P_1\}}$ is the counting number of the smallest element $d_1 = \min\{P_1\}$ in $\{[n, \alpha_1]\} = \{P_1\}$, and so $\tau_1 - \sigma_1 \geq 0$.

Next, it is easy to compute α_2 and $\gcd(n, \alpha_1, \alpha_2)$ as follows:

(b3) $\alpha_2 = \alpha_1 + (\alpha_2 - \alpha_1)$.

(b4) $\gcd(n, \alpha_1, \alpha_2) = \gcd(n, \alpha_1, \alpha_2 - \alpha_1)$.

(c) Moreover, to finish this step, it is enough to consider the following:

(c1) If $d_2 = \gcd(n, \alpha_1, \alpha_2) = 1$, then the problem is completely solved.

(c2) If $d_2 = \gcd(n, \alpha_1, \alpha_2) > 1$, then take the next step.

The general case is proved by induction. Suppose we have shown that the algorithm for the k -th step is given by (A.5.4). In order to find such an algorithm for $(k+1)$ -th step, it

suffices to prove the algorithm given by (A.5.9) later in the next case [III], which can be computable as follows. For the proof of the (k+1)-th step, we may assume that $d_k > 1$, otherwise there is nothing to prove.

[III] The (k+1)-th step: Let $d_k = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_k) > 1$. Note by the k-th step that $\mu_{k,q_k} = d_k$ from (A.5.5) and (A.5.6).

In this step, the aim is to find an elementary computational algorithm for $\alpha_{k+1} - \alpha_k$, in construction of the Euclidean divisor sequence $\{P_{k+1}\} = \{[d_k : \alpha_{k+1} - \alpha_k]\}$.

Then, it is enough to consider the following in order:

- (a) Let σ_k be the counting number of $d_k = \min\{P_k\}$ in $\{[d_{k-1} : \alpha_k - \alpha_{k-1}]\} = \{P_k\}$, and τ_k be the counting number of the element d_k in E . First, find σ_k and τ_k , respectively. It is clear that $\tau_k \geq \sigma_k$.
- (b) Next, compute $\alpha_{k+1} - \alpha_k$ by the result of (a) and the fundamental algorithm.
- (c) If $d_{k+1} = 1$, then the algorithm is completely finished.
If $d_{k+1} > 1$, then take the next step.

Now, let us solve (a), (b) and (c) in order.

(a) We are going to compute σ_k and τ_k , respectively.

(a1) Then, $d_k = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_k) = \mu_{k,q_k}$ in $\{[d_{k-1} : \alpha_k - \alpha_{k-1}]\} = \{P_k\}$, for coincidence of notation in (A.5.5). Then, by the fundamental algorithm, we can compute σ_k as follows:

Let μ_{k,s_k} be defined by the second smallest element in $\{P_k\}$, which is easy to be found, for coincidence of notation in (A.5.5). First, it is trivial to compute σ_k such that $\mu_{k,s_k} = \sigma_k \mu_{k,q_k}$ for $1 \leq s_k < q_k$.

(a2) It is clear that τ_k is directly countable in E .

(b) In order to compute $\alpha_{k+1} - \alpha_k$, there are two cases:

(b1) $\tau_k = \sigma_k$ and (b2) $\tau_k > \sigma_k$.

(b1) Let $\tau_k = \sigma_k$. Then, observe by Definition A.3 that $\alpha_{k+1} - \alpha_k < d_k$.

We compute $\alpha_{k+1} - \alpha_k$ as follows:

- (*) $\alpha_{k+1} - \alpha_k =$ the largest among the elements in E
any of which is less than $d_k = \mu_{k+1,1} = a_{q_1+q_2+\dots+q_k+1}$.

(b2) Let $\tau_k > \sigma_k$. Then, observe by Definition A.3 that $\alpha_{k+1} - \alpha_k > d_k$.
So, by the fundamental algorithm,

$$\alpha_{k+1} - \alpha_k = \gamma_k d_k + \text{the second largest element in } P_{k+1},$$

where $\max\{P_{k+1}\} = d_k$ and γ_k is the number of the element $d_k = \min\{P_k\}$ in $\{P_k\}$.

Observe that $\gamma_k = \tau_k - \sigma_k$, and also that the second largest among the elements in P_{k+1} is the same as the largest among the elements in E , any of which is less than $d_k = \min\{P_k\}$.

Find the largest among the elements in E , each of which is less than $d_k = \min\{P_k\}$.

Therefore, $\alpha_{k+1} - \alpha_k$ can be computed as follows:

- (**) $\alpha_{k+1} - \alpha_k = (\tau_k - \sigma_k)d_k +$ the largest among the elements in E ,
any of which is less than d_k .

Note that $d_k =$ the largest among the elements in $\{P_{k+1}\} = \{[d_k : \alpha_{k+1} - \alpha_k]\}$ with $d_k < \alpha_{k+1} - \alpha_k$.

The algorithm for the (k+1)-th step: For either of two cases (b1) and (b2), an elementary computational algorithm formula for $\{P_{k+1}\} = \{[d_k : \alpha_{k+1} - \alpha_k]\}$ with $\alpha_{k+1} - \alpha_k$ can be represented as follows, at the same time:
(A.5.9)

$$\begin{aligned} \sigma_k &= \frac{\min\{a_i \in E : a_i > d_k\}}{d_k} \quad \text{and} \\ \tau_k &= \text{the counting number of the element } d_k \text{ in } E, \\ \alpha_{k+1} - \alpha_k &= (\tau_k - \sigma_k)d_k + \max\{a_i \in E : a_i < d_k\} \quad \text{with} \\ d_{k+1} &= \gcd(d_k, \alpha_{k+1} - \alpha_k) \in E. \end{aligned}$$

Note that $\sigma_k = \frac{\min\{a_i \in E : a_i > d_k\}}{d_k} = \frac{\text{the 2nd smallest element in } \{P_k\}}{\min\{P_k\}}$ is the counting number of the smallest element $d_k = \min\{P_k\}$ in $\{P_k\}$ for $k \geq 1$, and so $\tau_k - \sigma_k \geq 0$.

Next, it is easy to compute α_{k+1} and $d_{k+1} = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{k+1})$ as follows:

(b3) $\alpha_{k+1} = \alpha_k + (\alpha_{k+1} - \alpha_k)$.

(b4) $d_{k+1} = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{k+1}) = \gcd(d_k, \alpha_{k+1} - \alpha_k)$ for $k \geq 1$.

(c) Moreover, to finish this step, it is enough to consider the following:

(c1) If $d_{k+1} = 1$, then the problem is completely solved.

(c2) If $d_{k+1} > 1$, then take the next step.

Thus, the proof of the theorem is completely finished. \square

Remark A.5.2. (a) Assuming that $k = 1$ in the above computational algorithm formula (A.5.4) for the generalized k -th step, by the convenience of notation, write $d_0 = n$, and then $\sigma_0 = 0$. As we have seen in (A.5.3), let τ_0 be the counting number of the element $d_0 = n$ in E when E is viewed as a finite sequence. Thus, if $k = 1$, then the elementary computational algorithm formula for $\alpha_1 - \alpha_0$ with d_1 can be viewed as follows: Note that $d_0 = n$ and $\sigma_0 = 0$. Also, we write $\alpha_0 = 0$, if necessary.
(A.5.4)*

$\begin{aligned} \sigma_0 &= 0 \quad \text{and} \\ \tau_0 &= \text{the counting number of the element } d_0 \text{ in } E, \\ \alpha_1 - \alpha_0 &= (\tau_0 - \sigma_0)d_0 + \max \{a_i \in E : a_i < d_0\} \quad \text{with} \\ d_1 &= \gcd(d_0, \alpha_1 - \alpha_0) \in E. \end{aligned}$

(b) A computational method for solving an existence of the standard Puiseux expansion being equivalent to any given finite sequence of all positive integers can be found by the same algorithm as we have used in Theorem A.5.

Appendix B

The β -algorithm for finding a one-to-one function from Family(2) onto Family(4)

§ B.0. Introduction

In Appendix B, the aim is to find an algorithm for computing a one-to-one function from Family(2) onto Family(4) in Definition A.1, called the β -algorithm for notation.

§B.1. The terminology and notations in preparation for finding the β -algorithm

We use the same terminology and notations as it has been in Definition A.1 and Definition A.2 of Appendix A.

§ B.2 and § B.3. The β -algorithm(The algorithm for finding a one-to-one function between Family(2) and Family(4)) and its examples

§ B.2. The 1st half of the β algorithm(Theorem B.2)

Theorem B.2(The 1st half of the β algorithm: an algorithm for finding a one-to-one function from Family(4) into Family(2)).

Assumptions *As in Family(4) of Definition A.1, for any $f \in \text{Family}(0)$ let $(f \circ \tau_\xi)_{\text{divisor}}$ be the divisor of $f \circ \tau_\xi$ given by*

$$(B.2.1) \quad (f \circ \tau_\xi)_{\text{divisor}} = V^{(\xi)}(f) + \sum_{i=1}^{\xi} e_i E_i,$$

where each e_i is the multiplicity of $f \circ \tau_\xi$ along E_i for $1 \leq i \leq \xi$ and $e_{i+1} > e_i$. Note that $\tau_\xi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_\xi : M^{(\xi)} \rightarrow \mathbb{C}^2$ is the composition of a finite number ξ of successive blow-ups π_i at the origin in \mathbb{C}^2 , which is needed only to get the standard resolution of the singularity of $V(f)$.

Conclusions *To compute the standard Puiseux expansion $C(t)$ with $f \stackrel{\text{multiseq}}{\sim} C(t)$ at the origin in \mathbb{C}^2 is to find **The 1st half of the β algorithm**, assuming that the standard Puiseux expansion $C_r(t)$ of the r -type satisfies an equation for some r in (B.2.2):*

$$(B.2.2) \quad C(t) = C_r(t) := \begin{cases} y = t^n \\ z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_r}, \end{cases}$$

where (i) $2 \leq n < \alpha_1 < \alpha_2 < \cdots < \alpha_r$ for a positive integer r and
(ii) $n > d_1 > d_2 > \cdots > d_r = 1$, and $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$ for $1 \leq i \leq r$.

(The 1st half of the β algorithm) *For notation, write $\lambda_r = \xi$. Then, by Sublemma A.2.1 of Definition A.2(Theorem 14.0 of Part[C]) and by (B.2.2), there are exactly r exceptional curves of the first kind, written by $\{E_{\lambda_w} : 1 = \lambda_0 < \lambda_1 < \cdots < \lambda_w < \cdots < \lambda_r = \xi\}$, each of which has three distinct intersection points with other exceptional curves and the proper transform $V^{(\lambda_r)}(f)$ under τ_{λ_r} . It is clear that $n = e_1$. So, to find the desired solution, it suffices to compute α_w and e_{λ_w} for each w -th step on the positive integer $w = 1, 2, \dots, r$.*

Step 1 for (The 1st half of the β algorithm) *For this step, the problem is how to compute the Puiseux exponent α_1 and the coefficient e_{λ_1} .*

Note that $n = e_1$, and let S_1 be the set defined by

$$(B.2.3) \quad S_1 = \{e_i \in S_0 : e_i \text{ cannot be divisible by } e_1 \text{ for } i > 1\},$$

where $S_0 = \{e_i : i = 1, 2, \dots, \xi\}$ by (B.2.1).

Then, the computational algorithm formula for α_1 and the coefficient e_{λ_1} of E_{λ_1} can be represented as follows: $\text{lcm}(n, \alpha_1)$ = the least common multiple of n and α_1 .

$$(B.2.4) \quad n = e_1, \quad \alpha_1 = \min S_1 \quad \text{and} \quad e_{\lambda_1} = \text{lcm}(n, \alpha_1) = n_1 \beta_{1,1} d_1,$$

$$\text{where} \quad d_1 = \gcd(n, \alpha_1) \quad \text{with} \quad n = n_1 d_1 \quad \text{and} \quad \alpha_1 = \beta_{1,1} d_1.$$

Then, there are two subcases:

Subcase(1) If $d_1 = 1$, then the desired standard Puiseux expansion $C(t) = C_1(t)$ can be defined by $y = t^n$ and $z = t^{\alpha_1}$.

Subcase(2) If $d_1 > 1$, then take the next step.

Let $d_{w-1} = \gcd(n, \alpha_1, \alpha_2, \dots, \alpha_{w-1}) > 1$ with $w \geq 2$. Then, an elementary computational algorithm formula for α_w can be represented as follows:

Step w for (The 1st half of the β algorithm) Let $d_{w-1} > 1$ with $2 \leq w \leq r$.

$$\text{Consider } (f \circ \tau_{\lambda_r})_{\text{divisor}} - V^{(\lambda_r)}(f) - \sum_{i=1}^{\lambda_{w-1}} e_i E_i = \sum_{i=\lambda_{w-1}+1}^{\lambda_r} e_i E_i.$$

For this step, the problem is how to compute the α_w and the coefficient e_{λ_w} , assuming by the induction method that each α_k and the coefficients e_{λ_k} of E_{λ_k} have been already computed for any $k = 1, 2, \dots, w-1$.

Let $S_w = \{e_i - e_{\lambda_{w-1}} : \lambda_{w-1} + 1 \leq i \leq \lambda_r\}$.

For Step w, we have exactly two cases:

Case(i) $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1}$.

Case(ii) $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1}$.

Case(i) of Step w for (The 1st half of the β algorithm) Let $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1}$.

Then, the computational algorithm formula for α_w , d_w , and the coefficient e_{λ_w} of E_{λ_w} can be represented as follows: It can be proved by (14.0.3) of Theorem 14.0 of Part[C] that $e_{(\lambda_{w-1})}$ is divisible by d_{w-1} for each $w \geq 2$.

(B.2.5) First, compute a unique positive integer s with $e_{(\lambda_{w-1}+s)}$, as follows:

$$(i) \quad e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1}.$$

$$(ii) \quad e_{(\lambda_{w-1}+t)} - e_{(\lambda_{w-1})} = t d_{w-1} \text{ for each } t = 1, 2, \dots, s-1.$$

$$(iii) \quad e_{(\lambda_{w-1}+s)} - e_{(\lambda_{w-1})} \neq s d_{w-1}.$$

(B.2.6) By (B.2.5), compute α_w and d_w , and e_{λ_w} (if necessary), as follows:

$$\alpha_w - \alpha_{w-1} = e_{(\lambda_{w-1}+s)} - e_{(\lambda_{w-1})} \quad \text{with} \quad e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1},$$

$$\text{noting that } d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1}) \quad \text{with } d_{w-1} = d_w n_w \text{ and } \alpha_w - \alpha_{w-1} = d_w \hat{\Delta}_w,$$

$$\text{and } e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \hat{\Delta}_w \text{ by (A.3.4).}$$

After the computation is done for Case(i) of Step w, there are two subcases for Case(i):

Subcase(1) If $d_w = 1$, then the standard Puiseux expansion can be defined by $y = t^n$ and

$$z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_w}.$$

Subcase(2) If $d_w > 1$, take the next step, Step (w+1) for (The 1st half of the β algorithm).

Case(ii) of Step w for (The 1st half of the β algorithm) Let $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1}$.

Then, the computational algorithm formula for α_w , d_w and the coefficient e_{λ_w} of E_{λ_w} can be represented as follows:

$$(B.2.7) \quad \alpha_w - \alpha_{w-1} = e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} \quad \text{with} \quad e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1},$$

$$\text{noting that } d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1}) \quad \text{with } d_{w-1} = d_w n_w \text{ and } \alpha_w - \alpha_{w-1} = d_w \hat{\Delta}_w,$$

$$\text{and } e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \hat{\Delta}_w \text{ by (A.3.4).}$$

After the computation is done for Case(ii) of Step w, there are two subcases for Case(ii):

Subcase(1) If $d_w = 1$, then the standard Puiseux expansion can be defined by $y = t^n$ and

$$z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_w}.$$

Subcase(2) If $d_w > 1$, take the next step, Step (w+1) for (The 1st half of the β algorithm) \square .

Remark. Note by (B.2.4) of Step 1 and by Theorem 3.6 of Part[B] that $g_r = (z^{n_1} + \varepsilon_1 y^{\beta_{1,1}})^{d_1} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(0)} y^\alpha z^\beta$ where ε_1 is a unit $\mathbb{C}\{y, z\}$ and the $c_{\alpha, \beta}^{(0)}$ are some nonzero complex numbers such that $n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} d_1$.

Remark B.2.1 for Theorem B.2.

(1) After λ_{w-1} iterations of blow-ups, let $(v_{\lambda_{w-1}}, u_{\lambda_{w-1}})$ and $(v'_{\lambda_{w-1}}, u'_{\lambda_{w-1}})$ be the local coordinates for $M^{(\lambda_{w-1})}$ where $\pi_{\lambda_{w-1}} : M^{(\lambda_{w-1})} \rightarrow M^{(\lambda_{w-1}-1)}$ was defined to be the λ_{w-1} -th blow-up at some point of $M^{(\lambda_{w-1}-1)}$ with $u'_{\lambda_{w-1}} = 1/u_{\lambda_{w-1}}$ and $v'_{\lambda_{w-1}} = v_{\lambda_{w-1}} u_{\lambda_{w-1}}$. Note that $E_{\lambda_{w-1}} = \{v_{\lambda_{w-1}} = 0\} \cup \{v'_{\lambda_{w-1}} = 0\}$ is the λ_{w-1} -th exceptional curve of the first kind.

$$(B.2.7.1) \quad (g_r \circ \tau_{\lambda_{w-1}})_{total} = v_{\lambda_{w-1}}^{e_{(\lambda_{w-1})}} (g_r \circ \tau_{\lambda_{w-1}})_{proper} \quad \text{with } f = g_r \in \text{Family}(1),$$

$$(g_r \circ \tau_{\lambda_{w-1}})_{proper} = \{(1 + \varepsilon_{w-1} u_{\lambda_{w-1}})^{n_w} + v_{\lambda_{w-1}}^{\widehat{\Delta}_w}\}^{d_w}$$

$$+ \sum_{\alpha, \beta \geq 0} B_{\alpha, \beta}^{(w-1)} v_{\lambda_{w-1}}^{\alpha} (1 + \varepsilon_{w-1} u_{\lambda_{w-1}})^{\beta},$$

where (i) $e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \widehat{\Delta}_w$ for $2 \leq j \leq r$, and $e_{\lambda_1} = d_1 n_1 \beta_{1,1}$,

(ii) $\varepsilon_{w-1} = \varepsilon_{w-1}(1 + \varepsilon_{w-1} u_{\lambda_{w-1}}, v_{\lambda_{w-1}})$ is a unit in $\mathbb{C}\{1 + \varepsilon_{w-1} u_{\lambda_{w-1}}, v_{\lambda_{w-1}}\}$ and the $B_{\alpha, \beta}^{(w-1)}$ are some nonzero complex numbers,

(iii) $d_w = \gcd(n, \alpha_1, \dots, \alpha_w) = n_{w+1} n_{w+2} \cdots n_r \geq 2$,

(iv) $\widehat{\Delta}_w = \Delta_w(\beta_{w,k})_{k=1}^w - n_w n_{w-1} \Delta_{w-1}(\beta_{w-1,k})_{k=1}^{w-1} > 0$,

(v) $\gcd(n_w, \widehat{\Delta}_w) = 1$ and $n_w \alpha + \widehat{\Delta}_w \beta > n_w \widehat{\Delta}_w d_w$.

(2) Observe by (B.2.7.1) that the following remark may be necessary for the construction of the statement of Theorem B.2.

Case(i) of Step w for Solution : Note that there exists an integer $s > 0$ such that $(s - 1)n_w < \widehat{\Delta}_w < sn_w$ because $2 \leq n_w < \widehat{\Delta}_w$, and then $e_{(\lambda_{w-1}+s)} = e_{(\lambda_{w-1})} + d_w \widehat{\Delta}_w$.

Case(ii) of Step w for Solution : Note that there exists an integer $s > 0$ such that $(s - 1)\widehat{\Delta}_w < n_w \leq s\widehat{\Delta}_w$ because $1 \leq \widehat{\Delta}_w < n_w$, and then $e_{(\lambda_{w-1}+s)} = se_{(\lambda_{w-1})} + d_{w-1}$.

Example B.2.2 for Theorem B.2. As in Definition A.1, let $(f \circ \tau_{\xi})_{divisor}$ be the divisor of $f \circ \tau_{\xi}$ given by the following:

$$(B.2.8) \quad (f \circ \tau_{\xi})_{divisor} = V^{(\xi)}(f) + \sum_{i=1}^{\xi} e_i E_i \quad \text{with } \xi = 12,$$

where $e_1 = 45$, $e_2 = 60$, $e_3 = 120$, $e_4 = 180$, $e_5 = 186$, $e_6 = 372$, $e_7 = 555$, $e_8 = 930$, $e_9 = 933$, $e_{10} = 935$, $e_{11} = 1869$, $e_{12} = 2805$.

To find the standard Puiseux expansion $C_r(t)$ for the curve C such that $C_r(t) \equiv V(f)$ (multi. seq.), by **(The 1st half of the β algorithm for Theorem B.2)**, just compute the following:

Step 1: It is clear that $n = e_1 = 45$, and $\alpha_1 = e_2 = 60$.

In preparation for the computation in Step 2, $d_1 = \gcd(n, \alpha_1) = \gcd(45, 60) = 15 > 1$ because $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$ imply that $n_1 = 3$, $\beta_{1,1} = 4$, and $e_{\lambda_1} = n_1 \beta_{1,1} d_1 = 180$ by (A.3.4).

Step 2: Since $e_{\lambda_1+1} - e_{\lambda_1} = 186 - 180 = 6 < 15 = d_1$, by Case(ii) of Step 2, $\alpha_2 - \alpha_1 = e_{\lambda_1+1} - e_{\lambda_1} = 6$. So, it is clear by Step 1 that $\alpha_2 = \alpha_1 + 6 = 60 + 6 = 66$.

In preparation for the computation in Step 3, $d_2 > 1$ and $e_{\lambda_2} = 1155$ because of the following computations:

$d_2 = \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(15, 6) = 3 > 1$ because $d_1 = n_2 d_2$ and $\alpha_2 - \alpha_1 = \widehat{\Delta}_2 d_2$ imply that $n_2 = 5$ and $\widehat{\Delta}_2 = 2$, and so $e_{\lambda_2} = n_2 \cdot e_{\lambda_1} + d_2 n_2 \widehat{\Delta}_2 = 5 \cdot 180 + 3 \cdot 5 \cdot 2 = 930$ by (A.3.4).

Step 3: Since $e_{\lambda_2+1} - e_{\lambda_2} = 933 - 930 = 3 = d_2$, by Case(i) of Step 3, $\alpha_3 - \alpha_2 = e_{(\lambda_2+s_2)} - e_{\lambda_2} = 5$ because $e_{(\lambda_2+s_2)}$ with $s_2 > 0$ is defined to be the first appearing integer which cannot be divisible by d_2 . Note by (14.0.3) of Theorem 14.0 of Part[C] that $e_{(\lambda_j)}$ is divisible by d_j . So, it is clear by Step 2 that $\alpha_3 = \alpha_2 + 5 = 66 + 5 = 71$.

So, the standard Puiseux expansion for $C(t)$ such that $C(t) \equiv V(f)$ (multi. seq.) can be given by $y = t^{45}$ and $z = t^{60} + t^{66} + t^{71}$, noting that $d_3 = 1$ because $d_2 = 3$.

§ **B.3. The 2nd half of the β algorithm(Theorem B.3)**

Theorem B.3(The 2nd half of the β algorithm: an algorithm for finding a function from Family(4) onto Family(2)).

Assumptions (1) Let the standard Puiseux expansion $C_r(t)$ of the r -th type for the curve C in Family(2) be given by

$$(B.3.1) \quad C(t) = C_r(t) := \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_r}, \end{cases}$$

where $2 \leq n < \alpha_1 < \alpha_2 < \cdots < \alpha_r$ and
 $n > d_1 > d_2 > \cdots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

(2) Without any need of proof, we may assume by Theorem 1.6 of Part[A] that there exists the standard Puiseux polynomial $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type in Family(1) such that $V(g_r) \equiv C_r(t)$ (multi. seq.).

(3) To compute $(g_r \circ \tau_{\lambda_r})_{\text{divisor}}$ of Family(2) with $V(g_r) \equiv C_r(t)$ (multi. seq.) directly, for brevity of notation we may assume by Theorem 1.6 of Part[A] that $(g_r \circ \tau_{\lambda_r})_{\text{divisor}}$ satisfies the same kind of properties and notations in Definition A.3.2.

Conclusions By (A.3.2) and (A.3.3) of Definition A.3.1, the problem is how to write explicit algorithm for finding the sequence $\{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}} \equiv \{e_i : i = 1, 2, \dots, \lambda_r\}$ as sequence, which consists of the coefficients of E_i for $(g_r \circ \tau_{\lambda_r})_{\text{divisor}}$.

Let $S = \{e_i : i = 1, 2, \dots, \lambda_r\}$. By Subdefinition A.2.1 of Definition A.2 and by Definition A.3, S is well-defined by $S = \text{Join}(B_1, B_2, \dots, B_r)$ with the following properties:

$$(B.3.2) \quad B_1 = \{b_{0,i} = e_i : i = 1, 2, \dots, \lambda_1\},$$

$$B_w = \{b_{w-1,i} = e_{\lambda_{w-1}+i} : i = 1, 2, \dots, (\lambda_w - \lambda_{w-1})\} \quad \text{for } w = 2, 3, \dots, r.$$

A desired algorithm(The 2nd half of the β algorithm for Theorem B.3) for finding the unique sequence $S = \text{Join}\{B_1, B_2, \dots, B_r\}$ is as follows:

In preparation for finding such a desired algorithm by (A.3.4), first the computation formulas for $\{d_1, n_1, \beta_{1,1}\}$ by (i), for $\{d_w, n_w, \hat{\Delta}_w : 2 \leq w \leq r\}$ by (ii), and for $\{e_{\lambda_w} : 1 \leq w \leq r\}$ by (iii) respectively, will be most applicable:

$$(B.3.3) \quad \begin{aligned} \text{(i)} \quad & d_1 = \gcd(n, \alpha_1) \text{ with } n = n_1 d_1 \text{ and } \alpha_1 = \beta_{1,1} d_1. \\ \text{(ii)} \quad & d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1}) \text{ with } d_{w-1} = n_w d_w \text{ and } \alpha_w - \alpha_{w-1} = \hat{\Delta}_w d_w. \\ \text{(iii)} \quad & e_{\lambda_1} = d_1 n_1 \Delta_1(\beta_{1,1}) \text{ and } e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \hat{\Delta}_w \text{ for } 2 \leq w \leq r. \end{aligned}$$

(The 2nd half of the β algorithm)

Step 1 for (The 2nd half of the β algorithm) To find B_1 , let $h_0 = (z^{n_1} + y^{\beta_{1,1}})$ and $H_0 = \prod_{i=1}^{d_1} (z^{n_1} + c_i y^{\beta_{1,1}})$ where all the c_i are the nonzero distinct complex numbers for $1 \leq i \leq d_1$.

Let τ_{λ_1} be the composition of a finite number λ_1 of successive blow-ups at the origin in \mathbb{C}^2 , which is the standard resolution of the isolated singularity of $V(h_0)$.

To find B_1 , it suffices to compute $(H_0 \circ \tau_{\lambda_1})_{\text{divisor}}$ under τ_{λ_1} because of the following:

$$(B.3.4) \quad (H_0 \circ \tau_{\lambda_1})_{\text{divisor}} = V^{(\lambda_1)}(H_0) + \sum_{i=1}^{\lambda_1} \ell_i E_i \quad \text{where } B_1 = \{e_i = \ell_i : i = 1, 2, \dots, \lambda_1\}.$$

Step w for (The 2nd half of the β algorithm) To find B_w ($2 \leq w \leq r$), let $h_{w-1} = v(u^{n_w} + v^{\hat{\Delta}_w})$, and let $H_{w-1} = \prod_{j=1}^{e_{\lambda_{w-1}}} (v + a_j u^{2n_w}) \prod_{i=1}^{d_w} (u^{n_w} + c_i v^{\hat{\Delta}_w})$ where all the a_j are the nonzero distinct complex numbers for $1 \leq j \leq e_{\lambda_{w-1}}$ and all the c_i are the nonzero distinct complex numbers for $1 \leq i \leq d_w$.

Let μ_{w,m_w} be the composition of a finite number m_w of successive blow-ups at $(u, v) = (0, 0)$ which are the standard resolution of the isolated singularity of both $V(h_{w-1})$ and $V(H_{w-1})$.

For a better representation, H_{w-1} can be also replaced by $H_{w-1} = v^{e_{\lambda_w-1}} \{(u^{n_w} + v^{\widehat{\Delta}_w})^{d_w} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(w-1)} v^\alpha u^\beta\}$ where the $c_{\alpha, \beta}^{(w-1)}$ are some nonzero complex numbers such that $n_w \alpha + \widehat{\Delta}_w \beta > n_w \widehat{\Delta}_w d_w$, if necessary.

To find B_w , it suffices to compute $(H_{w-1} \circ \mu_{m_w})_{divisor}$ by Theorem 3.6 of Part[B] and (B.2.7.1) because of the following:

$$(B.3.5) \quad (H_{w-1} \circ \mu_{m_w})_{divisor} = V^{(m_w)}(H_{w-1}) + \sum_{i=1}^{m_w} \ell_i E_i,$$

where $B_w = \{e_{\lambda_{w-1}+i} = \ell_i : i = 1, 2, \dots, \lambda_w - \lambda_{w-1}\}$ with $m_w = \lambda_w - \lambda_{w-1}$.

Note by Proposition 14.1 of Part[C] that $\tau_{\lambda_r} = \mu_{1, m_1} \circ \mu_{2, m_2} \circ \dots \circ \mu_{r, m_r}$ with $\lambda_r = m_1 + m_2 + \dots + m_r$ and that $e_{\lambda_{w-1}}$, n_w , $\widehat{\Delta}_w$ and d_w were already known.

Example B.3.1 for Theorem B.3. Let the parametrization for the Puiseux expansion be given by the following:

$$(B.3.6) \quad C(t) := \begin{cases} y = t^{45} \\ z = t^{60} + t^{65} + t^{71}. \end{cases}$$

This is the standard Puiseux expansion $C_3(t)$ of the 3rd type because of the computations (a) and (b):

- (a) $n < \alpha_1 < \alpha_2 < \alpha_3$ where $n = 45$, $\alpha_1 = 60$, $\alpha_1 = 65$ and $\alpha_3 = 71$.
- (b) $n = 45 > d_1 = 15 > d_2 = 5 > d_3 = 1$ where $d_1 = \gcd(n, \alpha_1) = \gcd(45, 60) = 15$, $d_2 = \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(15, 5) = 5$ and $d_3 = \gcd(d_2, \alpha_3 - \alpha_2) = \gcd(5, 6) = 1$.

Then, (i) Since $d_1 = \gcd(n, \alpha_1) = 15$, then $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$ imply that $n_1 = 3$ and $\beta_{11} = 4$.

(ii) Since $d_2 = \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(15, 5) = 5$, then $d_1 = d_2 n_2$ and $\alpha_2 - \alpha_1 = d_2 \widehat{\Delta}_2$ imply that $n_2 = 3$ and $\widehat{\Delta}_2 = 1$.

(iii) Since $d_3 = \gcd(d_2, \alpha_3 - \alpha_2) = \gcd(5, 6) = 1$, then $d_2 = d_3 n_3$ and $\alpha_3 - \alpha_2 = d_3 \widehat{\Delta}_3$ imply that $n_3 = 5$ and $\widehat{\Delta}_3 = 6$.

Also, $e_{\lambda_1} = n_1 \beta_{1,1} d_1 = 180$, $e_{\lambda_2} = n_2 e_{\lambda_1} + n_2 \widehat{\Delta}_2 d_2 = 540 + 15 = 555$, $e_{\lambda_3} = n_3 e_{\lambda_2} + n_3 \widehat{\Delta}_3 d_3 = 2775 + 30 = 2805$.

Therefore, we have the following:

(i) $H_0 = (z^{n_1} + y^{\beta_{1,1}})^{d_1} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(0)} y^\alpha z^\beta = (z^3 + y^4)^{15} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(0)} y^\alpha z^\beta$ where the $c_{\alpha, \beta}^{(0)}$ are some nonzero complex numbers such that $3\alpha + 4\beta = n_1 \alpha + \beta_{1,1} \beta > n_1 \beta_{1,1} d_1 = 180$.

(ii) $H_1 = v_1^{e_{\lambda_1}} \{(u_1^{n_2} + v_1^{\widehat{\Delta}_2})^{d_2} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(1)} v_1^\alpha u_1^\beta\} = v_1^{180} \{(u_1^3 + v_1)^5 + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(1)} v_1^\alpha u_1^\beta\}$ where the $c_{\alpha, \beta}^{(1)}$ are nonzero complex numbers such that $3\alpha + \beta = n_2 \alpha + \widehat{\Delta}_2 \beta > n_2 \widehat{\Delta}_2 d_2 = 15$.

(iii) $H_2 = v_2^{e_{\lambda_2}} \{(u_2^{n_3} + v_2^{\widehat{\Delta}_3})^{d_3} + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(2)} v_2^\alpha u_2^\beta\} = v_2^{555} \{(u_2^5 + v_2^6) + \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(2)} v_2^\alpha u_2^\beta\}$ where the $c_{\alpha, \beta}^{(2)}$ are nonzero complex numbers such that $5\alpha + 6\beta = n_3 \alpha + \widehat{\Delta}_3 \beta > n_3 \widehat{\Delta}_3 d_3 = 30$.

Following the same notations and properties as in (The 2nd half of the β algorithm for Theorem B.3), compute

$$(H_0 \circ \tau_{\lambda_1})_{divisor} \quad \text{and} \quad (H_{j-1} \circ \mu_{m_j})_{divisor} \quad \text{for } j = 2, 3, \quad \text{respectively.}$$

Using the blow-up process directly, then we can compute the following:

$B_1 = \{e_1, e_2, e_3, e_4\} = \{45, 60, 120, 180\}$, $B_2 = \{e_5, e_6, e_7\} = \{185, 370, 555\}$, and $B_3 = \{e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}\} = \{560, 561, 1122, 1683, 2244, 2805\}$.

Appendix [C]

The γ -algorithm for finding a one-to-one function from Family(2) onto Family(5)

§ C.0. Introduction

In Appendix [C], the aim is to find an algorithm for computing a one-to-one function from Family(2) of Definition A.1 onto Family(5) of Definition A.2, called the γ -algorithm for notation, which must be proved to be well-defined by Theorem C.2 and Theorem C.3.

§C.1. The terminology and notations in preparation for finding the β -algorithm

We use the same terminology and notations as it has been in Definition A.1, Definition A.2 and Definition A.3 of Appendix A.

Theorem C.1. *As we have seen in the definition of a singular exceptional curve of the first kind in Subdefinition A.2.4 and Subdefinition A.2.5 of Definition A.2, a singular part of the divisor in Subdefinition A.2.5 is well-defined. Rigorously speaking, it will be proved by two theorems, Theorem C.2 and Theorem C.3 that we can compute The γ -algorithm for finding a one-to-one function between Family(2) and Family(5).*

§C.2 and §C.3. The γ -algorithm(The algorithm for finding a one-to-one function between Family(2) and Family(5)) and its examples)

§ C.2. The 1st half of The 3rd algorithm(Theorem C.2)

Theorem C.2(The 1st half of the γ -algorithm: an algorithm for finding a one-to-one function from Family(2) into Family(5)).

Assumptions (1) *Let the standard Puiseux expansion $C_r(t)$ of the r -th type for the curve C in Family(2) be given by*

$$(C.2.1) \quad C_r(t) := \begin{cases} y = t^n, \\ z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_r}, \end{cases}$$

where $2 \leq n < \alpha_1 < \alpha_2 < \cdots < \alpha_r$ and

$n > d_1 > d_2 > \cdots > d_r = 1$ with $d_i = \gcd(n, \alpha_1, \dots, \alpha_i)$, $1 \leq i \leq r$.

(2) *Without any need of proof, we may assume by Theorem 1.6 of Part[A] that we can compute the standard Puiseux polynomial $g_r \in \mathbb{C}\{y, z\}$ of the recursive r -type in Family(1) such that $V(g_r) \equiv C_r(t)$ (multi. seq.).*

Conclusions *By Definition A.2, it is clear that Family(4) and Family(4)_{seq} can be identified and that Family(5) and Family(5)_{seq} can be identified.*

Also, $S_1 = \{(g_r \circ \tau_{\lambda_r})_{\text{singular part of the divisor}}\}_{\text{seq.}} \in \text{Family}(4)_{\text{seq}}$ is a proper subsequence of a finite sequence $S = \{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}} \in \text{Family}(5)_{\text{seq}}$.

To solve the problem is how to find an algorithm for computing S_1 directly, without computing S completely.

Let $S_1 = \text{Join}(T_1, T_2, \dots, T_r)$, satisfying the following: Write $\lambda_0 = 1$, if necessary.

$$(C.2.2) \quad \begin{aligned} T_1 &= \{e_i : i = 1, s_0\} \quad \text{with } 1 < s_0 \leq \lambda_1, \\ T_j &= \{e_{\lambda_{j-1}+i} : i = 1, s_{j-1}\} \quad \text{for } 2 \leq j \leq r \text{ and } 1 < s_{j-1} \leq \lambda_j - \lambda_{j-1}. \end{aligned}$$

A desired algorithm(The 1st half of the γ -algorithm for Theorem C.2) for finding the sequence $S_1 = \text{Join}(T_1, T_2, \dots, T_r)$ is as follows:

The arithmetic computation formula for $\{d_1, n_1, \beta_{1,1}\}$ by (i), and $\{d_w, n_w, \hat{\Delta}_w : 2 \leq w \leq r\}$ by (ii), and $\{e_{\lambda_w} : 1 \leq w \leq r\}$ by (iii) in (C.2.2) will be most helpful for finding such an algorithm:

$$(C.2.3) \quad \begin{aligned} \text{(i)} \quad & d_1 = \gcd(n, \alpha_1) \text{ with } n = n_1 d_1 \text{ and } \alpha_1 = \beta_{1,1} d_1. \\ \text{(ii)} \quad & d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1}) \text{ with } d_{w-1} = n_w d_w \text{ and } \alpha_w - \alpha_{w-1} = \hat{\Delta}_w d_w. \\ \text{(iii)} \quad & e_{\lambda_1} = d_1 n_1 \Delta_1(\beta_{1,1}) \text{ and } e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \hat{\Delta}_w \text{ for } 2 \leq w \leq r. \end{aligned}$$

(The 1st half of the γ -algorithm for Theorem C.2)

Step 1 for the 1st half of the γ -algorithm Compute e_1 and e_{s_0} such that $e_1 = n$ and $e_{s_0} = \alpha_1$.

Compute $d_1 = \gcd(n, \alpha_1)$ with $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$, and then $e_{\lambda_1} = n_1 \beta_{1,1} d_1$.

Then, there are two subcases:

Subcase(1) If $d_1 = 1$, it is clear that $S_1 = T_1 = \{e_1, e_{s_0}\} = \{n, \alpha_1\}$.

Subcase(2) If $d_1 > 1$, take the next step.

Remark We can compute an integer s_0 such that $(s_0 - 1)n < \alpha_1 < s_0 n$, if necessary.

Assuming that $d_1 > 1$, for $w = 2, 3, \dots, r$, an elementary computational algorithm formula with α_w can be represented as follows:

Step w for the 1st half of the γ -algorithm Let $d_{w-1} > 1$ with $2 \leq w \leq r$.

Then, there are two cases:

Case(i) of Step w $d_{w-1} < \alpha_w - \alpha_{w-1} (\iff e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1})$.

Case(ii) of Step w $d_{w-1} > \alpha_w - \alpha_{w-1} (\iff e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1})$.

Case(i) of Step w for the 1st half of the γ -algorithm Let $d_{w-1} < \alpha_w - \alpha_{w-1}$.

Then, the algorithm for finding a subsequence $T_w = \{e_{(\lambda_{w-1}+1)}, e_{(\lambda_{w-1}+s_{w-1})}\}$ of S_1 can be uniquely represented as follows:

$$(C.2.4) \quad \begin{aligned} e_{(\lambda_{w-1}+1)} &= e_{(\lambda_{w-1})} + d_{w-1}, \\ e_{(\lambda_{w-1}+s_{w-1})} &= e_{(\lambda_{w-1})} + (\alpha_w - \alpha_{w-1}) \quad \text{without computing an integer } s_{w-1}. \end{aligned}$$

After the computation is done for Case(i) of Step w, there are two subcases for Case(i):

Subcase(1) If $d_w = 1$, then the computation of $S_1 = \text{Join}(T_1, T_2, \dots, T_w)$ is finished by $T_w = \{e_{(\lambda_{w-1}+1)}, e_{(\lambda_{w-1}+s_{w-1})}\} = \{e_{(\lambda_{w-1})} + d_{w-1}, e_{(\lambda_{w-1})} + (\alpha_w - \alpha_{w-1})\}$.

Subcase(2) If $d_w > 1$, take the next step, Step (w+1) for the 1st half of the γ -algorithm.

Remark Note by (C.2.3) that $d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1})$ with $d_{w-1} = n_w d_w$ and $\alpha_w - \alpha_{w-1} = \widehat{\Delta}_w d_w$. We can compute a unique integer s_{w-1} such that $(s_{w-1} - 1)n_w < \widehat{\Delta}_w < s_{w-1}n_w$ because $d_{w-1} < \alpha_w - \alpha_{w-1}$.

Case(ii) of Step w for the 1st half of the γ -algorithm Let $d_{w-1} > \alpha_w - \alpha_{w-1}$.

Then, the algorithm for finding a subsequence $T_w = \{e_{(\lambda_{w-1}+1)}, e_{(\lambda_{w-1}+s_{w-1})}\}$ of S_1 can be uniquely represented as follows:

$$(C.2.5) \quad \begin{aligned} e_{(\lambda_{w-1}+1)} &= e_{(\lambda_{w-1})} + (\alpha_w - \alpha_{w-1}), \\ e_{(\lambda_{w-1}+s_{w-1})} &= s_{w-1}e_{(\lambda_{w-1})} + d_{w-1} \quad \text{with an integer } s_{w-1} \text{ in (C.2.5.1)}. \end{aligned}$$

Noting that $d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1})$ with $d_{w-1} = n_w d_w$ and $\alpha_w - \alpha_{w-1} = \widehat{\Delta}_w d_w$,

$$(C.2.5.1) \quad (s_{w-1} - 1)\widehat{\Delta}_w < n_w \leq s_{w-1}\widehat{\Delta}_w \quad \text{for some } s_{w-1} \in \mathbb{N} \quad \text{because } 1 \leq \widehat{\Delta}_w < n_w.$$

After the computation is done for Case(ii) of Step w, there are two subcases for Case(ii):

Subcase(1) If $d_w = 1$, then the computation of $S_1 = \text{Join}(T_1, T_2, \dots, T_w)$ is finished by $T_w = \{e_{(\lambda_{w-1}+1)}, e_{(\lambda_{w-1}+s_{w-1})}\} = \{e_{(\lambda_{w-1})} + (\alpha_w - \alpha_{w-1}), s_{w-1}e_{(\lambda_{w-1})} + d_{w-1}\}$.

Subcase(2) If $d_w > 1$, take the next step, Step (w+1) for the 1st half of the γ -algorithm.

Example C.2.1 for the 1st half of the γ -algorithm in Theorem C.2:

Let the Puiseux expansion for $C_3(t)$ be given by

$$(C.2.6) \quad C_3(t) := \begin{cases} y = t^{45} \\ z = t^{60} + t^{65} + t^{71}. \end{cases}$$

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Prove by [I] that the above $C_3(t)$ is the standard Puiseux expansion, and by [II] that compute a finite sequence $S_1 = \{(g_3 \circ \tau_{\lambda_3})_{\text{singular part of the divisor}}\}_{\text{seq.}}$ directly. Note that S_1 is a subsequence of $S = \{(g_3 \circ \tau_{\lambda_3})_{\text{divisor}}\}_{\text{seq.}}$ where g_3 is the standard Puiseux polynomial $g_3 \in \mathbb{C}[y, z]$ of the recursive 3-type in Family(1) such that $V(g_3) \equiv C_3(t)$ (multi. seq.)

First, we compute The Arithmetic Computation Formula by [I], and next apply [I] to solve the problem by [II].

[I] By The Arithmetic Computation Formula, we can compute a finite sequence $\{(n_j, \hat{\Delta}_j) \in N^2 : j = 1, 2, 3, \}$ of pairs, each of which satisfies the following: Recall that $d_j = \gcd(d_{j-1}, \alpha_j - \alpha_{j-1})$ for $1 \leq j \leq 4$ where $d_0 = n$ and $\alpha_0 = 0$.

(1) Let $45 = n = d_1 n_1$ and $60 = \alpha_1 = d_1 \hat{\Delta}_1$ with $d_1 = \gcd(n, \alpha_1)$. Then, $d_1 = 15$, $n_1 = 3$ and $\beta_{1,1} = \hat{\Delta}_1 = 4$. Also, $e_{\lambda_1} = n_1 \beta_{1,1} d_1 = 180$.

(2) Let $15 = d_1 = d_2 n_2$ and $5 = \alpha_2 - \alpha_1 = d_2 \hat{\Delta}_2$ with $d_2 = \gcd(d_1, \alpha_2 - \alpha_1)$. Then, $d_2 = 5$, $n_2 = 3$ and $\hat{\Delta}_2 = 1$. Also, $e_{(\lambda_2)} = e_{(\lambda_1)} n_2 + d_2 n_2 \hat{\Delta}_2 = 180 \cdot 3 + 5 \cdot 3 \cdot 1 = 555$.

(3) Let $5 = d_2 = d_3 n_3$ and $6 = \alpha_3 - \alpha_2 = d_3 \hat{\Delta}_3$ with $d_3 = \gcd(d_2, \alpha_3 - \alpha_2)$. Then, $d_3 = 1$, $n_3 = 5$ and $\hat{\Delta}_3 = 6$. Also, $e_{(\lambda_3)} = e_{(\lambda_2)} n_3 + d_3 n_3 \hat{\Delta}_3 = 555 \cdot 5 + 1 \cdot 5 \cdot 6 = 2805$.

By (1), (2) and (3), it is clear that $n < \alpha_1 < \alpha_2 < \alpha_3$ and $n > d_1 > d_2 > d_3 = 1$, and so $C_3(t)$ is the standard Puiseux expansion.

Using an algorithm in Theorem C.2, to solve the problem is just to find $S_1 = \text{Join}(T_1, T_2, T_3)$ in the sense of Definition A.3 where $T_j = \{e_{\lambda_{j-1}+i} : i = 1, s_{j-1}\}$ for $1 \leq j \leq 3$ and $1 < s_{j-1} \leq \lambda_j - \lambda_{j-1}$. For notation, we write $\lambda_0 = 0$.

[II] For a complete solution of (C.2.2), using **the 1st half of the γ -algorithm** then it suffices to compute $S_1 = \text{Join}(T_1, T_2, T_3)$ easily where $T_j = \{e_{\lambda_{j-1}+i} : i = 1, s_{j-1}\}$ for $1 \leq j \leq 3$ and $1 < s_{j-1} \leq \lambda_j - \lambda_{j-1}$ by the following three steps: Note that $\lambda_0 = 0$ for notation.

Step 1: It is clear that $T_1 = \{e_1, e_{s_0}\} = \{45, 60\}$ where $e_1 = n = 45$ and $e_{s_0} = \alpha_1 = 60$.

Step 2: Note that $d_1 = 15 > 6 = \alpha_2 - \alpha_1$. So, by Case(ii) of Step 2, $e_{(\lambda_1+1)} = e_{\lambda_1} + \alpha_2 - \alpha_1 = 180 + 5 = 185$ and $e_{(\lambda_1+s_1)} = s_1 \cdot 180 + d_1 = 3 \cdot 180 + 15 = 555$ because $(s_1 - 1) \hat{\Delta}_2 < n_2 \leq s_1 \hat{\Delta}_2$ with $n_2 = 3$ and $\hat{\Delta}_2 = 1$ implies that $s_1 = 3$. So, $T_2 = \{e_{(\lambda_1+1)}, e_{(\lambda_1+s_1)}\} = \{185, 555\}$.

Step 3: Note that $d_2 = 5 < 6 = \alpha_3 - \alpha_2$. So, by Case(i) of Step 3, $e_{(\lambda_2+1)} = e_{\lambda_2} + d_2 = 555 + 5 = 560$ and $e_{(\lambda_2+s_2)} = e_{\lambda_2} + \alpha_3 - \alpha_2 = 555 + 6 = 561$. So, $T_3 = \{e_{(\lambda_2+1)}, e_{(\lambda_2+s_2)}\} = \{560, 561\}$.

Therefore, it can be found by the above three steps that $S_1 = \text{Join}(T_1, T_2, T_3)$.

§ C.3. The 2nd half of the γ -algorithm(Theorem C.3)

Theorem C.3(The 2nd half of the γ -algorithm: an algorithm for finding a function from Family(2) onto Family(5)).

Assumptions By using the same properties and notations as in Definition A.1, Definition A.2 and Definition A.3, let $S_1 = \{(g_r \circ \tau_{\xi})_{\text{singular part of divisor}}\}_{\text{seq.}}$ be a given finite sequence of $(g_r \circ \tau_{\xi})_{\text{singular part of divisor}}$, which is a subsequence of $S = \{(g_r \circ \tau_{\lambda_r})_{\text{divisor}}\}_{\text{seq.}} = \{e_i : 1 \leq i \leq \lambda_r\}$.

Let $S_1 = \text{Join}(T_1, T_2, \dots, T_r)$ where each T_i is a subsequence of two elements in S_1 , satisfying the following properties: Write $\lambda_0 = 1$, if necessary.

$$(C.3.1) \quad T_1 = \{e_i : i = 1, s_0\} \quad \text{with } 1 < s_0 \leq \lambda_1, \\ T_j = \{e_{\lambda_{j-1}+i} : i = 1, s_{j-1}\} \quad \text{for } 1 \leq j \leq r \text{ and } 2 < s_{j-1} \leq \lambda_j - \lambda_{j-1}.$$

Conclusions Given the sequence of singular part of divisor in (C.3.1), the problem is how to compute a desired algorithm(The 2nd half of the γ -algorithm for Theorem C.3) for finding the standard Puiseux expansion $C_r(t)$ in (C.2.1) such that $C_r(t)$ and $(g_r \circ \tau_{\lambda_r})_{\text{singular part of divisor}}$ have the same multiplicity sequence.

The arithmetic computation formula for $\{d_1, n_1, \beta_{1,1}\}$ by (i), and $\{d_w, n_w, \hat{\Delta}_w: 2 \leq w \leq r\}$ by (ii), and $\{e_{\lambda_w}: 1 \leq w \leq r\}$ by (iii) in (C.3.2) will be most helpful for finding such an algorithm:

$$(C.3.2) \quad \begin{aligned} \text{(i)} \quad & d_1 = \gcd(n, \alpha_1) \text{ with } n = n_1 d_1 \text{ and } \alpha_1 = \beta_{1,1} d_1. \\ \text{(ii)} \quad & d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1}) \text{ with } d_{w-1} = n_w d_w \text{ and } \alpha_w - \alpha_{w-1} = \hat{\Delta}_w d_w. \\ \text{(iii)} \quad & e_{\lambda_1} = d_1 n_1 \Delta_1(\beta_{1,1}) \text{ and } e_{\lambda_w} = e_{\lambda_{w-1}} n_w + d_w n_w \hat{\Delta}_w \text{ for } 2 \leq w \leq r. \end{aligned}$$

(The 2nd half of the γ -algorithm for Theorem C.3)

Step 1 for The 2nd half of the γ -algorithm Let $T_1 = \{e_1, e_{s_0}\}$ by (C.3.1). It is trivial that $n = e_1$ and $\alpha_1 = e_{s_0}$.

Compute $d_1 = \gcd(n, \alpha_1)$ with $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$, and then $e_{\lambda_1} = n_1 \beta_{1,1} d_1$.

Then, there are two subcases:

Subcase(1) If $d_1 = 1$, then the standard Puiseux expansion $C_1(t)$ can be computed by $y = t^n$ and $z = t^{\alpha_1}$.

Subcase(2) If $d_1 > 1$, take the next step, Step (2) for The 2nd half of the γ -algorithm.

Remark Note that there exists an integer s_0 such that $(s_0 - 1)n < \alpha_1 < s_0 n$ or $(s_0 - 1)n_1 < \beta_{1,1} < s_0 n_1$.

Assuming that $d_1 > 1$, for $w = 2, 3, \dots, r$, an elementary computational algorithm formula with α_w can be represented as follows:

Step w for The 2nd half of the γ -algorithm Let $d_{w-1} > 1$ with $2 \leq w \leq r$. By (C.3.1), consider

$$(C.3.3) \quad T_w = \{e_{\lambda_{w-1}+1}, e_{\lambda_{w-1}+s_{w-1}}\}, \text{ the } w\text{-th subsequence of } S_1.$$

Then, there are two cases:

Case(i) for Step w $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1}$.

Case(ii) for Step w $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1}$.

Case(i) of Step w for The 2nd half of the γ -algorithm Let $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} = d_{w-1}$.

Then, the algorithm for finding the exponent α_w of the standard Puiseux expansion and the coefficient e_{λ_w} of E_{λ_w} can be represented as follows:

$$(C.3.4) \quad \begin{aligned} d_{w-1} &= e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})}, \\ \alpha_w - \alpha_{w-1} &= e_{(\lambda_{w-1}+s_{w-1})} - e_{(\lambda_{w-1})}, \end{aligned}$$

noting that $d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1})$ with $d_{w-1} = n_w d_w$ and $\alpha_w - \alpha_{w-1} = \hat{\Delta}_w d_w$,

$$e_{\lambda_w} = n_w e_{\lambda_{w-1}} + n_w d_w \hat{\Delta}_w \text{ by (C.3.2).}$$

Remark Note that there exists an integer s_{w-1} such that $(s_{w-1} - 1)n_w < \hat{\Delta}_w < s_{w-1} n_w$.

After the computation is done for Case(i) of Step w, there are two subcases for Case(i):

Subcase(1) If $d_w = 1$, then the standard Puiseux expansion can be defined by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \dots + t^{\alpha_w}$ with $\alpha_w - \alpha_{w-1} = e_{(\lambda_{w-1}+s_{w-1})} - e_{(\lambda_{w-1})}$.

Subcase(2) If $d_w > 1$, take the next step, Step (w+1) for The 2nd half of the γ -algorithm.

Case(ii) of Step w for The 2nd half of the γ -algorithm Let $e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})} < d_{w-1}$.

Then, the algorithm for finding the exponent α_w of the standard Puiseux expansion and the coefficient e_{λ_w} of E_{λ_w} can be represented as follows:

$$(C.3.5) \quad \begin{aligned} \alpha_w - \alpha_{w-1} &= e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})}, \\ d_{w-1} &= e_{(\lambda_{w-1}+s_{w-1})} - s_{w-1} e_{(\lambda_{w-1})} \text{ with an integer } s_{w-1} \text{ in (C.3.5.1)}. \end{aligned}$$

Note that $d_w = \gcd(d_{w-1}, \alpha_w - \alpha_{w-1})$ with $d_{w-1} = n_w d_w$ and $\alpha_w - \alpha_{w-1} = \hat{\Delta}_w d_w$,

$$(C.3.5.1) \quad (s_{w-1} - 1)\hat{\Delta}_w < n_w \leq s_{w-1}\hat{\Delta}_w, \text{ and } e_{\lambda_w} = n_w e_{\lambda_{w-1}} + n_w d_w \hat{\Delta}_w \text{ by (1.13.2).}$$

Remark Note that there exists an integer s such that $(s_{w-1} - 1)\widehat{\Delta}_w < n_w \leq s_{w-1}\widehat{\Delta}_w$ because $1 \leq \widehat{\Delta}_w < n_w$.

After the computation is done for Case(ii) of Step w , there are two subcases for Case(ii):

Subcase(1) If $d_w = 1$, then the standard Puiseux expansion can be defined by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_w}$ with $\alpha_w - \alpha_{w-1} = e_{(\lambda_{w-1}+1)} - e_{(\lambda_{w-1})}$.

Subcase(2) If $d_w > 1$, take the next step, Step $(w+1)$ for (The 2nd half of the γ -algorithm).

Example C.3.1 for The 2nd half of the γ -algorithm in Theorem C.3:

Using the same properties and notation as in (C.3.1), let $S_1 = \text{Join}(T_1, T_2, T_3)$ where $T_1 = \{e_1, e_{s_0}\} = \{45, 75\}$, $T_2 = \{e_{\lambda_1+1}, e_{\lambda_1+s_1}\} = \{231, 690\}$, $T_3 = \{e_{\lambda_2+1}, e_{\lambda_2+s_2}\} = \{1158, 1159\}$ (with $\lambda_i < \lambda_i + s_i \leq \lambda_{i+1}$ for $i = 0, 1, 2$).

(1) First, Prove that there is $g_3 \in \text{Family}(1)$ such that S_1 is given by

$$(C.3.6) \quad S_1 = \{(g_r \circ \tau_{\lambda_r})_{\text{singular part of the divisor}}\}_{\text{seq.}} = \text{Join}(T_1, T_2, T_3).$$

(2) Next, using an algorithm in Theorem C.3, the problem is to find the standard Puiseux expansion $C_3(t)$ of the 3-th type for the above irreducible plane curve C such that $C_3(t) \equiv V(g_3)$ (multi. seq.) where $C_3(t)$ is given by $y = t^n$ and $z = t^{\alpha_1} + t^{\alpha_2} + t^{\alpha_3}$.

For a complete solution for (C.3.1), using **The 2nd half of the γ -algorithm for Theorem C.3** then it suffices to compute $S_1 = \text{Join}(T_1, T_2, T_3)$ easily where $T_j = \{e_{\lambda_{j-1}+i} : i = 1, s_{j-1}\}$ for $1 \leq j \leq 3$ and $1 < s_{j-1} \leq \lambda_j - \lambda_{j-1}$ by the following three steps: Note that $\lambda_0 = 0$ for notation.

Step 1: It is clear that $n = e_1 = 45$, and $\alpha_1 = e_{s_0} = 75$.

In preparation for the computation in Step 2, $e_{\lambda_1} = 225$ because $d_1 = \gcd(n, \alpha_1) = 15 > 1$ with $n = n_1 d_1$ and $\alpha_1 = \beta_{1,1} d_1$ implies that $n_1 = 3$, $\beta_{1,1} = 5$ and $e_{\lambda_1} = n_1 \beta_{1,1} d_1 = 225$ by (C.3.2).

Step 2: Since $e_{\lambda_1+1} - e_{\lambda_1} = 231 - 225 = 6 < 15 = d_1$, by Case(ii) of Step 2, $\alpha_2 - \alpha_1 = e_{\lambda_1+1} - e_{\lambda_1} = 6$. So, it is clear by Step 1 that $\alpha_2 = \alpha_1 + 6 = 75 + 6 = 81$.

In preparation for the computation in Step 3, $e_{\lambda_2} = n_2 \cdot e_{\lambda_1} + d_2 n_2 \widehat{\Delta}_2 = 5 \cdot 225 + 3 \cdot 5 \cdot 2 = 1125 + 30$ by (C.3.2) because of the following computations:

$d_2 = \gcd(d_1, \alpha_2 - \alpha_1) = \gcd(15, 6) = 3 > 1$ with $d_1 = n_2 d_2$ and $\alpha_2 - \alpha_1 = \widehat{\Delta}_2 d_2$ implies that $n_2 = 5$ and $\widehat{\Delta}_2 = 2$, and so $e_{\lambda_2} = 1155$.

Step 3: Since $e_{\lambda_2+1} - e_{\lambda_2} = 1158 - 1155 = 3 = d_2$, by Case(i) of Step 3, $\alpha_3 - \alpha_2 = e_{(\lambda_2+s_2)} - e_{\lambda_2} = 1159 - 1155 = 4$. Then it is clear by Step 2 that $\alpha_3 = \alpha_2 + 4 = 81 + 4 = 85$.

So, the standard Puiseux expansion $C_3(t)$ such that $C_3(t) \equiv V(f)$ (multi. seq.) can be given by $y = t^{45}$ and $z = t^{75} + t^{81} + t^{85}$, because $d_2 = 1$ implies that $d_3 = \gcd(d_2, \alpha_3 - \alpha_2) = 1$.

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